

Competition in fund management and forward relative performance criteria*

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May 2, 2022

Abstract

In an Itô-diffusion market, two fund managers trade under relative performance concerns. For both the asset specialization and diversification settings, we analyze the passive and competitive cases. We measure the performance of the managers' strategies via forward relative performance criteria, leading to the respective notions of forward best-response criterion and forward Nash equilibrium. The motivation to develop such criteria comes from the need to relax various crucial, but quite stringent for this problem, existing assumptions - such as, the a priori choices of both the market model and the investment horizon, the commonality of the latter for both managers as well as the full a priori knowledge of the competitor's policies for the best-response case. We focus on locally riskless criteria and deduce the random forward equations. We solve the CRRA cases, thus also extending the related results in the classical setting. An important by-product of the work herein is the development of forward performance criteria for investment problems in Itô-diffusion markets under the presence of correlated random endowment process for both the perfectly and the incomplete market cases.

*A considerable part of this work is part of the Ph.D. Thesis (2017) of the second author ([29]) and first appeared in <https://ssrn.com/abstract=2870040>. The work has been presented at conferences and seminars (Oxford, Brown University and National and Technical University of Athens). The authors would like to thank the participants for their valuable comments and suggestions.

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1 Introduction

Relative performance is of tantamount importance in both the mutual and hedge fund management industries. It impacts a variety of factors spanning from a company's reputation and net cash inflows to the incentive structure and promotion schedule for its managers. Such facts have been very well documented in the finance practice and have been extensively studied in academic research (see, among others, [18, 20, 28, 36, 61]).

While classifying the various kinds patterns of relative performance among managers is rather complex (classification by sectors, asset riskiness, market conditions, business cycles and others), there is a prevailing dichotomy based on *asset specialization* or *asset diversification*. In the former, the competing managers specialize in distinct asset classes while, in the latter, they invest in common ones.

Asset specialization stems from a variety of reasons like familiarity with a certain sector, reduction of costs to enhance knowledge of new stocks, trading costs and constraints ([15]), liquidity costs ([66]), and ambiguity aversion ([17, 48]). The above evidence has been well established in the empirical literature ([21, 25, 33]) and has been incorporated in several theoretical models ([2, 27, 44, 65]; see also [7, 21, 33, 44, 45, 63, 64, 66]).

In asset diversification, the motivation is mainly to increase the net money inflows from clients ([10, 22]). This setting is also more suitable to model relative performance concerns against a given benchmark portfolio (typically, a mix of asset classes). In a different direction, asset diversification also occurs in delegated portfolio management, where the role of one manager is replaced by the client ([57]). In a related family of models, it also appears in the so-called "catching up with the Joneses" literature (see [1, 30]).

Relative performance has been also considered in terms of how each manager reacts to the performance of a competitor. This interaction can be *passive (best-response)* in that the manager takes the competitors' policies as given (arbitrary but fixed) and trades accordingly, without any further interaction (see, among others, [18, 36, 38]). On the other hand, interaction may be also *competitive*, when managers compete with each other dynamically while investing among the various accounts ([20, 28]).

Whether managers compete within the same or different asset classes and/or interact in a passive or competitive manner, there are common underlying assumptions that might limit the applicability of the existing theoretical studies. The aim herein is to revisit some of these assumptions, propose an alternative approach, study the related optimization problems and develop a comparative study with previous works. We are motivated to do so not only by theoretical and conceptual arguments but, also, by various empirical works on asset managers' investment strategies and their competition. As we explain below, these empirical studies point out, for example, to dependencies of the observed managers' behaviors to dynamically evolving factors, a dependency that cannot be explained in traditional settings without violating the time-consistent principle; see, among others, [32, 36], where the effects of the current, and possibly non-anticipated, phase of the market on the managers' behavior is discussed.

The first such assumption, present in the existed theoretical models, is that the market model is a priori chosen for the entire duration of the investment activity; for example, [8, 9, 13] choose log-normal models. This step is ubiquitous for solving all underlying terminal horizon expected utility problems under relative performance criteria. However, such model commitment might not be entirely realistic, as model error and model decay occur frequently. Of course, genuinely dynamic model revision may be incorporated (for example, in the context of adaptive control), but then, the intertemporal consistency is not guaranteed, as it is the case in adaptive control models. We also note that even in the popular robust case, widely used to remedy model uncertainty and ambiguity, the existed literature on optimal investment strategies imposes a-priori-chosen restrictions on the plausible family of models (see among others [41]). Similar restrictions are also present in filtering, in that the associated observation process is also pre-chosen. We comment, however, that robust control and filtering models give time-consistent solutions but do not allow for dynamic revision of the family of possible models and the filters, respectively. Furthermore, in the view of related empirical studies, assuming a pre-chosen and fixed

market model may be particularly problematic when analysing investments under relative performance concerns. For example, it is shown in [36] that managers' risk-taking attitude and relative performance concerns are heavily affected when the market becomes bull or bear, and such a change might not have been captured in the initial model. Alongside, [23] emphasizes the effect of evolving changes of market's dynamics to the managers' optimal investment, their competition and performance.

The second assumption is related to the choices of the investment horizon and corresponding investment criterion. Regarding the investment horizon, in all existing works, it is assumed that the horizon is i) a single one (finite or infinite), ii) a priori chosen and iii) common across competitors (see, for example, [9, 13, 39]). In practice, however, this is frequently not the case. While it is customary for managers to report their performance at common standardized time intervals (e.g. quarterly, annually), they almost always have internal sub-horizons that depend on company-related factors and which are themselves difficult to model (for example empirical study in [6] indicates that manager's past performance could have a delayed effect to the inflows and outflows and hence the effective investment horizon is not necessarily the end of a calendar year).

Furthermore, even if a common horizon is a priori chosen, the investment activity does not terminate at this specific pre-assigned time, as managers always roll their positions from one investment horizon to the next. One could then argue that managers apply the same (or very similar) goals for the upcoming period. This, however, is not supported by existing empirical evidence which shows that managers always adapt their goals in a stochastic and rather complex manner. Indeed, [30], using an equilibrium asset-pricing model that is based on [1], empirically supports the "catching up with the Joneses" behavior which implies that investors adjust their investment goals depending on the evolving wealth of their "neighbours". In a similar spirit, the so-called tournament hypothesis (introduced by [16] and also tested by [19]), states that managers' investment strategies are affected by their realized relative performance in the first half of the year, which alludes to stochastically changing risk aversion forward in time. We also note that, In addition, [43] shows that managers' strategies do change after they experience bad past performances, while [20] and [61] add termination risk and relative under- or out-performance's size respectively, to the factors that stochastically affect managers' investment goals. The above empirical evidence is also supported by the survey-based results of [14], according to which, managers change risk-taking attitude stochastically based on their past performances.

The aforementioned empirical evidence on the managers' behavior under relative performance concerns emphasizes the need of a model which investment goals are stochastically updated and are flexible with regards to investment horizon. Such modelling considerations might not be achieved in the classical utility maximization problem without violating time consistency.

The third assumption is related to managers' interaction. It is always assumed that each manager has full, and for the entire investment horizon, knowledge of the competitor's policy, in that she knows the stochastic process that models the investment policy she competes against. This modeling input is needed in order to solve all associated expected utility problems but as an assumption, it is quite unrealistic. Indeed, not only the manager cannot have such foreknowing skills for her competitor, the competitor herself might not a priori know how her own strategy will be changing as the market unfolds.

Herein, we propose an alternative framework aiming to remedy some of the aforementioned shortcomings without violating the time-consistency principle. As the analysis shows, this framework also allows us to work with general Itô-diffusion market model (see (3)) under rather weak structural assumptions. We also make no specific assumption about any pre-specified investment horizon, allowing for each manager to invest till personalized discretionary times. Finally, we make no assumption on a priori choosing the stochastic process that yields the competitor's policy in the best-response case. Rather, we allow (besides mild integrability conditions) this policy to be dynamically known to her competitor. For tractability, we only consider the case with two managers. The general case for N -managers as well as the mean field game limit are left for future research¹.

¹One may consider the case of infinite competitors and build the notion of a "forward" mean-field game (MFG). However, constructing such a notion is not immediate as various concepts might not "carry over", especially when there is common noise and/or the forward MFG performance process has its own volatility in a general Itô-diffusion setting. To date, the proper definition of a forward MFG has not been produced, and neither the convergence of the forward

The new framework is built on extensions of the so-called *forward* performance criteria. These criteria were introduced by one of the authors and M. Musiela in both discrete [50] and continuous time [53] and further developed by others (see [49, 51] and [67], as well as [3] and [62] for discrete settings). In the continuous time framework we work herein, they are modeled as stochastic processes, say $(U(x, t))_{t \geq 0}$, that continuously adapt to the market information, are (local) supermartingales along all admissible policies and (local) martingales along an optimal policy. These properties stem from the dynamic programming principle, the foundational result in stochastic optimization. We note, however, that contrary to the classical expected utility framework, an axiomatic construction of the forward criterion is still lacking. This is being currently investigated in [58].

To characterize forward criteria in such markets, a stochastic PDE was proposed in [52]. Depending on the choice of its volatility and structural parametrizations, various forms of $U(x, t)$ have been studied (see, among others, [5, 42, 60]). However, several questions remain open as the underlying stochastic optimization problems are ill-posed, fully non-linear and degenerate.

To build forward criteria that allow for interaction - passive or competitive - between two decision makers, we proceed as follows. Let us assume that each manager uses admissible policies α and β , generating wealths X_1^α and X_2^β .

For the case of best-response, we introduce the *best-response forward* criterion for manager 1 as a process $U_1(x_1, x_2, t; \beta)$ such that $U_1(X_1^\alpha, X_2^\beta, t; \beta)$ is a (local) supermartingale for each admissible policy α and becomes a (local) martingale, $U_1(X_1^{\alpha^*}, X_2^\beta, t; \beta)$, along an optimal α^* . We stress that in contrast to *all* classical cases, the competitor's policy process β is *not* pre-assumed. Rather it is being revealed in "real time" and, in turn, the performance criterion $U_1(x_1, x_2, t; \beta)$ adapts to it dynamically. As mentioned earlier, this differs from the assumptions in the classical settings; see, for example, [8]) where a manager has to fully know the strategy that her competitor is going to apply until the terminal time, in order to be able to solve the best-response problem. This assumption can be withdrawn in the forward setting. The best-response forward criterion for manager 2, $U_2(x_1, x_2, t; \alpha)$, is defined analogously, and with the competitor's policy process α also not a priori known.

For the case of competitive interaction between the managers, we introduce a *forward Nash equilibrium* criterion, consisting of two pairs $(U_1(x_1, x_2, t; \beta)_{t \geq 0}, (\alpha_t^*)_{t \geq 0})$ and $(U_2(x_1, x_2, t; \alpha)_{t \geq 0}, (\beta_t^*)_{t \geq 0})$ such that the processes $U_1(X_1^{\alpha^*}, X_2^\beta, t; \beta)$ and $U_1(X_1^\alpha, X_2^{\beta^*}, t; \alpha)$ are (local) supermartingales, while $U_1(X_1^{\alpha^*}, X_2^{\beta^*}, t; \alpha^*)$ and $U_2(X_1^{\alpha^*}, X_2^{\beta^*}, t; \beta^*)$ are (local) martingales.

For each kind of interaction, based on best response or on competition, we analyze both the asset specialization and the asset diversification cases. Herein, we focus on forward criteria that are locally riskless processes, namely, of the form

$$dU_1(x_1, x_2, t; \beta) = b_1(x_1, x_2, t; \beta) dt \quad \text{and} \quad dU_2(x_1, x_2, t; \alpha) = b_2(x_1, x_2, t; \alpha) dt, \quad (1)$$

for suitable adapted processes $(b_1(x_1, x_2, t; \beta))_{t \geq 0}$ and $(b_2(x_1, x_2, t; \alpha))_{t \geq 0}$. We choose this class because, in the absence of relative concerns, locally riskless forward criteria were the first to be extensively analyzed not only because of their tractability but, also, for the valuable intuition in terms of numéraire choice, time monotonicity of preferences, dependence on market performance, and others (see [52] for details).

Throughout, we model the market having one riskless bond and two risky securities representing proxies of two asset classes. Such proxies have been consistently used in the literature (see, for example, [9, 31, 37]). We model their prices as Itô-diffusion processes (cf. (3)) but we stress, once more, that their coefficients are not a priori chosen but, rather, become known gradually, infinitesimally in time, as the market evolves.

When managers invest in isolation, their wealths evolve as in (5) and in (36), (37), for the asset specialization and diversification cases, respectively. Under relative performance, the competitor's wealth needs to be incorporated. One way to do this was proposed in [8, 9, 10, 37]), which we also

finite game to the forward MFG. Formally, one may mimic the definitions herein (see also [29]) and the ones in [39] for the classical case, and calculate a special solution within the CARA functions (see [56]).

adopt herein. Namely, we introduce the *relative wealth* processes $(\tilde{X}_{1,t})_{t \geq 0}$ and $(\tilde{X}_{2,t})_{t \geq 0}$, with

$$\tilde{X}_1 := \frac{X_1}{X_2^{\theta_1}} \quad \text{and} \quad \tilde{X}_2 := \frac{X_2}{X_1^{\theta_2}},$$

where the *competition biases* $\theta_1, \theta_2 \in [0, 1]$ model the degree of relative performance considerations.

The limiting case $\theta_1 = 0$ (resp. $\theta_2 = 0$) expresses that manager 1 (resp. 2) is not at all concerned with the output of the opponent. The other limiting case, $\theta_1 = 1$ (resp. $\theta_2 = 1$) corresponds to the traditional relative performance in terms of a benchmark (such as S&P500 index, collection of index funds, and others; see, for example, the related discussion in [9, Section 1]).

The form of the relative state dynamics \tilde{X}_1 and \tilde{X}_2 , see (9), (11) for the asset specialization and (40), (42) for the asset diversification cases, prompts us to introduce “*personalized*” *fictitious* markets and define the relevant forward criteria within. Informationally, the original and these virtual markets do not differ but the forward criteria may have different characteristics, depending on the choice of the modified state variables.

The above choice of \tilde{X}_1 and \tilde{X}_2 suggests to develop criteria of the reduced scaled form, namely,

$$U_1(x_1, x_2, t; \beta) = V_1 \left(\frac{x_1}{x_2^{\theta_1}}, t; \beta \right) \quad \text{and} \quad U_2(x_1, x_2, t; \alpha) = V_2 \left(\frac{x_2}{x_1^{\theta_2}}, t; \alpha \right), \quad (2)$$

$\tilde{x}_1 = x_1/x_2^{\theta_1}$, $\tilde{x}_2 = x_2/x_1^{\theta_2}$, for suitable processes $(V_1(\tilde{x}_1, t; \beta))_{t \geq 0}$ and $(V_2(\tilde{x}_2, t; \alpha))_{t \geq 0}$. Other forms of relative criteria may be introduced depending on admissibility domains, type of risk preferences, etc. (see, for example, additive cases in [13, 24, 26, 39]).

In the asset specialization case, neither manager may invest in the asset class of the competitor. As a result, the market (be the original or the fictitious ones) is incomplete. Forward criteria for incomplete markets have been developed before but only when incompleteness comes exclusively from imperfectly correlated stochastic factors affecting the stocks’ dynamics (see, among others, [4, 40, 42, 60]). Herein, however, the kind of incompleteness is different, for it is generated by the specialization constraints. These constraints alter the relative wealth processes \tilde{X}_1 and \tilde{X}_2 in a way that the related dynamics may be interpreted as either including non-zero constrained allocations (cf. (9) and (11)) or, alternatively, having a stream with imperfectly correlated return (cf. (14) and (16)). The former interpretation is closer to the original formulation herein. The latter has a different scope. It shows how the forward criteria under asset specialization may be used to define analogous criteria for problems with (imperfectly correlated) random endowment process (also called stochastic income stream). This is a new class of forward criteria, not been considered so far.

We analyze both the best-response and the competition cases, and introduce the corresponding *best-response forward* relative performance criteria. The definitions extend the original ones in [49]. We, in turn, derive a random PDE (cf. (17)) that the (locally riskless) criterion is expected to satisfy. Its coefficients are adapted processes, and depend on both the market dynamics and the competitor’s policies. In general, equation (17) is not tractable unless for the homothetic class, which we solve. Nevertheless, its solution is used to derive and represent the optimal policies in a stochastic feedback form (see (18)).

When dynamic competition is allowed, this naturally leads to the new concept of *forward Nash equilibrium*, which we introduce in Definition 5. To derive the equilibrium policies, one needs to solve a system of equations, in general intractable due to interdependent nonlinearities. The homothetic case is solvable and we provide the relevant policies. In fact, their form resembles the ones in the log-normal and CRRA case studied in [9], but it is now derived, under the new criteria, in the general Itô-diffusion setting.

In the asset diversification case, both managers invest in a common market. Their relative performance concerns distort the original wealth processes (cf. (40) and (42)) which, as in the previous case, leads to two distinct personalized fictitious markets, each depending on the individual competition parameter and the policy of the opponent. Now, however, each of these markets is complete as investment is allowed in both stocks with modified risk premia. Forward criteria may, in turn, be defined as

in the asset specialization case and we focus again on locally riskless ones. The completeness of the markets enables us to extend the results of [52] and characterize both the relative performance and Nash equilibrium criteria, their optimal wealth and investment policies in full generality. The special case of homothetic criteria is also analyzed and as in the asset diversification case, the equilibrium strategies are in line with the simplified CRR-normal model of [9]. In other words, the related findings could be applied to much more general situations as long as we formulate the managers' investment goals with homothetic forward criteria.

Conceptually, the analyses of the asset specialization and the asset diversification cases are rather similar, in terms of the associated fictitious markets and the optimality criteria. The fundamental difficulty is in their (in)completeness which affects the tractability of the problem and the form of the optimal policies. A key difference is that the locally riskless forward criteria in the asset diversification case are always time-decreasing while, in the asset specialization case, they are not. We further elaborate on this later on.

We conclude the introductory section mentioning that an underlying assumption herein - which is also widely present in the classical literature - is that the managers have common information for both the market and the opponent's strategies; we refer the reader to [8, 11] for discussion of supporting arguments for this assumption. While the access to such information is much more realistic in our setting (as it occurs in "real-time" and does not require a priori knowledge of the entire process), the fact that both managers share common access to it is, in our view, a rather stringent requirement. As the focus herein is to develop the new, forward framework with relative performance, we also adopt this assumption. We provide ideas how to relax it and future research in this direction in section 4.

The paper is organized as follows. In section 2 we present the asset specialization case and analyze the forward best-response and the forward Nash equilibrium cases. In section 3, we analyze the asset diversification case while in section 4 we conclude and comment on possible extensions.

2 Asset specialization and forward competition

The market consists of one (locally) riskless asset and two risky securities, representing proxies of two distinct asset classes. The prices of the risky securities, $(S_{1,t})_{t \geq 0}$ and $(S_{2,t})_{t \geq 0}$ are Itô-diffusions solving

$$\frac{dS_1}{S_1} = \mu_1 dt + \sigma_1 dW_1 \quad \text{and} \quad \frac{dS_2}{S_2} = \mu_2 dt + \sigma_2 dW_2, \quad (3)$$

with $S_{1,0}, S_{2,0} > 0$. The processes $(W_{1,t})_{t \geq 0}, (W_{2,t})_{t \geq 0}$ are standard Brownian motions on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$, with correlation coefficient $\rho \in (-1, 1)$ and \mathcal{F}_t being the filtration generated by (W_1, W_2) . The market coefficients $(\mu_{i,t})_{t \geq 0}, (\sigma_{i,t})_{t \geq 0}$, $i = 1, 2$, are \mathcal{F}_t -adapted processes with values in \mathbb{R} and \mathbb{R}_+ , respectively. The riskless asset is a money market account $(B_t)_{t \geq 0}$ offering positive interest rate $(r_t)_{t \geq 0}$, an \mathcal{F}_t -adapted process.

We denote this *original* market by $\mathcal{M} = (B, S_1, S_2)$. The related market price of risk processes, $(\lambda_{1,t})_{t \geq 0}$ and $(\lambda_{2,t})_{t \geq 0}$, are given by

$$\lambda_1 = \frac{\mu_1 - r}{\sigma_1} \quad \text{and} \quad \lambda_2 = \frac{\mu_2 - r}{\sigma_2}, \quad (4)$$

and assumed to be bounded processes, $0 < c \leq \lambda_1, \lambda_2 \leq C < \infty$, $t \geq 0$, for some (possibly deterministic) constants c, C .

In this market environment, we consider two asset managers, indexed by $i = 1, 2$. They *specialize* in assets S_1 and S_2 , respectively, in that manager 1 (resp. 2) trades between the riskless asset and S_1 (resp. S_2). However, both managers have access to the common filtration $(\mathcal{F}_t)_{t \geq 0}$ (as for example in [11] and [12]).

We denote by $(X_{1,t})_{t \geq 0}, (X_{2,t})_{t \geq 0}$ the wealths of manager 1 and 2 and by $(\alpha_t)_{t \geq 0}$ and $(\beta_t)_{t \geq 0}$ the corresponding self-financing strategies in assets S_1 and S_2 . Then, (3) yields

$$\frac{dX_1}{X_1} = \sigma_1 \alpha (\lambda_1 dt + dW_1) \quad \text{and} \quad \frac{dX_2}{X_2} = \sigma_2 \beta (\lambda_2 dt + dW_2), \quad (5)$$

with $X_{i,0} = x_i > 0$, $i = 1, 2$; herein, X_1, X_2, α, β are expressed in discounted (by the riskless asset) units.

The set of admissible policies \mathcal{A}_1 and \mathcal{A}_2 , of manager 1 and 2, respectively, are defined for $(\pi_t)_{t \geq 0} = (\alpha_t)_{t \geq 0}, (\beta_t)_{t \geq 0}$, and $i = 1, 2$,

$$\mathcal{A}_i = \left\{ \pi : \pi_t \in \mathcal{F}_t, \mathbb{E} \left[\int_0^t \sigma_{i,s}^2 \pi_s^2 ds \right] < \infty \text{ and } X_i > 0, t > 0 \right\}. \quad (6)$$

The wealth positivity constraint is in accordance to what is frequently observed in the asset management industry (for instance, mutual funds cannot have negative wealth). The measurability of the individual investment policies reflects the access by both managers to the common information generated by \mathcal{F}_t (see section 4 for a discussion on this assumption).

We work with *relative wealth* processes with competition parameters $\theta_1, \theta_2 \in (0, 1]$ following the framework of [8, 9, 64]). Specifically, if manager 2 follows an arbitrary strategy $\beta \in \mathcal{A}_2$ generating wealth X_2 , the *relative wealth* of manager 1, $(\tilde{X}_{1,t})_{t \geq 0}$, is defined as

$$\tilde{X}_1 := \frac{X_1}{X_2^{\theta_1}}, \quad (7)$$

with X_1 and X_2 solving (5). Symmetrically, the relative wealth of manager 2, $(\tilde{X}_{2,t})_{t \geq 0}$, given an arbitrary strategy $\alpha \in \mathcal{A}_1$ of manager 1 generating wealth X_1 , is defined as

$$\tilde{X}_2 := \frac{X_2}{X_1^{\theta_2}}. \quad (8)$$

As discussed in the introduction, we introduce three new modeling elements. Firstly, while we make the structural model assumption (3), we do not pre-choose (at initial time) the processes $\mu_i, \sigma_i, i = 1, 2$. Secondly, in a similar manner, we do *not* assume that the competitors' policies, α and β , are a priori chosen stochastic processes. Rather, each manager learns the market coefficients and the opponent's strategy as time unfolds. Thirdly, there is no pre-chosen investment horizon.

The biased benchmark processes $(X_1^{\theta_2})_{t \geq 0}$ and $(X_2^{\theta_1})_{t \geq 0}$ solve, for $\theta_1, \theta_2 \in (0, 1]$,

$$\frac{dX_1^{\theta_2}}{X_1^{\theta_2}} = \sigma_1 \theta_2 \alpha_1 \left(\left(\lambda_1 - \frac{1}{2} (1 - \theta_1) \sigma_1 \alpha \right) dt + dW_1 \right)$$

and

$$\frac{dX_2^{\theta_1}}{X_2^{\theta_1}} = \sigma_2 \theta_1 \beta_2 \left(\left(\lambda_2 - \frac{1}{2} (1 - \theta_1) \sigma_2 \beta \right) dt + dW_2 \right).$$

In turn, the relative wealth \tilde{X}_1 satisfies

$$\frac{d\tilde{X}_1}{\tilde{X}_1} = \sigma_1 \alpha \left(\tilde{\lambda}_{1,1} dt + dW_1 \right) - \sigma_2 \theta_1 \beta \left(\tilde{\lambda}_{1,2} dt + dW_2 \right), \quad (9)$$

and $\tilde{X}_{1,0} = x_1/x_2^{\theta_1}$, $x_1, x_2 > 0$, with the processes $(\tilde{\lambda}_{1,1,t})_{t \geq 0}$ and $(\tilde{\lambda}_{1,2,t})_{t \geq 0}$,

$$\tilde{\lambda}_{1,1} := \lambda_1 - \rho \sigma_2 \theta_1 \beta \quad \text{and} \quad \tilde{\lambda}_{1,2} := \lambda_2 - \frac{1}{2} \sigma_2 (1 + \theta_1) \beta. \quad (10)$$

Symmetrically, the relative wealth \tilde{X}_2 satisfies

$$\frac{d\tilde{X}_2}{\tilde{X}_2} = -\sigma_1 \theta_2 \alpha \left(\tilde{\lambda}_{2,1} dt + dW_1 \right) + \sigma_2 \beta \left(\tilde{\lambda}_{2,2} dt + dW_2 \right), \quad (11)$$

and $\tilde{X}_{2,0} = \frac{x_2}{x_1^{\theta_2}}$, $x_1, x_2 > 0$, with the processes $(\tilde{\lambda}_{2,1,t})_{t \geq 0}$ and $(\tilde{\lambda}_{2,2,t})_{t \geq 0}$,

$$\tilde{\lambda}_{2,1} := \lambda_1 - \frac{1}{2} \sigma_1 (1 + \theta_2) \alpha \quad \text{and} \quad \tilde{\lambda}_{2,2} := \lambda_2 - \rho \sigma_1 \theta_2 \alpha. \quad (12)$$

We may now interpret the relative wealth dynamics (9) as follows. In the original market $\mathcal{M} = (B, S_1, S_2)$, manager 1 chooses (proportional) risky allocations $(\alpha, 0)$ in securities S_1 and S_2 , due to specialization. In the relative formulation, it is as if she invests in a “personalized” *fictitious* market² $\tilde{\mathcal{M}}_1^s := (B, \tilde{S}_{1,1}, \tilde{S}_{1,2})$ with (pseudo) stocks $(\tilde{S}_{1,1,t})_{t \geq 0}$, $(\tilde{S}_{1,2,t})_{t \geq 0}$ solving (in discounted units)

$$\frac{d\tilde{S}_{1,1}}{\tilde{S}_{1,1}} = \sigma_1 \left(\tilde{\lambda}_{1,1} dt + dW_1 \right) \quad \text{and} \quad \frac{d\tilde{S}_{1,2}}{\tilde{S}_{1,2}} = \sigma_2 \left(\tilde{\lambda}_{1,2} dt + dW_2 \right), \quad (13)$$

with modified Sharpe ratios $\tilde{\lambda}_{1,1}$ and $\tilde{\lambda}_{1,2}$ defined in (10). In this virtual market, the original specialization constraint is not binding, as the manager may now invest in both risky securities, $\tilde{S}_{1,1}$ and $\tilde{S}_{1,2}$, with respective proportional allocations $(\alpha, -\theta_1\beta)$, with only α being controlled by manager 1. The constrained allocation $-\theta_1\beta$ depends on both managers’ characteristics, statically on the bias parameter θ_1 (chosen by manager 1) and dynamically on β (chosen by manager 2).

Alternatively, we may view (9) as wealth dynamics in market $\tilde{\mathcal{M}}_1^s$ where manager 1 invests in the riskless security B and chooses ratio α to allocate in the fictitious stock $\tilde{S}_{1,1}$, while he receives a *process of random endowment* returns $(Y_{1,t})_{t \geq 0}$, namely,

$$\frac{d\tilde{X}_1}{\tilde{X}_1} = \sigma_1 \alpha \left(\tilde{\lambda}_{1,1} dt + dW_1 \right) + dY_1, \quad (14)$$

with

$$dY_1 = -\sigma_2 \theta_1 \beta \left(\tilde{\lambda}_{1,2} dt + dW_2 \right) = -\theta_1 \beta \frac{d\tilde{S}_{1,2}}{\tilde{S}_{1,2}},$$

with $Y_{1,0} = 0$. Note that Y_1 is driven only by W_2 and its dynamics do not depend on $\lambda_1, \sigma_1, \alpha$.

Analogous interpretations may be derived for manager 2, who now invests in the “personalized” fictitious market $\tilde{\mathcal{M}}_2^s := (B, \tilde{S}_{2,1}, \tilde{S}_{2,2})$ with (pseudo) stocks $(\tilde{S}_{2,1,t})_{t \geq 0}$ and $(\tilde{S}_{2,2,t})_{t \geq 0}$, solving

$$\frac{d\tilde{S}_{2,1}}{\tilde{S}_{2,1}} = \sigma_1 \left(\tilde{\lambda}_{2,1} dt + dW_1 \right) \quad \text{and} \quad \frac{d\tilde{S}_{2,2}}{\tilde{S}_{2,2}} = \sigma_2 \left(\tilde{\lambda}_{2,2} dt + dW_2 \right), \quad (15)$$

with modified Sharpe ratios $\tilde{\lambda}_{2,1}$ and $\tilde{\lambda}_{2,2}$ as in (12). We may then interpret (11) as the outcome of investing ratio β in stock $\tilde{S}_{2,2}$ while maintaining (ratio) allocation $-\theta_2\alpha$ in stock $\tilde{S}_{2,1}$. Alternatively,

$$\frac{d\tilde{X}_2}{\tilde{X}_2} = \sigma_2 \beta \left(\tilde{\lambda}_{2,2} dt + dW_2 \right) + dY_2, \quad (16)$$

with $(Y_{2,t})_{t \geq 0}$ solving

$$dY_2 = -\sigma_1 \theta_2 \alpha \left(\tilde{\lambda}_{2,1} dt + dW_1 \right) = -\theta_2 \alpha \frac{d\tilde{S}_{2,1}}{\tilde{S}_{2,1}},$$

with $Y_{2,0} = 0$.

Clearly, the personalized fictitious markets $\tilde{\mathcal{M}}_1^s$ and $\tilde{\mathcal{M}}_2^s$ do not coincide due to the asymmetry in both the competition parameters θ_1 and θ_2 , and the competitors’ allocations α and β . As the original market \mathcal{M} , the specialization constraints make both these markets *incomplete*. Note also that, in formationally, the markets $\mathcal{M}, \tilde{\mathcal{M}}_1^s$ and $\tilde{\mathcal{M}}_2^s$ do not differ but, conceptually, forward performance criteria are developed within $\tilde{\mathcal{M}}_1^s$ and $\tilde{\mathcal{M}}_2^s$.

²The superscript “s” corresponds to specialization.

2.1 Best-response forward relative performance criterion

Each manager invests between the riskless asset and the stock in which she specializes. She also competes with her opponent *passively*, in the sense that she observes and takes into account the competitor's policy but without interacting with him. In contrast to all existing settings, however, the competitor's policy is not a priori modeled, it is only taken to be a process in the admissible set \mathcal{A}_2 , and is being revealed by the competitor gradually, as time moves. To model, measure and optimize in this relative performance setting, we first introduce a suitable criterion. It extends the original forward criterion, proposed by Musiela and Zariphopoulou (see [49, 50]) and further developed by them and others (see, [35, 51, 52, 53, 54, 55, 67]).

Throughout, we will be working with the following set of random functions in the domain $\mathbb{D} = \mathbb{R}_+ \times \mathbb{R}_+$.

Definition 1 Let \mathcal{U} be the set of random functions $u(z, t)$, $(z, t) \in \mathbb{D}$, such that, for each $t \geq 0$ and \mathbb{P} -a.s., the mapping $z \rightarrow u(z, t)$ is strictly concave and strictly increasing, and $u(z, t) \in \mathcal{C}^{4,1}$.

Definition 2 Let policy $\beta \in \mathcal{A}_2$. An \mathcal{F}_t -adapted process $(V_1(\tilde{x}_1, t; \beta))_{t \geq 0}$, $\tilde{x}_1 \geq 0$, is called a best-response forward relative performance criterion for manager 1 if the following conditions hold:

- i) For each $t \geq 0$, $V_1(\tilde{x}_1, t; \beta) \in \mathcal{U}$ a.s.
- ii) For each $\alpha \in \mathcal{A}_1$, $V_1(\tilde{X}_1, t; \beta)$ is a (local) supermartingale, where \tilde{X}_1 solves (9) with α being used.
- iii) There exists $\alpha^* \in \mathcal{A}_1$, such that $V_1(\tilde{X}_1^*, t; \beta)$ is a (local) martingale, where \tilde{X}_1^* solves (9) with α^* being used.

Analogously, we define the best-response forward relative performance for manager 2, $V_2(\tilde{x}_2, t; \alpha)$, $\tilde{x}_2 \geq 0$ and $\alpha \in \mathcal{A}_1$, requiring that $V_2(\tilde{X}_2, t; \alpha)$ and $V_2(\tilde{X}_2^*, t; \alpha)$ are, respectively, a (local) supermartingale for any $\beta \in \mathcal{A}_2$ and a (local) martingale for an optimal $\beta^* \in \mathcal{A}_2$. The notational presence of β in V_1 and α in V_2 is self-evident.

Definition 2 implies in particular that the investment goals under relative forward performance criteria do not depend on any terminal time-horizon and are stochastically updated with the information generated by the stock price processes and the competitor's strategy. In sharp contrast to the classic expected utility maximization setting, the formation of these investment goals does not require the knowledge of competitor's strategy up to some specific time-horizon (see also the general form of the optimal strategy (18) and the corresponding one under the CRRA case (24)).

In the absence of competition and for Itô-diffusion markets, forward performance criteria have been constructed also as Itô-diffusion processes (cf. [51]). However, contrary to the classical expected utility case, their volatility process is an "investor-specific" modeling input. For a chosen volatility process, the supermartingality and martingality properties impose conditions on the drift of the forward criterion. Under enough regularity, these conditions lead to the forward performance SPDE (see [53]), which is a fully nonlinear infinite dimensional equation. Depending on whether the forward process is path-dependent or a deterministic functional of stochastic factors, the forward volatility can be chosen to be path- or state-dependent (see, for example, [34, 35, 42, 54, 55, 60]). In general, the underlying problems are inherently ill-posed and extra analysis is required to identify the viable initial conditions (see, for example, [12, 52]).

As mentioned in the introduction, we will work with *locally riskless* (no volatility) performance processes,

$$dV_1(\tilde{x}_1, t; \beta) = b_1(\tilde{x}_1, t; \beta) dt \quad \text{and} \quad dV_2(\tilde{x}_2, t; \alpha) = b_2(\tilde{x}_2, t; \alpha) dt,$$

for some suitably chosen \mathcal{F}_t -adapted processes $(b_1(\tilde{x}_1, t; \beta))_{t \geq 0}$ and $(b_2(\tilde{x}_2, t; \alpha))_{t \geq 0}$.

Next, we provide a characterization result for the best-response forward performance criterion $V_1(\tilde{x}_1, t; \beta)$. Similar results may be derived for manager 2 and are, thus, omitted. Throughout, it is assumed that $\rho^2 \neq 1$, as the case $\rho^2 = 1$ is more natural for the asset diversification setting.

Proposition 3 Let $\beta \in \mathcal{A}_2$, $\rho^2 \neq 1$, and $\tilde{\lambda}_{1,1}$ and $\tilde{\lambda}_{1,2}$ as in (10). Consider the random PDE

$$v_t - \frac{1}{2} \tilde{\lambda}_{1,1}^2 \frac{v_z^2}{v_{zz}} + \frac{1}{2} (1 - \rho^2) \theta_1^2 \sigma_2^2 \beta^2 z^2 v_{zz} + \left(\rho \tilde{\lambda}_{1,1} - \tilde{\lambda}_{1,2} \right) \theta_1 \sigma_2 \beta z v_z = 0, \quad (17)$$

for $(z, t) \in \mathbb{D}$, and assume that a solution $v(z, t) \in \mathcal{U}$ exists, for some admissible initial datum $v(z, 0) = V_1(z, 0; \beta)$. Furthermore, let the process $(\alpha_t^*)_{t \geq 0}$ be given by

$$\alpha^* = \alpha^*(\tilde{X}_1^*, t),$$

with the random function $\alpha^*(z, t)$, $(z, t) \in \mathbb{D}$, defined as

$$\alpha^*(z, t) = \frac{\tilde{\lambda}_{1,1}}{\sigma_1} R_1(z, t) + \rho \frac{\sigma_2}{\sigma_1} \theta_1 \beta, \quad (18)$$

with

$$R_1(z, t) := -\frac{v_z(z, t)}{z v_{zz}(z, t)}, \quad (19)$$

and $(\tilde{X}_{1,t}^*)_{t \geq 0}$ solving (5) with the control process α^* being used. If \tilde{X}_1^* is well defined and $\alpha^* \in \mathcal{A}_1$, then the process

$$V_1(\tilde{x}_1, t; \beta) := v(\tilde{x}_1, t),$$

$\tilde{x}_1 \geq 0$, is a locally riskless best-response forward relative performance criterion and the investment strategy α^* is optimal.

Proof. We first rewrite (9) as

$$\frac{d\tilde{X}_1}{\tilde{X}_1} = \sigma_1 \hat{\alpha} \left(\tilde{\lambda}_{1,1} dt + dW_1 \right) + \theta_1 \sigma_2 \beta \left(\left(\rho \tilde{\lambda}_{1,1} - \tilde{\lambda}_{1,2} \right) dt - \sqrt{1 - \rho^2} dW_1^\perp \right), \quad (20)$$

for W_1^\perp being a standard Brownian motion orthogonal to W^1 and the modified policy $(\hat{\alpha}_t)_{t \geq 0}$,

$$\hat{\alpha} := \alpha - \rho \theta_1 \frac{\sigma_2}{\sigma_1} \beta. \quad (21)$$

Assuming that $v(z, t) \in \mathcal{U}$, Ito's formula yields

$$\begin{aligned} dv(\tilde{X}_1, t) &= v_t(\tilde{X}_1, t) dt + \left(\frac{1}{2} \sigma_1^2 \hat{\alpha}^2 \tilde{X}_1^2 v_{zz}(\tilde{X}_1, t) + \tilde{\lambda}_{1,1} \hat{\alpha} \tilde{X}_1 v_z(\tilde{X}_1, t) \right) dt \\ &\quad + \left(\frac{1}{2} (1 - \rho^2) (\sigma_2 \theta_1 \beta)^2 \tilde{X}_1^2 v_{zz}(\tilde{X}_1, t) + \left(\rho \tilde{\lambda}_{1,1} - \tilde{\lambda}_{1,2} \right) \sigma_2 \theta_1 \beta \tilde{X}_1 v_z(\tilde{X}_1, t) \right) dt \\ &\quad + v_z(\tilde{X}_1, t) \left(\sigma_1 \tilde{\alpha} dW_1 - \sigma_2 \theta_1 \beta \sqrt{1 - \rho^2} dW_1^\perp \right). \end{aligned}$$

Note that for $v_{zz} < 0$, we have

$$\frac{1}{2} \sigma_1^2 \hat{\alpha}^2 \tilde{X}_1^2 v_{zz}(\tilde{X}_1, t) + \tilde{\lambda}_{1,1} \hat{\alpha} \tilde{X}_1 v_z(\tilde{X}_1, t) \leq -\frac{1}{2} \tilde{\lambda}_{1,1}^2 \frac{v_z^2}{v_{zz}},$$

with the maximum $\hat{\alpha}^*$ occurring at $\hat{\alpha}^* = -\frac{\tilde{\lambda}_{1,1}}{\sigma_1} \frac{v_z(\tilde{X}_1, t)}{\tilde{X}_1 v_{zz}(\tilde{X}_1, t)}$. The rest of the proof follows easily. ■

Discussion: Equation (17) is, in general, non-tractable due to the presence of the second-order linear term $\frac{1}{2} (1 - \rho^2) \theta_1^2 \sigma_2^2 \beta^2 z^2 v_{zz}$ (the first-order term $\theta_1 \left(\rho \tilde{\lambda}_{1,1} - \tilde{\lambda}_{1,2} \right) \sigma_2 \beta z v_z$ may be easily absorbed with a mere time-rescaling). Its form is random and evolves with the market and the competitor's policy forward in time.

Equations of similar structure also arise in expected utility problems in the classical setting when there is random endowment and/or labor income processes. To the best of our knowledge, they are also non-tractable and only general abstract results exist to date (see, among others, [46] and the more recent work [47]). In the forward case, an additional complication arises from the ill-posedness of the problem, for one also needs to specify the class of admissible initial conditions $V_1(\tilde{x}_1, 0; \beta)$. This is a rather challenging question, currently investigated by the authors. On the other hand, the CRRA class provides an example, showing that Definition 2 is not vacuous.

Despite its non-tractability, equation (17) demonstrates that the best-response criterion $V_1(\tilde{x}_1, t; \beta)$ is *endogenously* specified and depends on the current evolution of the market and the competitor's policy. Both these features are in contrast to their analogues in the classical cases (see for example [8] and [26]).

The optimal policy is constructed through the random feedback functional $\alpha^*(z, t)$, which consists of the "myopic"-type term $\frac{\tilde{\lambda}_{1,1}}{\sigma_1} R_1(z, t)$ and the linear term $\rho \frac{\sigma_2}{\sigma_1} \theta_1 \beta$. The first component resembles the one in the original forward setting but now with modified risk premium $\tilde{\lambda}_{1,1}$. It depends on the competitor's policy β through $\tilde{\lambda}_{1,1}$ and $R_1(z, t)$. If $\rho \neq 0$, it may become zero if there exist time(s), say t_0 , such that $\beta_{t_0} = \frac{\lambda_{1,t_0}}{\rho \sigma_{2,t_0} \theta_1}$.

In general, it is difficult to provide any qualitative conclusions on how $\alpha^*(z, t)$ is influenced by β but at least (18) highlights its endogeneity and that it is affected by the realized market performance, the competitor's policy, and the manager's realized performance. These characteristics are the outcome of the flexibility of the normative best-response forward criterion. We stress that empirical evidence strongly supports such features; see, for example, [36], [43] and the survey-based results of [14], for the effects of past performance by the manager and [61] for the impact of fund's size. The classical model in which the (terminal) risk tolerance is exogenously chosen does not seem to capture these phenomena, as argued in these papers.

Next we note that, in general, $V_1(\tilde{x}_1, t; \beta)$ may *not* be time-monotone (albeit being locally riskless). This can be seen from equation (17) when written as (recall $\rho^2 \neq 1$)

$$v_t + \frac{1}{2} (1 - \rho^2) \theta_1^2 v_{zz} (\sigma_2 \beta z - c_1) (\sigma_2 \beta z - c_2) = 0,$$

with $c_{1,2} = \frac{v_z}{\theta_1 v_{zz}} \frac{-(\rho \tilde{\lambda}_{1,1} - \tilde{\lambda}_{1,2}) \pm \sqrt{\Delta}}{1 - \rho^2}$, and the process $(\Delta_t)_{t \geq 0}$ given by $\Delta := \tilde{\lambda}_{1,1}^2 - 2\rho \tilde{\lambda}_{1,1} \tilde{\lambda}_{1,2} + \tilde{\lambda}_{1,2}^2 > 0$. We easily deduce that $c_1 c_2 < 0$ and the lack of time-monotonicity follows from the above equation and the assumed spatial concavity of v .

We recall that in the absence of competition ($\theta_1 = 0$), the analogous locally riskless criterion is given by $u(x_1, \int_0^t \lambda_1^2 ds)$, with u satisfying $u_t = \frac{1}{2} \frac{u_z^2}{u_{zz}}$, $(z, t) \in \mathbb{D}$. This process is always decreasing in time. The lack of time-monotonicity is one of the fundamental differences between the forward performance processes with and without competition, $V_1(\tilde{x}_1, t; \beta)$ and $u(x_1, \int_0^t \lambda_1^2 ds)$. We comment more on this in the next section.

If $\rho = 0$, then $\tilde{\lambda}_{1,1} = \lambda_1$ and equation (17) reduces to

$$v_t - \frac{1}{2} \lambda_1^2 \frac{v_z^2}{v_{zz}} + \frac{1}{2} \theta_1^2 \sigma_2^2 \beta^2 z^2 v_{zz} - \tilde{\lambda}_{1,2} \theta_1 \sigma_2 \beta z u_z = 0.$$

In turn, $\alpha^*(z, t) = -\frac{\lambda_1}{\sigma_1} \frac{v_z(z, t)}{z v_{zz}(z, t)}$, with v still depending on β through the coefficients in the reduced equation above.

If $\rho \neq 0$, relative performance concerns might lead to zero allocation in $\tilde{S}_{1,1}$, at time(s) t_0 such that $\frac{\tilde{\lambda}_{1,1,t_0}}{\sigma_{1,t_0}} R_1(z, t_0) + \rho \frac{\sigma_{2,t_0}}{\sigma_{1,t_0}} \theta_1 \beta_{t_0} = 0$.

2.1.1 The CRRA case

To provide further insights on the forward relative performance criteria and also compare them with the ones in the classical setting, we study the case of homothetic criteria for manager 1. We impose

no assumption on what criterion manager 2 might follow, we only use assume that she follows an arbitrary policy $\beta \in \mathcal{A}_2$.

Proposition 4 *Let policy $\beta \in \mathcal{A}_2$, $\rho^2 \neq 1$, and $\tilde{\lambda}_{1,1}$ and $\tilde{\lambda}_{1,2}$ as in (10). Let $\gamma_1 > 0$, $\gamma_1 \neq 0$, and $(\eta_{1,t})_{t \geq 0}$ be given by*

$$\eta_1 = \tilde{\lambda}_{1,1}^2 + 2 \left(\rho \tilde{\lambda}_{1,1} - \tilde{\lambda}_{1,2} \right) \theta_1 \sigma_2 \beta \gamma_1 - (1 - \rho^2) \theta_1^2 \sigma_2^2 \beta^2 \gamma_1^2. \quad (22)$$

Then, the process

$$V_1(\tilde{x}_1, t; \beta) = \frac{\tilde{x}_1^{1-\gamma_1}}{1-\gamma_1} e^{-\int_0^t \frac{1-\gamma_1}{2\gamma_1} \eta_1 ds}, \quad (23)$$

is a locally riskless best-response forward criterion and the investment policy

$$\alpha^* = \frac{1}{\gamma_1} \frac{\tilde{\lambda}_{1,1}}{\sigma_1} + \rho \theta_1 \frac{\sigma_2}{\sigma_1} \beta \quad (24)$$

is optimal.

Proof. We look for candidate criteria of the separable form $V_1(\tilde{x}_1, t; \beta) = \frac{\tilde{x}_1^{1-\gamma_1}}{1-\gamma_1} K$, where $(K_t)_{t \geq 0}$ is an \mathcal{F}_t -adapted process, differentiable in t with $K_0 = 1$. Using equation (17), the boundedness assumption on the Sharpe ratios and the admissibility of β , we easily conclude. ■

We may rewrite the process $(\eta_{1,t})_{t \geq 0}$ as

$$\eta_1 = (\lambda_{1,1} - \delta_1 \theta_1 \sigma_2 \beta)^2 + \left(\rho^2 (1 - \gamma_1)^2 + \gamma_1 (1 - \gamma_1 + \frac{1}{\theta_1}) - \delta_1^2 \right) \theta_1^2 \sigma_2^2 \beta^2, \quad (25)$$

with $(\delta_{1,t})_{t \geq 0}$ given by

$$\delta_1 = \gamma_1 \frac{\lambda_2}{\lambda_1} + \rho (1 - \gamma_1). \quad (26)$$

Similar expressions were derived in [9, Lemma 1] for the special case of log-normal markets for power utilities in the classical setting. Herein, we have analogous results for general \mathcal{F}_t -adapted processes η_1 and δ_1 and no time-horizon dependence. We also stress that *no* solutions of form (23), (25) and (26) may be derived in the classical expected (power) utility setting beyond the log-normal case.

The criterion $V_1(\tilde{x}_1, t; \beta)$ resembles its forward counterpart in the absence of relative performance ($\theta_1 = 0$), given by $u(x_1, t) = \frac{x_1^{1-\gamma_1}}{1-\gamma_1} e^{-\int_0^t \frac{1-\gamma_1}{2\gamma_1} \lambda_1^2 ds}$ (see [52]), which is however always time-monotone.

Rewriting (24) as

$$\alpha^* = \frac{1}{\gamma_1} \frac{\lambda_1}{\sigma_1} + \rho \theta_1 \left(1 - \frac{1}{\gamma_1} \right) \frac{\sigma_2}{\sigma_1} \beta, \quad (27)$$

we see that depending on the sign of the various terms, manager 1 might invest more or less in the risky asset under relative performance concerns. For example, for $\rho > 0$, $\frac{\sigma_2}{\sigma_1} > 0$, and a long competitor's strategy, $\beta > 0$, we have $\rho \theta_1 \left(1 - \frac{1}{\gamma_1} \right) \frac{\sigma_2}{\sigma_1} \beta > 0$ if $\gamma_1 < 0$, while $\rho \theta_1 \left(1 - \frac{1}{\gamma_1} \right) \frac{\sigma_2}{\sigma_1} \beta < 0$ if $0 < \gamma_1 < 1$. These results are also consistent with the ones in [9] where market coefficients are constant. Herein, however, these results hold even in more general Itô-diffusion settings.

Finally, if the market price of risk λ_1 increases, the position on the familiar asset always increases even with relative performance concerns. This is consistent with the fact that when the performance of the asset the manager invests in improves, she tends to increase her position to it (see for instance [16, 20, 36]). The process λ_1 usually refers to the manager's active-management *ability* (see among others [59]). We note that unless we work in log-normal models, such behavior does not always result in the classical expected utility settings.

Symmetric results are deduced for manager 2 if her competitor follows policy $\alpha \in \mathcal{A}_1$. Namely, for $\gamma_2 > 0$, $\gamma_2 \neq 1$, and $(\eta_{2,t})_{t \geq 0}$ with

$$\eta_2 := \tilde{\lambda}_{2,2}^2 + 2 \left(-\tilde{\lambda}_{2,1} + \rho \tilde{\lambda}_{1,1} \right) \sigma_1 \theta_2 \alpha \gamma_2 - (1 - \rho^2) (\sigma_1 \theta_2 \alpha)^2 \gamma_2^2, \quad (28)$$

the process $(V_2(\tilde{x}_2, t; \alpha))_{t \geq 0}$ given by

$$V_2(\tilde{x}_2, t; \alpha) = \frac{\tilde{x}_2^{1-\gamma_2}}{1-\gamma_2} e^{-\int_0^t \frac{1-\gamma_2}{2\gamma_2} \eta_2 ds}, \quad (29)$$

is a locally riskless best-response forward criterion and the investment policy

$$\beta^* = \frac{1}{\gamma_2} \frac{\tilde{\lambda}_{2,2}}{\sigma_2} + \rho \theta_2 \frac{\sigma_1}{\sigma_2} \alpha = \frac{1}{\gamma_2} \frac{\lambda_2}{\sigma_2} + \rho \theta_2 \left(1 - \frac{1}{\gamma_2} \right) \frac{\sigma_2}{\sigma_1} \alpha \quad (30)$$

is optimal.

Finally, we may construct a best-response (locally riskless) forward criterion for the limiting cases $\gamma_1 = 0$ and/or $\gamma_2 = 0$. Looking for a candidate process of the additive form $V_1(\tilde{x}_1, t; \beta) = \log \tilde{x}_1 + K$, for a suitable process $(K_t)_{t \geq 0}$, equation (17) yields

$$V_1(\tilde{x}_1, t; \beta) = \log \tilde{x}_1 + \int_0^t \left(\frac{1}{2} \tilde{\lambda}_{1,1}^2 - (\rho \tilde{\lambda}_{1,1} - \tilde{\lambda}_{1,2}) \theta_1 \sigma_2 \beta + \frac{1}{2} (1 - \rho^2) \theta_1^2 (\sigma_2 \beta)^2 \right) ds,$$

with optimal policy $\alpha^* = \frac{\tilde{\lambda}_{1,1}}{\sigma_1} + \rho \frac{\sigma_2}{\sigma_1} \theta_1 \beta$. Similar results can be produced for the case $\gamma_2 = 0$.

2.2 Forward Nash equilibrium

The asset managers not only trade between the riskless account and the respective specialized risky asset but, also, interact dynamically with each other. Then, the individual best-response problems lead conceptually to a pure-strategy Nash game. We call the equilibrium of this game a *forward Nash equilibrium* and propose the following definition for its analysis.

We recall the modified risk premia $\tilde{\lambda}_{1,1}(\beta)$, $\tilde{\lambda}_{1,2}(\beta)$ and $\tilde{\lambda}_{2,1}(\alpha)$, $\tilde{\lambda}_{2,2}(\alpha)$ (cf. (10) and (12)), highlighting their dependence on the competitor's policies.

Definition 5 *A forward Nash equilibrium consists of two pairs of \mathcal{F}_t -adapted processes,*

$(V_1(\tilde{x}_1, t; \beta^)_{t \geq 0}, (\alpha_t^*)_{t \geq 0})$ and $((V_2(\tilde{x}_2, t; \alpha^*)_{t \geq 0}, (\beta_t^*)_{t \geq 0})$, $\tilde{x}_1, \tilde{x}_2 > 0$, $t \geq 0$, with the following properties:*

- i) The processes $\alpha^* \in \mathcal{A}_1$ and $\beta^* \in \mathcal{A}_2$,*
- ii) The processes $V_1(\tilde{x}_1, t; \beta^*)$, $V_2(\tilde{x}_2, t; \alpha^*) \in \mathcal{U}$.*
- iii) For $\alpha \in \mathcal{A}_1$, $V_1(\tilde{X}_1, t; \beta^*)$ is a (local) super-martingale and $V_1(\tilde{X}_1^*, t; \beta^*)$ is a (local) martingale where \tilde{X}_1 and \tilde{X}_1^* solve (9) with $\tilde{\lambda}_{1,1} = \tilde{\lambda}_{1,1}(\beta^*)$ and $\tilde{\lambda}_{1,2} = \tilde{\lambda}_{1,2}(\beta^*)$, and with α and α^* being, respectively, used.*
- iiii) For $\beta \in \mathcal{A}_2$, $V_2(\tilde{X}_2, t; \alpha^*)$ is a (local) super-martingale and $V_2(\tilde{X}_2^*, t; \alpha^*)$ is a (local) martingale where \tilde{X}_2 and \tilde{X}_2^* solve (11) with $\tilde{\lambda}_{2,1} = \tilde{\lambda}_{2,1}(\alpha^*)$ and $\tilde{\lambda}_{2,2} = \tilde{\lambda}_{2,2}(\alpha^*)$, and with β and β^* being, respectively, used.*

If, under appropriate integrability conditions, processes $(V_1(\tilde{X}_1, t; \beta^*))_{t \geq 0}$ and $(V_1(\tilde{X}_1^*, t; \beta^*))_{t \geq 0}$ are, respectively, a true supermartingale and a true martingale then, for any $\alpha \in \mathcal{A}_1$,

$$\mathbb{E} \left[V_1(\tilde{X}_1^*, t; \beta^*) \right] = \mathbb{E} [V_1(\tilde{x}_1, 0)] \geq \mathbb{E} \left[V_1(\tilde{X}_1^1, t; \beta^*) \right].$$

Analogously,

$$\mathbb{E} \left[V_2(\tilde{X}_2^*, t; \alpha^*) \right] = \mathbb{E} [V_2(\tilde{x}_2, 0)] \geq \mathbb{E} \left[V_2(\tilde{X}_2, t; \alpha^*) \right].$$

In other words, no unilateral deviation in strategy by either manager will result in an increase in the expected utility of her relative performance metric.

From Proposition 3 and, in particular, the best-response strategy (18) and analogous results for the optimal policy β^* , it follows that the candidate forward Nash equilibrium strategies should satisfy the system of equations

$$\begin{cases} \alpha^* = \frac{\tilde{\lambda}_{1,1}(\beta^*)}{\sigma_1} R_1^* \left(\tilde{X}_1^*, t; \beta^* \right) + \rho \theta_1 \frac{\sigma_2}{\sigma_1} \beta^* \\ \beta^* = \frac{\tilde{\lambda}_{2,2}(\alpha^*)}{\sigma_2} R_2^* \left(\tilde{X}_2^*, t; \alpha^* \right) + \rho \theta_2 \frac{\sigma_1}{\sigma_2} \alpha^*, \end{cases} \quad (31)$$

where $\left(R_{1,t}^* \left(\tilde{X}_1^*, t; \beta^* \right) \right)_{t \geq 0}$ and $\left(R_{2,t}^* \left(\tilde{X}_2^*, t; \alpha^* \right) \right)_{t \geq 0}$ are defined as

$$R_1^* \left(\tilde{X}_1^*, t; \beta^* \right) = - \frac{v_{1,z}(\tilde{X}_1^*, t)}{\tilde{X}_1^* v_{1,zz}(\tilde{X}_1^*, t)} \quad \text{and} \quad R_2^* \left(\tilde{X}_2^*, t; \alpha^* \right) = - \frac{v_{2,z}(\tilde{X}_2^*, t)}{\tilde{X}_2^* v_{2,zz}(\tilde{X}_2^*, t)},$$

with $v_1(z, t)$ and $v_2(z, t)$, $(z, t) \in \mathbb{D}$, solving

$$v_{1,t} - \frac{1}{2} \tilde{\lambda}_{1,1}^2(\beta^*) \frac{v_{1,z}^2}{v_{1,zz}} + \frac{1}{2} (1 - \rho^2) \sigma_2^2 \theta_1^2 \beta^{*2} z^2 v_{1,zz} + \left(\rho \tilde{\lambda}_{1,1}(\beta^*) - \tilde{\lambda}_{1,2}(\beta^*) \right) \sigma_2 \theta_1 \beta^* z v_{1,z} = 0 \quad (32)$$

and

$$v_{2,t} - \frac{1}{2} \tilde{\lambda}_{2,2}^2(\alpha^*) \frac{v_{2,z}^2}{v_{2,zz}} + \frac{1}{2} (1 - \rho^2) \sigma_1^2 \theta_2^2 \alpha^{*2} z^2 v_{2,zz} + \left(-\tilde{\lambda}_{2,1}(\alpha^*) + \rho \tilde{\lambda}_{2,2}(\alpha^*) \right) \sigma_1 \theta_2 \alpha^* z v_{2,z} = 0. \quad (33)$$

System (31) is in general non-tractable because of the highly non-linear terms $R_1^* \left(\tilde{X}_1^*, t; \beta^* \right)$ and $R_2^* \left(\tilde{X}_2^*, t; \alpha^* \right)$.

2.2.1 The CRRA cases

For tractability and to highlight the differences between the forward approach and the classical setting, we examine the case of homothetic criteria for *both* managers.

Proposition 6 *Let $\gamma_1, \gamma_2 > 0$ with $\gamma_1, \gamma_2 \neq 1$, and assume that*

$$\delta := \gamma_1 \gamma_2 - \rho^2 \theta_1 \theta_2 (1 - \gamma_1)(1 - \gamma_2) \neq 0. \quad (34)$$

Consider the processes $(\alpha_t^)_{t \geq 0}$, $(\beta_t^*)_{t \geq 0}$ given by*

$$\alpha^* = \frac{\gamma_2 \lambda_1 - \rho \theta_1 (1 - \gamma_1) \lambda_2}{\sigma_1 \delta} \quad \text{and} \quad \beta^* = \frac{\gamma_1 \lambda_2 - \rho \theta_2 (1 - \gamma_2) \lambda_1}{\sigma_2 \delta}. \quad (35)$$

Let also $(\eta_{1,t}^)_{t \geq 0}$ and $(\eta_{2,t}^*)_{t \geq 0}$ be given by (22) and (28) when β^* and α^* are, respectively, used and $(V_1(\tilde{x}_1, t; \beta^*))_{t \geq 0}$ and $(V_2(\tilde{x}_2, t; \alpha^*))_{t \geq 0}$ defined as*

$$V_1(\tilde{x}_1, t; \beta^*) = \frac{\tilde{x}_1^{1-\gamma_1}}{1-\gamma_1} e^{-\int_0^t \frac{1-\gamma_1}{2\gamma_1} \eta_1^* ds}, \quad V_2(\tilde{x}_2, t; \alpha^*) = \frac{\tilde{x}_2^{1-\gamma_2}}{1-\gamma_2} e^{-\int_0^t \frac{1-\gamma_2}{2\gamma_2} \eta_2^* ds}.$$

Then, the pair of processes $((V_1(\tilde{x}_1, t; \beta^), \alpha^*)$ and $(V_2(\tilde{x}_2, t; \alpha^*), \beta^*)$ constitutes a forward Nash equilibrium.*

Proof. From (27) and (30), we deduce that the candidate strategies $(\alpha_t^*)_{t \geq 0}$, $(\beta_t^*)_{t \geq 0}$ must solve the system

$$\begin{cases} \alpha^* - \left(1 - \frac{1}{\gamma_1}\right) \rho \theta_1 \frac{\sigma_2}{\sigma_1} \beta^* = \frac{1}{\gamma_1} \frac{\lambda_1}{\sigma_1} \\ - \left(1 - \frac{1}{\gamma_2}\right) \rho \theta_2 \frac{\sigma_1}{\sigma_2} \alpha^* + \beta^* = \frac{1}{\gamma_2} \frac{\lambda_2}{\sigma_2}. \end{cases}$$

Using that its determinant is given by $\frac{\delta}{\gamma_1 \gamma_2}$, with δ as in (34) and, by assumption, $\delta \neq 0$, we easily deduce (35). Furthermore, $\alpha^* \in \mathcal{A}_1$ and $\beta^* \in \mathcal{A}_2$, given the assumption on bounded λ_1 and λ_2 . The rest of the proof follows easily. ■

In the special case $\rho = 0$, the forward Nash equilibrium strategies simplify to,

$$\alpha^* = \frac{1}{\gamma_1} \frac{\lambda_1}{\sigma_1} \quad \text{and} \quad \beta_2^* = \frac{1}{\gamma_2} \frac{\lambda_2}{\sigma_2},$$

which are the optimal policies without competition. Note, however, that the associated forward Nash criteria still depend on the other manager's strategy through the processes η_1^* and η_2^* .

Continuing the discussion in §2.1.1., we mention that the forward Nash equilibrium investment strategies have the same form as those of the classical setting in a log-normal market (see [9, Proposition 1]). Hence, we may generalize *all* comparative statics of [9] in the general Itô-diffusion market setting herein. Furthermore, allowing stochastic coefficients to the model makes the equilibrium investment strategies rich enough to include some observed empirical evidence, which cannot be captured by the constant investment strategies (induced by the constant coefficients). For example, we readily get that α^* is increasing with respect to λ_2 if $\gamma_1 > 1$ (and both managers have long position on their market with $\delta, \rho > 0$). This is consistent with the fact that managers may increase (resp. decrease) their risky positions when their competitors have a better (resp. worse) performance (see among others [16, 19, 36]).

3 Asset diversification and forward competition

In this section we impose the situation where both managers invest in the same market $\mathcal{M} = (B, S_1, S_2)$, with S_1, S_2 solving (3) and without any trading constraints. This case is particularly popular when managers aim to beat the same benchmark. The managers have relative performance concerns and may interact passively or competitively. As in the asset specialization case, we incorporate these concerns by working with relative wealth processes with competition parameters θ_1, θ_2 . We measure the performance of their strategies using forward best response and forward Nash equilibrium criteria, respectively. We define them as in Definitions 2 and 5, and we also work with locally riskless processes.

Using (3), the (discounted by the bond) wealth processes $(X_{1,t})_{t \geq 0}$ and $(X_{2,t})_{t \geq 0}$, $t \geq 0$, satisfy

$$\frac{dX_1}{X_1} = \sigma_1 \alpha_1 (\lambda_1 dt + dW_1) + \sigma_2 \alpha_2 (\lambda_2 dt + dW_2) \quad (36)$$

and

$$\frac{dX_2}{X_2} = \sigma_1 \beta_1 (\lambda_1 dt + dW_1) + \sigma_2 \beta_2 (\lambda_2 dt + dW_2), \quad (37)$$

with $X_{1,0} = x_1 > 0$ and $X_{2,0} = x_2 > 0$, and α_1, α_2 (resp. β_1, β_2) being the fractions of wealth X_1 (resp. X_2) invested in asset classes S_1 and S_2 , respectively. The set \mathcal{A} of admissible policies $\alpha = (\alpha_1, \alpha_2)$ and $\beta = (\beta_1, \beta_2)$ is defined similarly to (6).

For $\theta_1, \theta_2 \in (0, 1]$, $\alpha = (\alpha_1, \alpha_2)$ and $\beta = (\beta_1, \beta_2)$, the biased benchmark processes $\left(X_{1,t}^{\theta_2}\right)_{t \geq 0}$ and $\left(X_{2,t}^{\theta_1}\right)_{t \geq 0}$ solve

$$\frac{dX_{1,t}^{\theta_2}}{X_{1,t}^{\theta_2}} = \theta_2 \sigma_1 \alpha_1 (\lambda_1 dt + dW_1) + \theta_2 \sigma_2 \alpha_2 (\lambda_2 dt + dW_2) + \frac{1}{2} \theta_2 (\theta_2 - 1) C_1(\alpha) dt$$

with the process $(C_{1,t}(\alpha))_{t \geq 0}$,

$$C_1(\alpha) := \sigma_1^2 \alpha_1^2 + 2\rho \sigma_1 \sigma_2 \alpha_1 \alpha_2 + \sigma_2^2 \alpha_2^2. \quad (38)$$

Similarly,

$$\frac{dX_{2,t}^{\theta_1}}{X_{2,t}^{\theta_1}} = \theta_1 \sigma_1 \beta_1 (\lambda_1 dt + dW_1) + \theta_1 \sigma_2 \beta_2 (\lambda_2 dt + dW_2) + \frac{1}{2} \theta_1 (\theta_1 - 1) C_2(\beta) dt,$$

with the process $(C_{2,t}(\beta))_{t \geq 0}$,

$$C_2(\beta) = \sigma_1^2 \beta_1^2 + 2\rho \sigma_1 \sigma_2 \beta_1 \beta_2 + \sigma_2^2 \beta_2^2. \quad (39)$$

Direct calculations yield that the relative wealths processes $\tilde{X}_1 := \frac{X_1}{X_{2,t}^{\theta_1}}$ and $\tilde{X}_2 := \frac{X_2}{X_{1,t}^{\theta_2}}$ satisfy

$$\begin{aligned} \frac{d\tilde{X}_1}{\tilde{X}_1} &= \sigma_1 \alpha_1 \left(\tilde{\lambda}_{1,1} dt + dW_1 \right) + \sigma_2 \alpha_2 \left(\tilde{\lambda}_{1,2} dt + dW_2 \right) \\ &\quad - \sigma_1 \theta_1 \beta_1 (\lambda_1 dt + dW_1) - \sigma_2 \theta_1 \beta_2 (\lambda_2 dt + dW_2) + \frac{1}{2} \theta_1 (1 + \theta_1) C_2(\beta) dt, \end{aligned} \quad (40)$$

with the processes $(\tilde{\lambda}_{1,1,t})_{t \geq 0}$ and $(\tilde{\lambda}_{1,2,t})_{t \geq 0}$,

$$\tilde{\lambda}_{1,1} := \lambda_1 - \theta_1 (\sigma_1 \beta_1 + \rho \sigma_2 \beta_2) \quad \text{and} \quad \tilde{\lambda}_{1,2} := \lambda_2 - \theta_1 (\rho \sigma_1 \beta_1 + \sigma_2 \beta_2). \quad (41)$$

Similarly,

$$\begin{aligned} \frac{d\tilde{X}_2}{\tilde{X}_2} &= \sigma_1 \beta_1 \left(\tilde{\lambda}_{2,1} dt + dW_1 \right) + \sigma_2 \beta_2 \left(\tilde{\lambda}_{2,2} dt + dW_2 \right) \\ &\quad - \sigma_1 \theta_2 \alpha_1 (\lambda_1 dt + dW_1) - \sigma_2 \theta_2 \alpha_2 (\lambda_2 dt + dW_2) + \frac{1}{2} \theta_2 (1 + \theta_2) C_1(\alpha) dt, \end{aligned} \quad (42)$$

with the processes $(\tilde{\lambda}_{2,1,t})_{t \geq 0}$ and $(\tilde{\lambda}_{2,2,t})_{t \geq 0}$,

$$\tilde{\lambda}_{2,1} := \lambda_1 - \theta_2 (\sigma_1 \alpha_1 + \rho \sigma_2 \alpha_2) \quad \text{and} \quad \tilde{\lambda}_{2,2} := \lambda_2 - \theta_2 (\rho \sigma_1 \alpha_1 + \sigma_2 \alpha_2). \quad (43)$$

As in the asset specialization case, we may interpret (40) as the wealth of a manager who invests in the personalized fictitious market $\tilde{\mathcal{M}}_1^d := (B, \tilde{S}_{1,1}, \tilde{S}_{1,2})$ with (pseudo) stocks $\tilde{S}_{1,1}, \tilde{S}_{1,2}$ solving

$$\frac{d\tilde{S}_{1,1}}{\tilde{S}_{1,1}} = \sigma_1 \left(\tilde{\lambda}_{1,1} dt + dW_1 \right) \quad \text{and} \quad \frac{d\tilde{S}_{1,2}}{\tilde{S}_{1,2}} = \sigma_2 \left(\tilde{\lambda}_{1,2} dt + dW_2 \right),$$

with $\tilde{\lambda}_{1,1}$ and $\tilde{\lambda}_{1,2}$ given in (41), while receiving returns from a *random endowment process* $(Y_{1,t})_{t \geq 0}$,

$$\frac{d\tilde{X}_1}{\tilde{X}_1} = \alpha_1 \sigma_1 \left(\tilde{\lambda}_{1,1} dt + dW_1 \right) + \alpha_2 \sigma_2 \left(\tilde{\lambda}_{1,2} dt + dW_2 \right) + dY_{1,t}$$

with

$$dY_1 = -\theta_1 \sigma_1 \beta_1 (\lambda_1 dt + dW_1) - \theta_1 \sigma_2 \beta_2 (\lambda_2 dt + dW_2) + \frac{1}{2} \theta_1 (1 + \theta_1) C_2(\beta) dt$$

and $Y_{1,0} = 0$.

Similarly, manager 2 invests in a personalized fictitious market $\tilde{\mathcal{M}}_2^d := (B, \tilde{S}_{2,1}, \tilde{S}_{2,2})$ with (pseudo) stocks $\tilde{S}_{2,1}, \tilde{S}_{2,2}$ solving

$$\frac{d\tilde{S}_{2,1}}{\tilde{S}_{2,1}} = \sigma_2 \left(\tilde{\lambda}_{2,1} dt + dW_1 \right) \quad \text{and} \quad \frac{d\tilde{S}_{2,2}}{\tilde{S}_{2,2}} = \sigma_2 \left(\tilde{\lambda}_{2,2} dt + dW_2 \right),$$

with $\tilde{\lambda}_{2,1}$ and $\tilde{\lambda}_{2,2}$ given in (43), and

$$\frac{d\tilde{X}_2}{\tilde{X}_2} = \sigma_1\beta_1 \left(\tilde{\lambda}_{2,1}dt + dW_1 \right) + \sigma_2\beta_2 \left(\tilde{\lambda}_{2,2}dt + dW_2 \right) + dY_2$$

with

$$dY_2 = -\theta_2\sigma_1\alpha_1 (\lambda_1dt + dW_1) - \theta_2\sigma_2\alpha_2 (\lambda_2dt + dW_2) + \frac{1}{2}\theta_2 (1 + \theta_2) C_1(\alpha) dt,$$

and $Y_{2,0} = 0$.

The personalized fictitious markets $\tilde{\mathcal{M}}_1^d$ and $\tilde{\mathcal{M}}_2^d$ are both *complete*, in contrast to their counterparts $\tilde{\mathcal{M}}_1^s$ and $\tilde{\mathcal{M}}_2^s$ in the asset specialization case. This completeness makes the underlying problems tractable, as we discuss next.

3.1 Best-response forward relative performance criteria

In analogy to the asset diversification case, we apply Definition 2 to define the best-response forward performance criteria, denoted with a slight abuse of notation by $(V_1(\tilde{x}_1, t; \beta))_{t \geq 0}$ and $(V_2(\tilde{x}_2, t; \alpha))_{t \geq 0}$, $\tilde{x}_1 = \frac{x_1}{x_1^{\theta_1}}$, $\tilde{x}_2 = \frac{x_2}{x_2^{\theta_2}}$, where α, β stand for arbitrary policies of the competitors. Because of symmetry, we only analyze the quantities pertinent to manager 1. We provide complete characterization of her relative forward criterion, the optimal investment and the optimal wealth processes under relative performance concerns.

Following [52], we first define two auxiliary functions, $u_1 : \mathbb{D} \rightarrow \mathbb{R}_+$ and $h_1 : \mathbb{R} \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$. Function u_1 solves

$$u_{1,t} = \frac{1}{2} \frac{u_{1,z}^2}{u_{1,zz}}, \quad (44)$$

with initial condition given by

$$(u_1'(z, 0))^{(-1)} := \int_{0^+}^{\infty} z^{-y} d\nu_1(y), \quad (45)$$

for a finite positive Borel measure ν_1 .

The function h_1 is defined as $h_1(z, t) := (u_{1,z})^{(-1)}(e^{-z+\frac{t}{2}}, t)$ (spatial inverse). It solves $h_{1,t} + \frac{1}{2}h_{1,zz} = 0$ with $h_1(z, 0) = \int_{0^+}^{\infty} e^{yz} d\nu_1(y)$, and is given by

$$h_1(z, t) = \int_{0^+}^{\infty} e^{yz - \frac{1}{2}y^2t} d\nu_1(y). \quad (46)$$

Let also $R_1 : \mathbb{D} \rightarrow \mathbb{R}_+$,

$$R_1(z, t) := -\frac{u_{1,z}(z, t)}{zu_{1,zz}(z, t)} = \frac{h_{1,z}(h_1^{(-1)}(z, t), t)}{h_1(h_1^{(-1)}(z, t), t)}, \quad (47)$$

with the latter equality following from the definition of h_1 .

The functions u_1, h_1 and R_1 were introduced in [52], and used to construct in full generality the locally riskless forward criteria in the absence of competition ($\theta_1 = 0$); we refer the reader therein for details, and especially for the assumptions on measure ν_1 .

Finally, we consider the processes $(A_{1,t})_{t \geq 0}$ and $(M_{1,t})_{t \geq 0}$ defined, for $\tilde{\lambda}_{1,1}, \tilde{\lambda}_{1,2}$ as in (41), as

$$A_1 := \frac{1}{1 - \rho^2} \int_0^t \left(\tilde{\lambda}_{1,1}^2 - 2\rho\tilde{\lambda}_{1,1}\tilde{\lambda}_{1,2} + \tilde{\lambda}_{1,2}^2 \right) ds \quad \text{and} \quad M_1 := \int_0^t \tilde{\lambda}_{1,1} dW_1 + \int_0^t \tilde{\lambda}_{1,2} dW_2. \quad (48)$$

Next, we present the main result in the asset diversification case.

Proposition 7 Let policy $\beta = (\beta_1, \beta_2) \in \mathcal{A}$ and $C_2(\beta)$ as in (39), and define $(B_{1,t})_{t \geq 0}$ as

$$B_1 := e^{\frac{1}{2}\theta_1(1-\theta_1)\int_0^t C_2(\beta)ds}. \quad (49)$$

Let the processes A_1 and M_1 be as in (48), $u_1(\tilde{x}_1, 0)$ as in (45) with $u_1(z, t)$ solving (44), and introduce the process $(H_{1,t})_{t \geq 0}$,

$$\begin{aligned} H_1 &:= \frac{h_{1,z} \left(h_1^{(-1)}(\tilde{x}_1, 0) + A_1 + M_1, A_1 \right)}{h_1 \left(h_1^{(-1)}(\tilde{x}_1, 0) + A_1 + M_1, A_1 \right)} \\ &= \frac{\int_{0^+}^{\infty} e^{yh_1^{(-1)}(\tilde{x}_1, 0)} y d\tilde{\nu}_{1,t}(y)}{\int_{0^+}^{\infty} e^{yh_1^{(-1)}(\tilde{x}_1, 0)} d\tilde{\nu}_{1,t}(y)}, \end{aligned} \quad (50)$$

with

$$d\tilde{\nu}_{1,t}(y) = e^{y(1-\frac{\theta_1}{2})A_1 + yM_1} d\nu_{1,t}(y). \quad (51)$$

The following assertions hold:

i) The process $(V_1(\tilde{x}_1, t; \beta))_{t \geq 0}$, given by

$$V_1(\tilde{x}_1, t; \beta) = u_1 \left(\frac{\tilde{x}_1}{B_1}, A_1 \right), \quad (52)$$

with $V_1(\tilde{x}_1, 0; \beta) = u_1(\tilde{x}_1, 0)$ is the unique locally riskless best-response forward criterion with such initial condition. For each $\beta \in \mathcal{A}$ and $\tilde{x}_1 > 0$, $V_1(\tilde{x}_1, t; \beta)$ is time-decreasing.

ii) The optimal wealth process $(\tilde{X}_{1,t}^*)_{t \geq 0}$ is given by

$$\begin{aligned} \tilde{X}_1^* &= B_1 h_1 \left(h_1^{(-1)}(\tilde{x}_1, 0) + A_1 + M_1, A_1 \right) \\ &= B_1 \int_{0^+}^{\infty} e^{yh_1^{(-1)}(\tilde{x}_1, 0)} d\tilde{\nu}_{1,t}(y), \end{aligned} \quad (53)$$

with $\tilde{\nu}_{1,t}$ as in (51). iii) Let $\alpha^*(z, t) = (\alpha_1^*(z, t), \alpha_2^*(z, t))$, $(z, t) \in \mathbb{D}$, be defined as

$$\alpha_1^*(z, t) = \frac{\tilde{\lambda}_{1,1} - \rho\tilde{\lambda}_{1,2}}{(1-\rho^2)\sigma_1} B_1 R_1 \left(\frac{z}{B_1}, A_1 \right) + \theta_1 \beta_1 \quad (54)$$

and

$$\alpha_2^*(z, t) = \frac{-\rho\tilde{\lambda}_{1,1} + \tilde{\lambda}_{1,2}}{(1-\rho^2)\sigma_2} B_1 R_1 \left(\frac{z}{B_1}, A_1 \right) + \theta_1 \beta_2.$$

Then, the optimal investment processes $(\alpha_{1,t}^*)_{t \geq 0}$, $(\alpha_{2,t}^*)_{t \geq 0}$ are given in the feedback form,

$$\alpha_1^* = \alpha_1^*(\tilde{X}_1^*, A_1) \quad \text{and} \quad \alpha_2^* = \alpha_2^*(\tilde{X}_1^*, A_1), \quad (55)$$

and in closed form,

$$\alpha_1^* = \frac{\tilde{\lambda}_{1,1} - \rho\tilde{\lambda}_{1,2}}{(1-\rho^2)\sigma_1} H_1 + \theta_1 \beta_1 \quad (56)$$

and

$$\alpha_2^* = \frac{-\rho\tilde{\lambda}_{1,1} + \tilde{\lambda}_{1,2}}{(1-\rho^2)\sigma_2} H_1 + \theta_1 \beta_2,$$

with H_1 as in (50).

Proof. Let $\hat{\alpha}_1 := \alpha_1 - \theta_1\beta_1$, $\hat{\alpha}_2 := \alpha_2 - \theta_1\beta_2$. Then, the state dynamics (40) can be written as

$$\frac{d\tilde{X}_1}{\tilde{X}_1} = \hat{\alpha}_1\sigma_1 \left(\tilde{\lambda}_{1,1}dt + dW_1 \right) + \hat{\alpha}_2\sigma_2 \left(\tilde{\lambda}_{1,2}dt + dW_2 \right) + \frac{1}{2}\theta_1(1 - \theta_1)C_2(\beta) dt.$$

Defining the auxiliary process $(\hat{X}_{1,t})_{t \geq 0}$ by

$$\hat{X}_1 = \frac{\tilde{X}_1}{B_1},$$

we have that

$$\frac{d\hat{X}_1}{\hat{X}_1} = \hat{\alpha}_1\sigma_1 \left(\tilde{\lambda}_{1,1}dt + dW_1 \right) + \hat{\alpha}_2\sigma_2 \left(\tilde{\lambda}_{1,2}dt + dW_2 \right), \quad (57)$$

with $\hat{X}_{1,0} = \tilde{X}_{1,0} = \tilde{x}_1$. We are, then, in the complete market framework of [52, Section 3] and we deduce that if $u_1 : \mathbb{D} \rightarrow \mathbb{R}_+$ solves (44) and satisfies (45), then the process $u_1 \left(\hat{X}_1, A_1 \right)$ is a supermartingale for any $\left((\hat{\alpha}_{1,t})_{t \geq 0}, (\hat{\alpha}_{2,t})_{t \geq 0} \right)$ and becomes a martingale for $(\hat{\alpha}_{1,t}^*)_{t \geq 0}, (\hat{\alpha}_{2,t}^*)_{t \geq 0}$ given by

$$\hat{\alpha}_1^* = -\frac{\tilde{\lambda}_{1,1} - \rho\tilde{\lambda}_{1,2}}{(1 - \rho^2)\sigma_1} \hat{R}^* \quad \text{and} \quad \hat{\alpha}_2^* = \frac{-\rho\tilde{\lambda}_{1,1} + \tilde{\lambda}_{1,2}}{(1 - \rho^2)\sigma_2} \hat{R}^*,$$

with $\left(\hat{R}_t^* \right)_{t \geq 0} = -\frac{u_{1,z}(\hat{X}_1^*, A_1)}{\hat{X}_1^* u_{1,zz}(\hat{X}_1^*, A_1)}$, where \hat{X}_1^* solves (57) with $(\hat{\alpha}_1^*, \hat{\alpha}_2^*)$ being used. Following the analysis in [52], we deduce that the optimal process $\left(\hat{X}_{1,t}^* \right)_{t \geq 0}$ is given in closed form by $\hat{X}_1^* = h_1 \left(h_1^{(-1)}(\hat{x}_1, 0) + A_1 + M_1, A_1 \right)$ and (53) follows. Furthermore, from the definition of h_1 we deduce that

$$\hat{\alpha}_1^* = \frac{\tilde{\lambda}_{1,1} - \rho\tilde{\lambda}_{1,2}}{(1 - \rho^2)\sigma_1} \frac{h_{1,z} \left(h_1^{(-1)}(\hat{x}_1, 0) + A_1 + M_1, A_1 \right)}{h_1 \left(h_1^{(-1)}(\hat{x}_1, 0) + A_1 + M_1, A_1 \right)}$$

and, similarly,

$$\hat{\alpha}_2^* = \frac{\tilde{\lambda}_{1,1} - \rho\tilde{\lambda}_{1,2}}{(1 - \rho^2)\sigma_1} \frac{h_{1,z} \left(h_1^{(-1)}(\hat{x}_1, 0) + A_1 + M_1, A_1 \right)}{h_1 \left(h_1^{(-1)}(\hat{x}_1, 0) + A_1 + M_1, A_1 \right)}.$$

We easily deduce that $\hat{\alpha}_1^*, \hat{\alpha}_2^* \in \mathcal{A}$ as well as the rest of the assertions for the optimal wealth and optimal policies.

To establish the time monotonicity of $V_1(\tilde{x}_1, t; \beta)$, observe that, for each $\beta \in \mathcal{A}$ and $\tilde{x}_1 > 0$,

$$\frac{d}{dt} V_1(\tilde{x}_1, t; \beta) = -\frac{1}{2}\theta_1(1 - \theta_1) \frac{C_2(\beta)}{B_1(\beta)} u_{1,x} \left(\frac{\tilde{x}_1}{\tilde{B}_1}, A_1 \right) + \frac{\tilde{\lambda}_{1,1}^2 - 2\rho\tilde{\lambda}_{1,1}\tilde{\lambda}_{1,2} + \tilde{\lambda}_{1,2}^2}{1 - \rho^2} u_{1,t} \left(\frac{\tilde{x}_1}{\tilde{B}_1}, A_1 \right) < 0$$

as $\theta_1 < 1$, $C_2 > 0$, $u_{1,x} > 0$ and $u_{1,t} < 0$. ■

Remark 8 We note that the above best-response forward performance differs from the one introduced in [51] given by $u(x/Y_t, Z_t)$, where $(Y_t)_{t \geq 0}$ is a traded benchmark and $(Z_t)_{t \geq 0}$ a “market-view” process. This process is not locally riskless and its state variable is the individual wealth, and not the relative one.

Similar results may be derived for manager 2. Let manager 1 follow an arbitrary policy, say $\alpha = (\alpha_1, \alpha_2) \in \mathcal{A}$. If we choose $V_2(\tilde{x}_2, 0; \alpha) = \int_{0^+}^{\infty} \tilde{x}_2^{-y} d\nu_2(y)$, for a suitable positive Borel measure ν_2 , we deduce that the unique locally riskless best-response forward criterion is given by

$$V_2(\tilde{x}_2, t; \alpha) = u_2 \left(\frac{\tilde{x}_2}{B_2}, A_2 \right),$$

with u_2 solving (44) with $u_2(z, 0) = V_2(\tilde{x}_2, 0; \alpha)$, $(B_{2,t})_{t \geq 0} = e^{\frac{1}{2}\theta_2(1-\theta_2)\int_0^t C_1(\alpha) ds}$, $(A_{2,t})_{t \geq 0} := (1 - \rho^2) \int_0^t (\tilde{\lambda}_{2,1}^2 - 2\rho\tilde{\lambda}_{2,1}\tilde{\lambda}_{2,2} + \tilde{\lambda}_{2,2}^2) ds$ with $\tilde{\lambda}_{2,1}$, $\tilde{\lambda}_{2,2}$ and $C_1(\alpha)$ as in (43) and (38).

Furthermore, if $(M_{2,t})_{t \geq 0} := \int_0^t \tilde{\lambda}_{2,1} dW_1 + \int_0^t \tilde{\lambda}_{2,2} dW_2$, $h_2(z, t) := u_{2,z}^{(-1)}(e^{-z+\frac{t}{2}}, t)$ and $(H_2)_{t \geq 0}$ defined as

$$H_2 := \frac{h_{2,z} \left(h_2^{(-1)}(\tilde{x}_2, 0) + A_2 + M_2, A_2 \right)}{h_2 \left(h_2^{(-1)}(\tilde{x}_2, 0) + A_2 + M_2, A_2 \right)}, \quad (58)$$

then, the optimal wealth $(\tilde{X}_{2,t}^*)_{t \geq 0}$ is given by $\tilde{X}_2^* = B_2 h_2 \left(h_2^{(-1)}(\tilde{x}_2, 0) + A_2 + M_2, A_2 \right)$ and the policies $(\beta_{1,t}^*)_{t \geq 0}$, $(\beta_{2,t}^*)_{t \geq 0}$, with

$$\beta_1^* = \frac{\tilde{\lambda}_{2,1} - \rho\tilde{\lambda}_{2,2}}{(1 - \rho^2)\sigma_2} H_2 + \theta_2 \alpha_1, \quad \beta_2^* = \frac{-\rho\tilde{\lambda}_{2,1} + \tilde{\lambda}_{2,2}}{(1 - \rho^2)\sigma_2} H_2 + \theta_2 \alpha_2 \quad (59)$$

are optimal.

Replacing $\tilde{\lambda}_{1,1}$, $\tilde{\lambda}_{1,2}$, $\tilde{\lambda}_{2,1}$, $\tilde{\lambda}_{2,2}$ in (41) and (43), yields the simplified forms (recall that $\alpha = (\alpha_1, \alpha_2)$ and $\beta = (\beta_1, \beta_2)$) in the original market dynamics,

$$\begin{cases} a_1^* = \frac{\lambda_1 - \rho\lambda_2}{(1 - \rho^2)\sigma_1} H_1(\tilde{x}_1, \beta) + (1 - H_1(\tilde{x}_1, \beta)) \theta_1 \beta_1 \\ a_2^* = \frac{-\rho\lambda_1 + \lambda_2}{(1 - \rho^2)\sigma_1} H_1(\tilde{x}_1, \beta) + (1 - H_1(\tilde{x}_1, \beta)) \theta_1 \beta_2, \end{cases} \quad (60)$$

and

$$\begin{cases} \beta_1^* = \frac{\lambda_1 - \rho\lambda_2}{(1 - \rho^2)\sigma_1} H_2(\tilde{x}_2, \alpha) + (1 - H_2(\tilde{x}_2, \alpha)) \theta_2 \alpha_1, \\ \beta_2^* = \frac{-\rho\lambda_1 + \lambda_2}{(1 - \rho^2)\sigma_1} H_2(\tilde{x}_2, \alpha) + (1 - H_2(\tilde{x}_2, \alpha)) \theta_2 \alpha_2. \end{cases} \quad (61)$$

Discussion: The best-response forward criterion (rewritten with more explicit notation) is given by the locally riskless process $u_1 \left(\frac{\tilde{x}_1}{B_1(\beta, \rho, \theta_1)}, A_1(\lambda, \sigma, \beta; \rho, \theta_1) \right)$. This representation and the fact that processes A_1, B_1 depend on competitor's policy imply that the stochastic updating of the manager's investment goals are affected by the competitor's actions (a well-evidenced empirical result as we have previously argued). Note also that the process B_1 is higher than 1 and depends only on the competitor's policy β , the correlation ρ , and the competition parameter θ_1 . It is increasing in θ_1 , when $\theta_1 \in (0, \frac{1}{2})$, and decreasing when $\theta_1 \in (\frac{1}{2}, 1)$, with maximum discounting at $\theta_1 = \frac{1}{2}$. The discounting vanishes at the limiting values $\theta_1 = 0, 1$. For $\rho^2 \neq 1$, the process $C_2(\beta_1, \beta_2; \rho) > 0$ is jointly convex in (β_1, β_2) and achieves a global minimum at $(0, 0)$. The process $A_1(\lambda, \sigma, \beta, \theta_1)$ is non-decreasing in time and represents a stochastic time change. Furthermore, its time derivative is convex in the competition parameter θ_1 .

The case when manager 2 uses policies $(\beta_{1,t}^0)_{t \geq 0}$, $(\beta_{2,t}^0)_{t \geq 0}$ with $\beta_1^0 = -\frac{\lambda_1 - \rho\lambda_2}{(1 - \rho^2)\sigma_1}$, $\beta_2^0 = -\frac{\lambda_1 - \rho\lambda_2}{(1 - \rho^2)\sigma_2}$ requires special attention. Therein, the modified risk premia vanish at all times, $\tilde{\lambda}_{1,1} = \tilde{\lambda}_{1,2} = 0$, and thus the "personalized" fictitious market \mathcal{M}_1^d becomes *worthless*. In turn, $A_1 = M_1 = 0$, and $\alpha_1^* = \theta_1 \beta_1^0$ and $\alpha_2^* = \theta_1 \beta_2^0$. Therefore, the optimal risky strategy is to simply follow fraction θ_1 of this specific competitor's strategy. This yields $\tilde{X}_1^* = B_1 \tilde{x}_1$, with $B_1 := e^{\frac{1}{2}\theta_1(1-\theta_1)(1-\rho^2)\int_0^t (\lambda_1 + \lambda_2)^2 ds}$ and, thus, $\left(V_1 \left(\tilde{X}_1^*, t; \beta^0 \right) \right)_{t \geq 0} = u_1(\tilde{x}_1, 0)$. This is intuitively pleasing and consistent with the fact that, in a worthless market, the performance criterion should *not* change with time (provided all quantities are expressed in discounted units).

The optimal policy is given both via a feedback and in closed form (cf. (54) and (56)). The feedback control depends on wealth only through the random function $R_1(z, t)$, which is the relative risk tolerance associated with $u_1(z, t)$. Using the results in [52], we deduce that $R_1(z, t)$, and thus $\alpha_1(z, t)$ and $\alpha_2(z, t)$, are decreasing in time and non-increasing in z .

3.1.1 The CRRA case

Let the measure in (45) be a Dirac, $\nu_1(dy) = \delta_{\frac{1}{\gamma_1}}$, $\gamma_1 > 0$. Then, $h_1(z, t) = e^{\frac{z}{\gamma_1} - \frac{1}{2}\left(\frac{1}{\gamma_1}\right)^2 t}$ and $H_1(\tilde{x}_1, \beta) = \frac{1}{\gamma_1}$ (cf. (46) and (50)). Criterion (52) becomes

$$V_1(\tilde{x}_1, t; \beta) = \frac{1}{1 - \gamma_1} \left(\frac{\tilde{x}_1}{B_1} \right)^{1 - \gamma_1} e^{-\frac{1}{2} \frac{1 - \gamma_1}{\gamma_1} A_1},$$

and, this is the unique locally riskless homothetic criterion associated with γ_1 . The optimal policies and optimal wealth processes are given by (56) and (53),

$$\begin{aligned} \alpha_1^* &= \frac{1}{\gamma_1} \frac{\lambda_1 - \rho\lambda_2}{(1 - \rho^2)\sigma_1} + \left(1 - \frac{1}{\gamma_1}\right) \theta_1 \beta_1, \\ \alpha_2^* &= \frac{1}{\gamma_1} \frac{-\rho\lambda_1 + \lambda_2}{(1 - \rho^2)\sigma_2} + \left(1 - \frac{1}{\gamma_1}\right) \theta_1 \beta_2 \end{aligned}$$

and $\tilde{X}_1^* = \tilde{x}_1 e^{\frac{1}{\gamma_1} \left(1 - \frac{1}{2\gamma_1}\right) A_1 + \frac{1}{\gamma_1} M_1} B_1$. We recall that there is no assumption for the preferences of manager 2, only that she follows an arbitrary policy $\beta \in \mathcal{A}$.

Similarly, let manager 1 follow policy $\alpha = (\alpha_1, \alpha_2)$. If $\nu_2(dy) = \delta_{\frac{1}{\gamma_2}}$, for $\gamma_2 > 0$, $\gamma_2 \neq 1$, the unique locally riskless best-response forward criterion for manager 2 with initial condition $V_2(\tilde{x}_2, 0; \alpha) = \frac{\tilde{x}_2^{1 - \gamma_2}}{1 - \gamma_2}$ is given by

$$V_2(\tilde{x}_2, t; \alpha) = \frac{1}{1 - \gamma_2} \left(\frac{\tilde{x}_2}{B_2} \right)^{1 - \gamma_2} e^{-\frac{1}{2} \frac{1 - \gamma_2}{\gamma_2} A_2},$$

and the optimal policies and optimal wealth by

$$\beta_1^* = \frac{1}{\gamma_2} \frac{\lambda_1 - \rho\lambda_2}{(1 - \rho^2)\sigma_1} + \left(1 - \frac{1}{\gamma_2}\right) \theta_2 a_1, \quad \beta_2^* = \frac{1}{\gamma_2} \frac{-\rho\lambda_1 + \lambda_2}{(1 - \rho^2)\sigma_2} + \left(1 - \frac{1}{\gamma_2}\right) \theta_2 a_2 \quad (62)$$

and $\tilde{X}_{2,t}^* = \tilde{x}_2 e^{\frac{1}{\gamma_2} \left(1 - \frac{1}{2\gamma_2}\right) A_2 + \frac{1}{\gamma_2} M_2} B_2$.

3.2 Forward Nash equilibrium

The forward Nash equilibrium is defined as in Definition 5. To find the equilibrium strategies $(\alpha_t^*)_{t \geq 0}$, $(\beta_t^*)_{t \geq 0}$ one then needs to solve the non-linear system (cf. (60) and (61)),

$$\begin{cases} \alpha_1^* = \frac{\lambda_1 - \rho\lambda_2}{(1 - \rho^2)\sigma_1} H_1(\tilde{x}_1, \beta^*) + (1 - H_1(\tilde{x}_1, \beta^*)) \theta_1 \beta_1^* \\ \alpha_2^* = \frac{-\rho\lambda_1 + \lambda_2}{(1 - \rho^2)\sigma_2} H_1(\tilde{x}_1, \beta^*) + (1 - H_1(\tilde{x}_1, \beta^*)) \theta_1 \beta_2^* \\ \beta_1^* = \frac{\lambda_1 - \rho\lambda_2}{(1 - \rho^2)\sigma_1} H_2(\tilde{x}_2, \alpha^*) + (1 - H_2(\tilde{x}_2, \alpha^*)) \theta_2 \alpha_1^* \\ \beta_2^* = \frac{-\rho\lambda_1 + \lambda_2}{(1 - \rho^2)\sigma_2} H_2(\tilde{x}_2, \alpha^*) + (1 - H_2(\tilde{x}_2, \alpha^*)) \theta_2 \alpha_2^*. \end{cases} \quad (63)$$

The system is in general difficult to solve unless for special cases, one of which is examined next.

3.2.1 The CRRA case

We derive explicit solutions when both managers have homothetic forward criteria using (62) and (63).

Proposition 9 *Let $\gamma_1, \gamma_2 > 0$ with $\gamma_1, \gamma_2 \neq 1$, and assume that $\gamma_1 \gamma_2 - \theta_1 \theta_2 (1 - \gamma_1)(1 - \gamma_2) \neq 0$. Then, the Nash equilibrium strategies $(\alpha_t^*)_{t \geq 0}$, $(\beta_t^*)_{t \geq 0}$ are given as*

$$\alpha_1^* = c_\alpha \frac{\lambda_1 - \rho\lambda_2}{\sigma_1(1 - \rho^2)} \quad \text{and} \quad \alpha_2^* = c_\alpha \frac{\lambda_2 - \rho\lambda_1}{\sigma_2(1 - \rho^2)} \quad (64)$$

$$\beta_1^* = c_\beta \frac{\lambda_1 - \rho\lambda_2}{\sigma_1(1 - \rho^2)} \quad \text{and} \quad \beta_2^* = c_\beta \frac{\lambda_2 - \rho\lambda_1}{\sigma_2(1 - \rho^2)} \quad (65)$$

where the constants c_α and c_β are defined as

$$c_\alpha := \frac{\gamma_2 + \theta_1(\gamma_1 - 1)}{\gamma_1\gamma_2 - \theta_1\theta_2(1 - \gamma_1)(1 - \gamma_2)}, \quad c_\beta := \frac{\gamma_1 + \theta_2(\gamma_2 - 1)}{\gamma_1\gamma_2 - \theta_1\theta_2(1 - \gamma_1)(1 - \gamma_2)}.$$

Proof. Taking into account (62), we get that system (63) becomes linear. Assumption $\gamma_1\gamma_2 - \theta_1\theta_2(1 - \gamma_1)(1 - \gamma_2) \neq 0$ guarantees that the determinant is different than zero and hence the system admits a unique solution. Simple calculations imply (64) and (65) and standing assumptions on λ_1 and λ_2 yield the admissibility of the equilibrium investment strategies. ■

Similarly to the asset specialization setting, the Nash equilibrium strategies (64) and (65) have the same form as the ones in the log-normal market and backward utility maximization criteria (see [9, Proposition 2]). All conclusions in [9] hold for the general Itô-diffusion setting we assume herein.

4 Conclusions and extensions

We have studied portfolio allocations of two fund managers when they incorporate relative performance concerns. We have looked at the asset specialization and asset diversification settings in an Itô-diffusion market. For both these cases, we have considered the best response and the Nash equilibria. We studied these issues in a new framework we introduce herein that is based on forward performance criteria. These criteria allow for “real-time” updating of both the model coefficients and the competitor’s policies as well as for flexible horizons. Thus, we considerably generalize the existing work on the subject by allowing i) a considerably more general market model, ii) no a priori modeling of the competitor’s policy and iii) flexible investment horizons. Next, we discuss some possible extensions.

i) Multi-frequencies: In all cases herein, we have assumed that model selection, trading and relative/competitive performance valuation are all aligned and, furthermore, that they all occur continuously in time. In reality, however, these three fundamental attributes are not synchronized. A more realistic scenario would allow trading to take place more frequently than model selection, and relative performance evaluation to occur less frequently than trading. Note that the most extreme case is in the classical expected utility problem in which the terminal utility is specified only once, at initial time, with no further risk preference adjustment.

With regards to the relative frequency of trading and model selection, it is more realistic to assume that the model is selected for some trading period ahead, say a week, and that within this week, trading takes place in discrete or continuous time. When relative performance is involved, the distinct scales of time evolution are more critical, for each fund manager typically announces her performance at discrete times and not continuously.

ii) Information about market and competitors: Information availability and acquisition for both the market and the competitor’s behavior and performance are of tantamount importance. In the existing literature it is assumed that both managers have full access to both the market(s) and risk preferences. While we relax the requirement that neither the model dynamics nor the competitor’s input (risk preferences, chosen policy and investment horizon) need to be a priori modeled, we do assume that any information - acquired in real time - about them is available to both managers, together with their relative bias parameters. These assumptions are partially supported by existing results; see, for example, [37], where it is argued that managers acquire such information from the realized, and publicly available, returns of their peers.

However, several “under-specification” issues remain open, especially in terms of the manager’s risk preferences, specialized knowledge and past performance. For example, it might be more realistic to assume that at the end of each relative evaluation period, each fund manager receives information about the performance of the other and, right after, formulates a view about the possible upcoming performance till the end of the next evaluation period. This will partially address the absence of

complete information under asset specialization. In this case, injecting personal views could lead to a forward Black-Litterman type criterion under competition.

iii) Beyond locally riskless and reduced form relative performance/competition criteria: Herein, we worked with criteria that are, from the one hand, locally riskless processes and, from the other, of the “homogeneous” scaling (7) and (8). In general, relative performance concerns might be modeled, at the level of the criterion, by arbitrary \mathcal{F}_t -adapted processes, say $C_1(x_2, t)$ and $C_2(x_1, t)$. These processes might then model, in a more refined way, the competition dependence on past performance of the competitors, market conditions and time in a more realistic way.

In a different direction, the forward criteria might have volatility, which would capture uncertainty about the model dynamics and/or the competitor’s beliefs and policies. We will then work with criteria of the form

$$dU_1(x_1, x_2, t) = b_{1,t}(x_1, C_1(x_2, t), t) dt + a_{1,1,t}(x_1, C_1(x_2, t), t) dW_{1,t} + \alpha_{1,2,t}(x_1, C_1(x_2, t), t) dW_{2,t}$$

and

$$dU_2(x_1, x_2, t) = b_{2,t}(C_2(x_1, t), x_2, t) dt + a_{2,1,t}(C_2(x_1, t), x_2) dW_{1,t} + a_{2,2,t}(C_2(x_1, t), x_2, t) dW_{2,t},$$

with the volatilities $(a_{1,1,t}, a_{1,2,t})_{t \geq 0}$ and $(a_{2,1,t}, a_{2,2,t})_{t \geq 0}$ being adapted and manager-specific input processes. Proceeding as in [53] we would then obtain a stochastic PDE (rather than a random one) with coefficients depending on the evolving market dynamics and the competitor’s policies. As in the absence of relative concerns, these equations will be ill-posed and degenerate with little, if any, tractability. In turn, the systems related to the forward Nash equilibria (cf. (31) and (63)) would be systems of such infinite dimensional equations.

In a different direction, relative forward criteria may be modeled as discrete or a combination of discrete and continuous-time processes for different, possible nested, time regimes, associated with distinct frequencies as discussed above. For discrete processes, predictability is a natural assumption (see [3] as well as [62] for a binomial model and adaptive market parameter selection).

References

- [1] A. B. Abel. Asset prices under habit formation and catching up with the Joneses. *The American Economic Review*, 80(2):38–42, 1990.
- [2] A. G. Ahearne, W. L. Grier, and F. E. Warnock. Information costs and home bias: An analysis of US holdings of foreign equities. *Journal of International Economics*, 62(2):313–336, March 2004.
- [3] B. Angoshtari, T. Zariphopoulou, and X. Y. Zhou. Predictable forward performance processes: The binomial case. *SIAM Journal on Control and Optimization*, 58(1):327–347, 2020.
- [4] M. Anthropelos. Forward exponential performances: Pricing and optimal risk sharing. *SIAM Journal on Financial Mathematics*, 5(1):626–655, 2014.
- [5] L. Avanesyan, M. Shkolnikov, and R. Sircar. Construction of forward performance processes in stochastic factor models and an extension of Widder’s theorem. To appear in *Finance and Stochastics*, available at <https://arxiv.org/pdf/1805.04535.pdf>, 2018.
- [6] B. M. Barber, X. Huang, and T. Odean. Which Factors Matter to Investors? Evidence from Mutual Fund Flows. *The Review of Financial Studies*, 29(10):2600–2642, 2016.
- [7] N. Barberis and M. Huang. Stocks as lotteries: The implications of probability weighting for security prices. *The American Economic Review*, 98(5):2066–2100, 2008.
- [8] S. Basak and D. Makarov. Strategic asset allocation in money management. *Journal of Finance*, 69(1):179–217, 02 2014.

- [9] S. Basak and D. Makarov. Competition among portfolio managers and asset specialization. Paris December 2014 Finance Meeting EUROFIDAI - AFFI Paper, September 2015.
- [10] S. Basak, A. Pavlova, and A. Shapiro. Optimal asset allocation and risk shifting in money management. *The Review of Financial Studies*, 20(5):1583–1621, 2007.
- [11] J. B. Berk and J. H. van Binsbergen. Measuring skill in the mutual fund industry. *Journal of Financial Economics*, 118(1):1–20, 2015.
- [12] F. Berrier, C. Rogers, and M. Tehranchi. A characterization of forward utility functions. Preprint, May 2009.
- [13] J. Bielagk, A. Lionnet, and G. D. Reis. Equilibrium pricing under relative performance concerns. *SIAM Journal on Financial Mathematics*, 8(1):435–482, 2017.
- [14] A. Bodnaruk and A. Simonov. Loss-averse preferences, performance, and career success of institutional investors. *The Review of Financial Studies*, 29(11):3140–3176, 2016.
- [15] M. J. Brennan. The optimal number of securities in a risky asset portfolio when there are fixed costs of transacting: Theory and some empirical results. *Journal of Financial and Quantitative Analysis*, 10(03):483–496, September 1975.
- [16] K. Brown, W. V. Harlow, and L. Starks. Of tournaments and temptations: An analysis of managerial incentives in the mutual fund industry. *The Journal of Finance*, 51(1):85–110, 1996.
- [17] H. H. Cao, T. Wang, and H. H. Zhang. Model uncertainty, limited market participation, and asset prices. *The Review of Financial Studies*, 18(4):1219–1251, 2005.
- [18] H.-l. Chen and G. G. Pennacchi. Does prior performance affect a mutual fund’s choice of risk? theory and further empirical evidence. *Journal of Financial and Quantitative Analysis*, 44(04):745–775, August 2009.
- [19] J. Chevalier and G. Ellison. Risk taking by mutual funds as a response to incentives. *Journal of Political Economy*, 105(6):1167–1200, 1997.
- [20] J. Chevalier and G. Ellison. Career concerns of mutual fund managers. *The Quarterly Journal of Economics*, 114(2):389–432, 1999.
- [21] J. D. Coval and T. J. Moskowitz. Home bias at home: Local equity preference in domestic portfolios. *The Journal of Finance*, 54(6):2045–2073, 1999.
- [22] D. Cuoco and R. Kaniel. Equilibrium prices in the presence of delegated portfolio management. *Journal of Financial Economics*, 101(2):264–296, 2011.
- [23] X. Dong, S. Feng, and R. Sadka. Liquidity risk and mutual fund performance. *Management Science*, 65(3):1020–1041, 2019.
- [24] G.-E. Espinosa and N. Touzi. Optimal investment under relative performance concerns. *Mathematical Finance*, 25(2):221–257, 2015.
- [25] M. A. Ferreira, A. Keswani, A. F. Miguel, and S. B. Ramos. The determinants of mutual fund performance: A cross-country study. *Review of Finance*, 17(2):483–525, 2013.
- [26] C. Frei and G. D. Reis. A financial market with interacting investors: does an equilibrium exist? *Mathematics and Financial Economics*, 4:161–182, 2011.
- [27] K. R. French and J. M. Poterba. Investor diversification and international equity markets. *American Economic Review*, 81(2):222–226, May 1991.

- [28] S. Gallaher, R. Kaniel, and L. T. Starks. Madison avenue meets wall street: Mutual fund families, competition and advertising. Working paper, January 2006.
- [29] T. Geng. Essays on forward portfolio theory and financial time series modeling. *Ph.D. Thesis, University of Texas at Austin*, 2017.
- [30] J.-P. Gómez, R. Priestley, and F. Zapatero. Implications of keeping-up-with-the-Joneses behavior for the equilibrium cross section of stock returns: International evidence. *The Journal of Finance*, 64(6):2703–2737, 2009.
- [31] Z. He and A. Krishnamurthy. Intermediary asset pricing. *American Economic Review*, 103(2):732–770, April 2013.
- [32] J. Huang, C. Sialm, and H. Zhang. Risk shifting and mutual fund performance. *The Review of Financial Studies*, 24(8):2575–2616, 2011.
- [33] M. Kacperczyk, C. Sialm, and L. Zheng. On the industry concentration of actively managed equity mutual funds. *The Journal of Finance*, 60(4):1983–2011, 2005.
- [34] S. Källblad, J. Oblój, and T. Zariphopoulou. Dynamically consistent investment under model uncertainty: The robust forward criteria. *Finance and Stochastics*, 22:879–918, 2018.
- [35] N. E. Karoui and M. Mrad. An exact connection between tow solvable SDEs and a nonlinear utility stochastic PDE. *SIAM Journal on Financial Mathematics*, 4(1):697–736, 2014.
- [36] A. Kempf, S. Ruenzi, and T. Thiele. Employment risk, compensation incentives, and managerial risk taking: Evidence from the mutual fund industry. *Journal of Financial Economics*, 92(1):92–108, April 2009.
- [37] R. S. J. Koijen. The cross-section of managerial ability, incentives, and risk preferences. *The Journal of Finance*, 69(3):1051–1098, 2014.
- [38] J. L. Koski and J. Pontiff. How are derivatives used? Evidence from the mutual fund industry. *Journal of Finance*, 54(2):791–816, 1999.
- [39] D. Lacker and T. Zariphopoulou. Mean field and N-agent games for optimal investment under relative performance criteria. *Mathematical Finance*, 29(4):1003–1038, 2019.
- [40] T. Leung, R. Sircar, and T. Zariphopoulou. Forward indifference valuation of American options. *Stochastics*, 84(5-6):741–770, 2012.
- [41] B. Li, P. Luo, and D. Xiong. Equilibrium strategies for alpha-maxmin expected utility maximization. *SIAM Journal on Financial Mathematics*, 10(2):394–429, 2019.
- [42] G. Liang and T. Zariphopoulou. Representation of homothetic forward performance processes in stochastic factor models via ergodic and infinite horizon BSDE. *SIAM Journal on Financial Mathematics*, 8(1):344–372, 2017.
- [43] A. W. Lynch and D. K. Musto. How investors interpret past fund returns. *The Journal of Finance*, 58(5):2033–2058, 2003.
- [44] R. C. Merton. A simple model of capital market equilibrium with incomplete information. *The Journal of Finance*, 42(3):483–510, 1987.
- [45] T. Mitton and K. Vorkink. Equilibrium underdiversification and the preference for skewness. *The Review of Financial Studies*, 20(4):1255–1288, 2007.
- [46] O. Mostovyi. Optimal investment with intermediate consumption and random endowment. *Mathematical Finance*, 27(1):96–114, 2017.

- [47] O. Mostovyi and M. Sirbu. Optimal investment and consumption with labor income in incomplete markets. *Annals of Applied Probability*, 30(2):747–787, 04 2020.
- [48] S. Mukerji and J.-M. Tallon. Ambiguity aversion and incompleteness of financial markets. *The Review of Economic Studies*, 68(4):883–904, 2001.
- [49] M. Musiela and T. Zariphopoulou. Investments and forward utilities. Technical report, University of Texas at Austin, 2006.
- [50] M. Musiela and T. Zariphopoulou. *Derivative pricing, investment management and the term structure of exponential utilities: The case of binomial model*. Princeton University Press, 2009.
- [51] M. Musiela and T. Zariphopoulou. Portfolio choice under dynamic investment performance criteria. *Quantitative Finance*, 9(2):161–170, 2009.
- [52] M. Musiela and T. Zariphopoulou. Portfolio choice under space-time monotone performance criteria. *SIAM Journal on Financial Mathematics*, 1(1):326–365, 2010.
- [53] M. Musiela and T. Zariphopoulou. Stochastic partial differential equations and portfolio choice. *Contemporary Quantitative Finance*, pages 195–215, 2010.
- [54] S. Nadtochiy and M. Tehranchi. Optimal investment for all time horizons and martin boundary of space-time diffusions. *Mathematical Finance*, 27(2):438–470, 2017.
- [55] S. Nadtochiy and T. Zariphopoulou. A class of homothetic forward investment performance process with non-zero volatility. *Inspired by Finance: The Musiela Festschrift*, pages 475–505, 2014.
- [56] G. D. Reis and V. Platonov. Forward utilities and mean-field games under relative performance concerns, 2020. Sumbitted for publication, available at <https://arxiv.org/abs/2005.09461>.
- [57] K. Ron, S. Tompaidis, and T. Zhou. Impact of managerial commitment on risk taking with dynamic fund flows. Preprint, April 2016.
- [58] H. S. S. Kou and T. Zariphopoulou. Axiomatic construction of the forward performance criterion. work in progress, 2021.
- [59] W. F. Sharpe. Mutual fund performance. *The Journal of Business*, 39:119–119, 1965.
- [60] M. Shkolnikov, R. Sircar, and T. Zariphopoulou. Asymptotic analysis of forward performance processes in incomplete markets and their ill-posed HJB equations. *SIAM Journal on Financial Mathematics*, 7(1):588–618, 2016.
- [61] E. R. Sirri and P. Tufano. Costly search and mutual fund flows. *The Journal of Finance*, 53(5):1589–1622, 1998.
- [62] M. S. Strub and X. Y. Zhou. Evolution of the Arrow–Pratt measure of risk-tolerance for predictable forward utility processes. *Finance and Stochastics*, 25:331–358, 2021.
- [63] R. Uppal and T. Wang. Model misspecification and underdiversification. *The Journal of Finance*, 58(6):2465–2486, 2003.
- [64] J. H. van Binsbergen, M. W. Brandt, and R. S. J. Koijen. Optimal decentralized investment management. *The Journal of Finance*, 63(4):1849–1895, 2008.
- [65] S. Van Nieuwerburch and L. Veldkamp. Information immobility and the home bias puzzle. *The Journal of Finance*, 64(3), 2009.
- [66] W. Wagner. Systemic liquidation risk and the diversity-diversification trade-off. *The Journal of Finance*, 66(4):1141–1175, 2011.

- [67] G. Žitković. A dual characterization of self-generation and exponential forward performances. *Annals of Applied Probability*, 19(6):2176–2210, 2009.