TIME-CONSISTENT PENSION FUND MANAGEMENT IN STOCHASTICALLY CHANGING MARKETS AND EVOLVING HORIZONS

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Abstract. We propose an alternative model for a pension fund's investment policy that produces time-consistent portfolio strategies. Unlike the optimization problem faced by an individual member, a pension fund does not have a fixed terminal (retirement) time because new members continuously join and existing members retire over time. Consequently, setting an optimization goal for a specific future time can result in time-inconsistent investment strategies after any given time horizon. To address this challenge, we introduce a forward continuous-time model in which the investment criteria are defined by an adapted process that behaves as a supermartingale for any admissible strategy and becomes a true martingale under the optimal strategy. This model allows the pension fund manager to dynamically adjust risk aversion, beliefs, and the investment environment. We derive the necessary conditions for characterizing the solution through random partial differential equations and investigate whether well-known initial utility functions, such as the exponential utility, can provide explicit expressions for the optimal portfolio.

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1. Introduction

Discussion. Pension funds are not typical institutional investors. In principle, they have two main special characteristics that distinguish them from other large investors in the markets. First, they can not apply self-financing portfolios, since they do get exogenously given contributions from their members, but also they have to pay the liabilities. In fact both contributions and liabilities are stochastic and hence in the wealth process of a pension fund there exists an exogenously given (usually non-replicable) stochastic process.

In addition, the majority of pension funds invest in a collective manner for several generations. This holds not only for the defined benefit (DB) pension schemes, but also for the defined contribution

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(DC) ones. The direct consequence of this fact is that there is no specific time horizon that determines the investment goal of such a pension fund. Indeed, if the investment strategy were only for a single generation, the pension fund could form a target-dated fund where the time target is the actuarial estimation of the retirement year of this particular generation. However, for the collective investment schemes generation come and go in the pension fund and hence the placement of a specific time-horizon could lead to a sub-optimal investment strategies.

In this work, we propose a novel investment modeling for a pension fund, where there is a stochastic random endowment that stems from the contributions and liabilities and the investment criterion does not place an a-priori time horizon.

Under the continuous-time models, a standard model of an investor's goals that is linked with her risk preferences is through the excepted utility at a given terminal time. In other words, the optimal asset allocation problems can be formulated as classical stochastic optimization problems, where the controlled process is investor's wealth invested in risky assets and the optimization criterion is the conditional expectation of a wealth functional (utility).

A seminal well-known work on this issue is the so-called Merton's problem [Mer71], which has been extended to several different directions. Merton solves the investor's optimization problem with setting up the Hamilton-Jacobi-Bellman equation (HJB), a classic method to transform a stochastic optimization problem into a PDE, whose solution yields the optimal allocation.

This problem however works backward in time, in the sense that the investment goal is set at a given terminal time T and the solution is obtained from T to zero. According to the backward in time optimization process, the key element is the a-priori choice of both the horizon and the associated (deterministic) utility function. Intuitively, under the assumption that the investor sets her terminal utility at T, she moves backward in time, restructuring her portfolio according to the respective information at each point of time. As already mentioned this approach is not suitable for a pension fund.

Contributions. In this work, we propose and analyze the investment criteria of both Collective DC and DB pension funds through the notion of the *forward performance criterion* (forward utility process) targeted on the fund's surplus. In particular, we consider a pension fund as an investor with a stochastic endowment process where the optimization objective of her investment is set in a forward manner. For the later we use the notion of *forward performance criterion* that was introduced by Musiela and Zariphopoulou [MZ06] and [MZ08] and extended in [MZ09a] and [MZ10c].

While the investment problem and the risky asset are considered in a continuous-time basis, our proposed model imposes a sequence of discrete times which stand for the times generations retire and new ones enters the fund. At that time-setting, the fund's surplus is defined as the difference between the wealth that is created by investments and contributions minus the running fund's liabilities. Besides the fund's population, at each discrete time, the fund manager could update her model's parameters allowing in that way a dynamic updating of the model's dynamics.

Financial market is incomplete market, in that the risky asset is correlated with contributions that cannot be hedged out). In fact, the model of forward performance criterion allows us to impose a stock market with parameters that are stochastically changes through each period between the entrance and exit of generations. Hence, not only we deal with the absence of time horizon, but at the same time we generalize the market model.

Under families of power and exponential initial utility, we derive the optimal investment strategies for both the collective DC (CDC) and the collective DB (CDB) pension settings. Although the existence of the stochastic endowment does not allow explicit solution when the market is incomplete with stochastic coefficients, we do provide analytic formulas for the optimal investment strategies.

Based on the analytic expression of the optimal strategies we make a several predictions in terms of the models parameters. For example, we show that higher correlation between market and contribution, higher contributions' volatility, higher running population and lower market's volatility result in higher position in the risky asset.

Related literature. Based on the works of Musiela and Zariphopoulou, there is a continuity on the literature of forward preferences, for example, dual characterization by Zitcovic [Zit09], discrete-time binomial model by Angoshtari [AZZ20], maturity-independent risk measures by Zariphopoulou and Zitkovic [ZZ10], indifference valuation in discrete-time binomial model by Zariphopoulou et al. [MSZ16], indifference pricing and risk-sharing by [Ant14], pricing and hedging equity-linked life insurance by Chong [Cho19].

Relatively to fund management, Anthropelos et al. in [AGZ22] measure the performance of the managers' strategies via forward relative performance criteria, leading to the respective notions of forward best-response criterion and forward Nash equilibrium. Solving the CRRA cases, they develop a forward performance criteria for investment problems in Ito-diffusion markets under the presence of correlated random endowment process for both the perfectly and the incomplete market cases. At the case of pension fund management, Hillairet et al. in [HKM22], propose an optimal pair investment/pension policy for a Pay-As-You-Go (PAYG) pension scheme, where a social planner decides both the investment and the payment (to the pensioners) polices. Under a setting that is linked to DB pension schemes where liabilities are determined as a function of an exogenously-given minimum guarantee payment, they find the optimal strategies. Also, Hin Ng et al. in [HNC24], consider the optimal investment strategy of an individual employee who enrolls in a DC pension scheme during the accumulation phase. Preferences are given by forward utility preferences. They consider that the employee receives a stochastic salary, and under the assumption of an incomplete market, employee evaluates her pension fund's performance by the ratio of the fund value to her salary, ensuring that her pre-retirement habits could be consistent after her retirement.

Structure of the paper. Section 2 introduces the general setup of the market and after a brief description of the classic backward optimization problem, we present the main ideas and results of the forward performance criteria for individual investor. In Section 3, we introduce our main model of a collective defined contribution pension fund and apply it to a CRRA initial utility function, where we also provide some quantitative discussion. Section 4 is dedicated to a slightly more general model, which allows the application to both CDB and CDC schemes and provides analytic formulas under CARA initial utility function. We conclude our work in In Section 5.

2. From the Backward to the Forward Optimization Problem

We now briefly discuss some indicative cases of the backward optimization, with and without random income, both in complete and incomplete markets.

2.1. Self-financing Portfolios.

As stated before, when there is no exogenous income or withdrawal of capital, the portfolio's restructure is financed by its current value. At the beginning of the investment period ($t = 0$), the investor has an initial wealth $X_0 = x > 0$. Her wealth process, denoted by $\{X_t\}_{t\geq 0}$ consists of investments on a risky asset (stock), whose price $\{S_t\}_{t\geq 0}$ is represented by the process

$$dS_t = \mu S_t dt + \sigma S_t dW_t, \tag{2.1}$$

where $\mu \in$ and $\sigma \in_+$ and a riskless asset $\{R_t\}_{t\geq 0}$,

$$dR_t = rR_t dt, \qquad r > 0. \tag{2.2}$$

The price process of the risky asset is driven by a Brownian motion $\{W_t\}_{t\geq 0}$ that introduces the stochasticity into our model, defined on a probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t, t \geq 0), \mathbb{P})$. The dynamics of wealth process $\{X_t\}_{t\geq 0}$ can be written in terms of the diffusion dS_t and dR_t since those are the components constituting the portfolio. From (2.1) and (2.2), the wealth process is an Itô drift-diffusion process

$$dX_t = \pi_t \frac{dS_t}{S_t} + (X_t - \pi_t)\frac{dR_t}{R_t} = [rX_t + (\mu - r)\pi_t]dt + \sigma\pi_t dW_t, \tag{2.3}$$

where $(\pi_t)_{t\geq 0}$ is the money invested in risky asset. The integral form of (2.3) is

$$X_t = X_0 + \int_0^t [rX_s + (\mu - r)\pi_s]ds + \sigma \int_0^t \pi_s dW_s.$$

Investor's main purpose is to maximize the expected value of her terminal utility, so her objective criteria is then

$$\sup_{\pi\in\mathcal{A}} \mathbb{E}_{\mathbb{P}}[U(X_T)],$$

over a set of admissible strategies

$$\mathcal{A} = \{ \ \mathcal{F}\text{-measurable } \pi : \mathbb{E}\left[\int_0^t \pi_s^2 ds\right] < \infty, \forall t > 0\}.$$

Note that admissibility requirements guarantees that on the optimal portfolio (if it exists) the wealth process is a true martingale under probability measure $\mathbb{P}$. The classic family of utility functions is given below.

Assumption 2.1. The utility function $U : domU \to \mathbb{R}$ is strictly increasing, strictly concave and continuously differentiable function, which satisfies the *Inada Conditions*

$$\frac{\partial U}{\partial x}|_{x=\infty} = \lim_{x\to\infty} \frac{\partial U}{\partial x}(x) = 0 \quad \text{and} \quad \frac{\partial U}{\partial x}|_{x=0} = \lim_{x\to 0} \frac{\partial U}{\partial x}(x) = +\infty,$$

where its effective domain is $domU := \{x : U(x) > -\infty\}$ and is not empty.

Maximizing of expected utility over a given set of admissible strategies yields the so-called *Value Function* (otherwise known as *Indirect Utility Function*) (see e.g. 11.1 in [Ber00]).

Now we give a demonstration of how a value function is determined. A descriptive way to show this is by considering a partition of the total period $[0, T]$ into non-overlapping periods $[T_n, T_{n+1}]$, where $T_N = T$, and at each change of interval we reconstruct the portfolio. Hence, we start from period $[T_{N-1}, T]$ where the maximization problem is formed as

$$\sup_{\pi} \mathbb{E}_{\mathbb{P}}[U(X_T^{\pi})|\mathcal{F}_{T_{N-1}}].$$

The solution of the above maximization problem, π^*_{N-1}, is a random vector depending on the stock market's information as well on the portfolio's value until T_{N-1}. Then, *backwardly in time*, we stand at $[T_{N-2}, T_{N-1}]$ and we solve the maximization problem

$$\sup_{\pi} \mathbb{E}_{\mathbb{P}}[U_{T_{N-1}}(X^{\pi}_{T_{N-1}})|\mathcal{F}_{T_{N-2}}].$$

At this point we need to clarify which is that function $U_{T_{N-1}}$ placed as the intermediate investment goal. The answer comes from the *Time Consistency Rule*, according to which, when an investment is optimal in $[0, T]$, then it is optimal for each $[t, t+1]$, $0 \le t \le T-1$. Namely, the maximization problem:

$$[T_{N-2}, T] : \quad \sup_{\pi} \mathbb{E}_{\mathbb{P}}[U(X_T^{\pi})|\mathcal{F}_{T_{N-2}}],$$

must give the same optimal solution as in the two above maximization problems.

This is the main idea of the so-called *Dynamic Programming Principle*[1] according to which the consistent utility function that has to be placed at the first period $[T_{N-1}, T]$ and which satisfies the time consistency rule is the value function

$$V_{T_{N-1}}(X_T) = \sup_{\pi} \mathbb{E}_{\mathbb{P}}[U(X_T)|\mathcal{F}_{T_{N-1}}],$$

and hence

$$[T_{N-2}, T_{N-1}] : \quad V_{T_{N-2}}(X_{T_{N-1}}) = \sup_{\pi} \mathbb{E}_{\mathbb{P}}[U(X_T)|\mathcal{F}_{T_{N-2}}]$$

[1]Bellman's optimality principle, initiated by Bellman [Bel57] and also called the Dynamic Programming Principle (DPP), is a fundamental principle in control theory: it formally means that if one has followed an optimal control decision until some arbitrary observation time, say t, then, given this information, it remains optimal to use it after t. The Dynamic Programming Principle states that for a controlled Markov process, i.e. a process which is independent of all movements in the past, we can divide the optimization problem into the sub-intervals $[t, \theta]$, $[\theta, T]$. We find the optimal control on $[t, T]$ by searching for an optimal control with the initial point $(\theta, X_\theta^{t;x})$ and then maximizing the expected value over the controls on $[t, \theta]$.

$$= \sup_{\pi} \mathbb{E}_{\mathbb{P}}[V_{T_{N-1}}(X_{T_{N-1}})|\mathcal{F}_{T_{N-2}}].$$

That is the process that we follow until $t = 0$ finding the optimal strategy π_0^*. The accurate definition of a value function follows.

Definition 2.2 (Value Function)**.** Let T be the time horizon, U the pre-specified terminal utility function and $\mathcal{A}$ the set of admissible investment strategies. The value function is defined as:

$$V(X, t; T) = \sup_{\pi \in \mathcal{A}} \mathbb{E}_{\mathbb{P}}[U(X_T)|\mathcal{F}_t].$$

Otherwise,

$$V(X_s^*, s; T) = \begin{cases} \mathbb{E}_{\mathbb{P}}[V(X_{s'}^*, s'; T)|\mathcal{F}_s], & t \leq s \leq s' < T \\ U(X_T^*), & s = T. \end{cases}$$

In words, the value function could be seen as a random field, pre-specified at the end of the horizon and becomes a true *martingale* at the optimum and a *supermartingale* otherwise. Traditionally, value function is also called as the optimal solution with fundamental properties the super-martingality for arbitrary investment strategy and martingality at an optimum, which serves as the intermediate (indirect) utility in the relevant market environment (see e.g. §3.3 of [Huy09]).

The martingality property stems from the natural requirement that, if the process is currently at an optimal state, one needs to seek for controls so that the same average performance level is preserved at all future times. Note that the above optimization problem does not impose any liabilities at any time until T. The solution could be divided into four steps. The first is to formulate the problem, we need to know how the value function is defined and to find an SDE that represents its state process. The second step is to use dynamic programming principle to derive a Hamilton-Jacobi-Bellman PDE, while the third step starts with differentiating and plug into the HJB equation the optimal investment. The fourth step is to try to solve PDE whose solution yields the value function and in turn the optimal investment strategy.

More precisely, we first apply Itô's Lemma for value function V (assuming for now that it is twice differentiable),

$$dV(X_t, t) = V_t dt + V_X dX_t + \frac{1}{2} V_{XX} (dX_t)^2$$

and get that

$$dV(X_t, t) = \left(V_t + V_X \Big(rX_t + (\mu - r)\pi_t \Big) + \frac{1}{2} V_{XX} \sigma^2 {\pi_t}^2 \right) dt + \sigma \pi_t V_X dW_t.$$

This immediately implies that $V(X_t, t)$ is itself an Itô drift-diffusion process, which should be a martingale at the optimal level π_t^*. Hence, the drift has to be negative for any strategy and zero at the optimal which produces the Hamilton-Jacobi-Bellman equation

$$V_t^* + V_X^* \Big(rX_t + (\mu - r)\pi_t \Big) + \frac{1}{2} V_{XX}^* \sigma^2 {\pi_t}^2 = 0 \tag{2.4}$$

that should hold at the optimal. Maximizing the drift with respect to π gives the optimal investment in risky asset

$$\pi_t^* = -\frac{(\mu - r)V_X^*}{\sigma^2 V_{XX}^*}. \tag{2.5}$$

In order to find the value function, we need to substitute π_t^* in Hamiltonian equation (2.4):

$$V_t^* + V_X^*\left(rX_t + (\mu - r)\Big(-\frac{(\mu - r)V_X^*}{\sigma^2 V_{XX}^*}\Big)\right) + \frac{1}{2}V_{XX}^*\sigma^2\Big(-\frac{(\mu - r)V_X^*}{\sigma^2 V_{XX}^*}\Big)^2 = 0$$

that yields the optimal value function PDE

$$V_t^* + rX_tV_X^* - \frac{(\mu - r)}{2\sigma^2}\frac{V_X^{*\,2}}{V_{XX}^*} = 0. \tag{2.6}$$

As for the boundary condition, we know that at the terminal time T the value function satisfies $V(x,T) = f(T)U(x)$, where f is a positive function of time. For example, Merton uses $U(x) = x^{1-\gamma}/(1-\gamma)$ (CRRA utility function), where $\gamma > 0$ is the so-called coefficient of relative risk aversion (that is generally defined as $\gamma = -V''/V'$). We conjugate the solution of the form

$$V^*(x,t) = f(t)\frac{x^{1-\gamma}}{1-\gamma}, \tag{2.7}$$

that gives

$$V_t^* = f_t\frac{x^{1-\gamma}}{1-\gamma}, \quad V_X^* = fx^{-\gamma}, \quad V_{XX}^* = -\gamma f x^{-\gamma-1}.$$

Substituting the guess solution in (2.6), we get the simple ODE for f

$$f' = \nu f, \qquad \text{where} \quad \nu = -(1-\gamma)\left(\frac{(\mu - r)^2}{2\gamma\sigma^2} + r\right),$$

where we could simply consider the boundary condition $f(T) = 1$. We readily derive the unique solution

$$f(t) = \begin{cases} e^{-\nu(T-t)}, & \nu \neq 0 \\ 1, & \nu = 0. \end{cases}$$

The induced value function is then

$$V^*(X_t, t) = \frac{X_t^{1-\gamma}}{1-\gamma}e^{-\nu(T-t)}, \quad (\nu \neq 0),$$

and the respective optimal investment in risky asset is

$$\pi_t^* = \frac{\mu - r}{\sigma^2\gamma}X_t, \quad \gamma \geq 0, \quad \gamma \neq 0. \tag{2.8}$$

Remark 2.3. The numerator of the ratio is the difference between the risky asset drift and the risk-free rate of return. In the case the numerator is negative, and therefore r is bigger than μ, the investor takes short position on the risky asset and invests all her wealth to the risk-free one. In the case $\mu > r$, the investor invests at least a fraction of her wealth on the risky asset. This fraction is occasionally called *Merton ratio* and a particular feature of it is that it tends to decrease with an increase in volatility.

Remark 2.4. For $\gamma = 1$, power utility (CRRA) function becomes the logarithmic utility function $U(x) = \ln(x)$ and the optimal investment in the risky asset is consistent by setting $\gamma = 1$ in (2.8).

The representation (2.5) gives the optimal asset allocation under the assumption that the portfolio is self-financing and there are no liabilities to be covered at any time $t \leq T$. Hence, the optimal portfolio depends only on the market's characteristics, namely at the characteristics of the risky assets whose returns generate the wealth process. At the following section we study the case of a self-financing portfolio in the presence of liabilities taking a position of a DB pension fund.

2.1.1. *Optimization of a DB pension fund.* In contrast to the Merton's optimization problem, Sundaresan and Zapatero [SZ97] introduced the idea of the optimization problem of a self-financing portfolio having an initial endowment taking also into account a particular kind of liabilities that have to be covered at the terminal time. They consider a pension fund that has a contractual obligation to satisfy the payoff of a pension plan at time T stemming from the weighted average of the latest wages

$$P_T = \int_0^T a_t C_t dt, \tag{2.9}$$

where the wage of an employee derives from the following stochastic process

$$dC_t = mC_t dt + uC_t dW_t, \tag{2.10}$$

where m, u are positive constants and $\{a_t\}_{t\geq 0}$ is a deterministic process that stands for the level that wage W_t contributes to the pension plan's obligation.

The crucial assumption that they made is that wages are perfectly correlated with the risky assets, namely both the process of the risky asset and the wage are driven by the same Brownian Motion $(W_t)_{t\geq 0}$, an assumption that makes the market complete. It is also quite usual in pension fund industry that managers charge performance-fee. This creates extra motive (in terms of higher compensation) to achieve a better performance. For this, we follow the related idea of Sundaresan and Zapatero and capture these incentives through the following utility maximization of the surplus

$$\sup_{\tilde{\pi}\in\mathcal{A}} \mathbb{E}[U(X_T - P_T)|\mathcal{F}_t], 0 \leq t \leq T.$$

Instead of ϵ_t that we have used in the previous section in order to define the surplus process in discrete time, from now on we use the notation

$$x_t = X_t - P_t.$$

Dynamics of the surplus process is

$$dx_t = \Big(rx_t + (\mu - r)\tilde{\pi}_t\Big)dt + \sigma\tilde{\pi}_t dW_t.$$

Similarly as in the case of a self-financing portfolio without liabilities, we use Ito's Lemma on $V(x_t, t)$

$$dV(x_t, t) = \left(V_t + V_x\Big(rx_t + (\mu - r)\tilde{\pi}_t\Big) + \frac{1}{2}V_{xx}\sigma^2\tilde{\pi}_t^2\right)dt + \sigma\tilde{\pi}_t V_x dW_t.$$

Following the same steps as before we conclude that value function satisfies again (2.6). Simple calculations gives that the optimal investment strategy under liabilities (in a complete market) is

$$\tilde{\pi}_t^* = \frac{\mu - r}{\sigma^2 \gamma} x_t = \frac{\mu - r}{\sigma^2 \gamma}(X_t - P_t) = \pi_t^* - \frac{\mu - r}{\gamma \sigma^2} P_t,$$

where π_t^* is the optimal investment under no liability (see (2.5)). This means that higher pension's liabilities makes the manager to invest less is the investment in risky asset. Since the wealth is formed only by the returns of the assets (there is no random income process), the manager has to cover all her liabilities through investment in risky asset. Since, however, there is perfect correlation between risky asset and liabilities lower position in risky asset can be seen as a diversification choice.

Note also that it is impossible to ensure with probability one that pension's liabilities will be covered unless the initial wealth is sufficiently high. In practice, contributions to the pension fund come dynamically through time in a non-perfectly correlated way. This is the subject of the following subsection, which provides our contribution to the field.

2.1.2. *Incomplete Markets.* Herein, the pension fund is not funded solely by the investment's return, but also from some random contributions through time. In contrast to Sundaresan and Zapatero [SZ97] who consider an initial constant endowment, we consider a random process as defined in [JR01], which represents the continuous inflows to the fund. Contributions' random process is a given (known and fixed) proportion of the wage, $y_t := hC_t$, where $h > 0$ is the constant amount of the wage $\{C_t\}_{t \geq 0}$ that is going to be contributed to the pension fund. The randomness of the contributions to the fund comes from the randomness of the stochastic wage. We consider the case of an *incomplete market*, where the pension fund is endowed with a stochastic income that cannot be replicated by trading the available securities. We argue that the usually imposed assumption of market's completeness is rather unrealistic. In order to incorporate the incompleteness in the model we consider an additional Brownian Motion W', correlated with $\rho(W, W') \in (-1, 1)$. Risky asset is given again by (2.1) and the contribution process is now

$$dy_t = m y_t dt + u y_t dW_t'. \tag{2.11}$$

The surplus evolves according to the equation

$$dx_t = (r x_t + \hat{\pi}_t(\mu - r) + m y_t)\, dt + (\sigma \hat{\pi}_t dW_t + u y_t dW_t'). \tag{2.12}$$

Applying Ito's formula on V we get that

$$dV(x_t, t) = V_t dt + V_x \Big(\Big(r x_t + \hat{\pi}_t(\mu - r) + m y_t \Big) dt + (\sigma \hat{\pi}_t dW_t + u y_t dW_t') \Big) + \frac{1}{2} V_{xx} (\sigma \hat{\pi}_t dW_t + u y_t dW_t')^2$$

and taking into account that $(dW_t)(dW_t') = \rho dt$, we get the HJB equation

$$V_t^* + V_x^* \Big(r x_t + (\mu - r)\hat{\pi}_t + m y_t \Big) + \frac{1}{2} V_{xx}^* \sigma^2 \hat{\pi}_t^2 + V_{xx}^* \rho \sigma \hat{\pi}_t u y_t + \frac{1}{2} V_{xx}^* u^2 y_t^2 = 0. \tag{2.13}$$

From the partial derivative with respect to $\hat{\pi}$, we get the optimal investment

$$\hat{\pi}_t^* = -\frac{(\mu - r)V_x^*}{\sigma^2 V_{xx}^*} - \rho\frac{u}{\sigma}y_t. \tag{2.14}$$

The optimal value function PDE, by replacing (2.14) in (2.13),is

$$V_t^* + rx_tV_x^* - \frac{(\mu - r)}{2\sigma^2}\frac{V_x^{*2}}{V_{xx}^*} + V_x^*\left(my_t - \rho\frac{u}{\sigma}(\mu - r)y_t\right) + \frac{1}{2}V_{xx}^*u^2y_t^2(1-\rho^2) = 0. \tag{2.15}$$

We see that the stochastic random correlated income makes the PDE of value function more involved than those of (2.4).

Remark 2.5. In principle, the optimal portfolio that satisfies consists of two individual terms. The first one identifies with the Merton's optimal portfolio (2.5) that depends on market's characteristics, and the second term depends on random income. Particularly, the higher the contribution to the fund is, the less the investment in risky assets is placed. When the pension liabilities increase, it means that contributions increase too, since both of them are increasing functions of the wage. At the case of positive correlation, the pension manager prefers to invest less in risky assets in order to cover her pension liabilities (as in the case of complete markets).

Therefore, when there is not a random income and hence the stochasticity comes from a unique Brownian Motion, optimal portfolio π_t^* identifies with Merton's fraction, satisfying (2.5) and depends only on the risky assets parameters. When there is a random income, optimal investments consists of one more term that depends on the random income (contributions) and its correlation with the market. At the case of incomplete markets, optimal investment satisfies (2.14). So, when the correlation is negative, $\rho \in (-1, 0)$, the investment in risky assets increases, while when the correlation is positive, $\rho \in (0, 1)$ the respective investment decreases.

2.2. **Introduction to Forward Utility Process.** So far, we have seen only backward utility optimization problems. At this section we give a brief introductory to an alternative optimization methodology established by Musiela and Zariphopoulou, [MZ06].

In the classical Von Neuman-Morgesntern utility maximization problem and its use for optimal portfolio choice (Merton 1971), we have assume a specific (terminal) time-horizon T, where the optimization criterion (i.e., the expected utility from the wealth possessed at time T is placed). For any $t < T$, the dynamic programming principle implies that the objective that should be placed at t is the value function (indirect utility). However, there is no extension of this theory after T. It is quite common in the literature on asset-liability management on pension funds to use optimization myopically, in the sense that the same optimization criterion (such as expected utility maximization) is placed at the end of each period. This most likely violates the time-consistency of investment process and results in sub-optimal portfolios. In the case of pension fund's investment, this sub-optimality is essentially transferred to pension members, whose contributions are not invested according to a promised optimality setting. As Musiela states in [Mus22], Merton's problem is in fact a single-period optimisation problem, in which the opportunity set is a result of applying self-financing strategies to the continuous-time asset dynamics and collecting the random variables generated by that procedure. Clearly, such an approach leads to some form of time-inconsistency.

Alternatively, we should place an optimization goal at any point in time $t > T$ that guarantees the time-consistency of the optimal investment strategies. This is crucial for pension funds, since they do not have an expiration day; new members enter while older retire. Hence, the classical (backward) formulation is not suitable for modelling their investment goals. Note also that, along with the market, the investment criteria should also be stochastic, since both risk aversion and subjective probability measures (beliefs) have to be updated through time. For this, Musiela and Zariphopoulou came up with the concept of *forward utility*, also called *forward performance*.

This concept of forward utility is quite different from the classical utility, as formulated in the Merton problem, where utility is defined only for a single point of time in the future. In the forward utility framework, utility is stochastic and there is no specific time horizon. Forward utility is assumed to be measurable with respect to the flow of information (filtration), a reasonable feature, since new information should be taken into account for the updating of risk preferences.

Another important difference with the classical recursive utility is that the forward utility embeds information about investment opportunities and as such is directly linked with the assumptions we are making about the asset dynamics. The concept of recursive utility is more general as it is not linked so strongly with the specification of investment opportunities. As in the Merton problem, the recursive utility may be defined exogenously from the opportunity set. This, however, is not the case for the forward utility.

The notion of forward performance criterion has studied by several authors and under many different situations. It was introduced by Musiela and Zariphopoulou in both discrete [MZ09a] and continuous time [MZ06], [MZ10c]. In fact, in [MZ06] they introduce a class of forward dynamic utilities and in [MZ08], they also analyze a portfolio management problem with exponential forward criteria. Under minimal model assumptions they explicitly construct the forward performance process and the associated optimal wealth and asset allocations. Furthermore, the above authors in [MZ09b] (also in [MZ10a] and [MZ10b]), propose the forward performance process as an alternative approach of measuring performance of self- nancing investment strategies, with flexibility with regards to the aforementioned a priori choices while preserving the natural optimality properties of the value function process (martingality at an optimum and supermartingality away from it). One of their works that is relative enough to our particular section is [MZ07], where they analyzed the investment and pricing problems of investors endowed with backward (BDU) and forward (FDU) dynamic exponential utilities. These utilities have similarities but also striking differences. These features are, in turn, inherited by the associated optimal policies, indifference prices, and risk monitoring strategies. They first observe that the backward and forward utilities are produced via a conditional expected criterion. They are both self-generating, in that they coincide with their implied value functions. Moreover, in the absence of exogenous cash flows, investors endowed with such utilities are indifferent to the investment horizons. Backward and forward dynamic utilities are constructed in entirely different ways. Backward utilities are first specified at a given future time T and, they are, subsequently, generated at previous to T times. Forward utilities are defined at present s and are, in turn, generated forward in time. The times T and s, at which the backward and forward utility data are determined, are the backward and forward normalization points.

2.2.1. *A Forward framework introduced by Musiela and Zariphopoulou.* This new class of performance criterion is characterized by a deterministic datum $u_0(x)$ at the beginning of the period, and a family of adapted random fields $U_t(x)$ such that: $U_0(x) = u_0(x)$, where $U_t(X_t)$ is a supermartingale for any investment strategy, while there exists a strategy at which U_t is a martingale. Thus, this criterion does not need the pre-specification of a time horizon. The adjective *forward* refers to the fact that the utility is constructed forward in time starting from zero.

In contrast to the backward framework, we may generalize the process of the risky asset

$$dS_t = S_t(\mu_t dt + \sigma_t dW_t),$$

and

$$dR_t = rR_t dt,$$

with the difference that the drift $\{\mu_t\}_{t\geq 0} \in \mathcal{A}$, volatility $\{\sigma_t\}_{t\geq 0} \in \mathcal{A}$, and interest rate $\{r_t\}_{t\geq 0} \in \mathcal{A}$, are now measurable processes. Furthermore, we define the $\mathcal{F}_t$-adapted process

$$\mu_t - r_t = \sigma_t \lambda_t,$$

the existence of which ensures the absence of arbitrage opportunities.

The investor uses the opportunity to invest in order to satisfy her impatience, her risk-aversion and her preferences to higher-than-lower wealth levels. Starting at $t = 0$ with an initial endowment x at time zero, she invests at future times $t > 0$ in all available assets following a *self-financing strategy.* Hence, the process $\{X_t, t \geq 0\}$ satisfies

$$dX_t = \pi_t(\mu_t - r_t)dt + \pi_t \sigma_t dW_t = \pi_t \sigma_t(\lambda_t dt + dW_t).$$

We now formally define the forward utility.

Definition 2.6 (Forward Utility Process)**.** Let $U(X_t, t)$ be an $\mathcal{F}_t$ - adapted process. We say that $U(X_t, t)$ is a *forward utility process*, if:

(1) $U(x, t)$ is an increasing and concave function of wealth x, $\forall t \geq 0$.
(2) For all $t \leq T$ and $\pi \in \mathcal{A}$, the induced wealth X satisfies :

$$\mathbb{E}_{\mathbb{P}}\left[U(X_T, T)|\mathcal{F}_t\right] \leq U(X_t, t) \quad \textit{supermartingale property.}$$

(3) For all $t \leq T$ there exists a self-financing strategy $\pi^* \in \mathcal{A}$ for which the induced wealth process X^* satisfies:

$$\mathbb{E}_{\mathbb{P}}\left[U(X_T^*, T)|\mathcal{F}_t\right] = U(X_t^*, t) \quad \textit{martingale property}$$

(4) $U(x, 0) = v_0(x)$.

A class of forward utilities. In [MZ06] a class of forward utilities defined for the construction of which we need to combine the stochastic input market with the utility input. The investor chooses an initial utility function $v_0(x)$ that represents her preferences for today $t = 0$. Then, the utility input is being constructed by solving the following partial differential equation:

$$\begin{cases} v_t v_{xx} = \frac{1}{2} v_x^2 \\ v(x,0) = v_0(x) \end{cases} \tag{2.16}$$

while the market input are given by

$$\begin{cases} dA = [\sigma(\lambda+\phi)-\delta]^2 dt \\ A_0 = 0 \end{cases}, \quad \begin{cases} dY = Y\delta(\lambda dt + dB) \\ Y_0 = 1 \end{cases}, \quad \begin{cases} dZ = Z\phi dB \\ Z_0 = 1 \end{cases}, \tag{2.17}$$

where A is an $\mathcal{F}_t$ - adapted process, with $\delta, \phi \in \mathbb{R}$, Y is $\mathcal{F}_t$ - adapted process called benchmark process and Z is an $\mathcal{F}_t$ exponential martingale. In order to construct the forward utility process, the above market relevant information is injected into the variational input $v(x,t)$.

Another major components in the definition of the optimal utility volume and the optimal portfolio is the *local risk tolerance*, defined by

$$r(x,t) = -\frac{v_x(x,t)}{v_{xx}(x,t)}. \tag{2.18}$$

and the *benchmarked-suborfinated risk tolerance* process defined as

$$R = r\left(\frac{X}{Y}, A\right). \tag{2.19}$$

Theorem 2.7. *(Musiela and Zariphopoulou, [MZ06]). Let A, Y, Z be defined in* (2.17) *and the function v that solves* (2.16)*. Then,*

(1) *The process $U(x,t)$ defined by*

$$U(x,t) = v\left(\frac{x}{Y_t}, A_t\right) Z_t$$

is a forward utility process, and

(2) *the optimal portfolio is*

$$\pi_f^* = Y \frac{\left((\frac{X^*}{Y} - R^*)\delta + R^*(\lambda+\phi) \right)}{\sigma},$$

where

$$R^* = r\left(\frac{X^*}{Y}, A\right). \tag{2.20}$$

Proof. We analytically provide the proof for thesis self-completion. For the function $U(x,t) = v(x/Y_t, A_t)Z_t$ to be forward utility process it must be super-martingale for each π and martingale for the optimal π^*.

We apply Ito's formula to $U(x,t)$ to obtain

$$\begin{aligned} dU &= d\left(v\left(\frac{X}{Y},A\right)Z\right) \\ &= dv\left(\frac{X}{Y},A\right)Z + v\left(\frac{X}{Y},A\right)dZ + d < v\left(\frac{X}{Y},A\right), Z > . \end{aligned}$$

Moreover,

$$dv\left(\frac{X}{Y},A\right) = v_x\left(\frac{X}{Y},A\right)d\left(\frac{X}{Y}\right) + v_t\left(\frac{X}{Y},A\right)dA + \frac{1}{2}v_{xx}\left(\frac{X}{Y},A\right)d < \frac{X}{Y} >$$

where

$$d\left(\frac{X}{Y}\right) = \left(\frac{1}{Y}\sigma\pi - \frac{X}{Y}\delta\right)((\lambda-\delta)dt + dB).$$

Consequently,

$$\begin{aligned} dv\left(\frac{X}{Y},A\right)Z = v_x\left(\frac{X}{Y},A\right)Z\left(\frac{1}{Y}\sigma\pi - \frac{X}{Y}\delta\right)((\lambda-\delta)dt + dB) + v_t\left(\frac{X}{Y},A\right)Z\left(\sigma(\lambda+\phi)-\delta\right)^2 dt \\ + \frac{1}{2}v_{xx}\left(\frac{X}{Y},A\right)Z\left(\frac{1}{Y}\sigma\pi - \frac{X}{Y}\delta\right)^2 dt. \end{aligned}$$

Also,

$$v\left(\frac{X}{Y},A\right)dZ = v\left(\frac{X}{Y},A\right)Z\phi dW = UZ\phi dB,$$

and

$$\begin{aligned} d < v\left(\frac{X}{Y},A\right), Z > &= v_x\left(\frac{X}{Y},A\right)d < \frac{X}{Y}, Z > \\ &= v_x\left(\frac{X}{Y},A\right)d\left(\frac{X}{Y}\right)dZ \\ &= v_x\left(\frac{X}{Y},A\right)\left(\frac{1}{Y}\sigma\pi - \frac{X}{Y}\delta\right)((\lambda-\delta)dt + dW)Z\phi dB \\ &= v_x\left(\frac{X}{Y},A\right)\left(\frac{1}{Y}\sigma\pi - \frac{X}{Y}\delta\right)Z\phi dt. \end{aligned}$$

Dropping the arguments in v_x, v_t, v_{xx} we can write:

$$\begin{aligned} dU &= v_xZ\left(\frac{1}{Y}\sigma\pi - \frac{X}{Y}\delta\right)((\lambda-\delta)dt + dB) + v_tZ\left(\sigma(\lambda+\phi)-\delta\right)^2 dt \\ &+ \frac{1}{2}v_{xx}Z\left(\frac{1}{Y}\sigma\pi - \frac{X}{Y}\delta\right)^2 dt + v_x\left(\frac{X}{Y},A\right)\left(\frac{1}{Y}\sigma\pi - \frac{X}{Y}\delta\right)Z\phi dt + U\phi dW \\ &= \left(v_x\frac{Z}{Y}\sigma\pi - v_x\frac{XZ}{Y}\delta + U\phi\right)dB \\ &+ v_xZ\left(\frac{1}{Y}\sigma\pi - \frac{X}{Y}\delta\right)((\lambda-\delta)dt + v_tZ(\sigma(\lambda+\phi)-\delta)^2 dt \\ &+ \frac{1}{2}v_{xx}Z\left(\frac{1}{Y}\sigma\pi - \frac{X}{Y}\delta\right)^2 dt + v_x\left(\frac{1}{Y}\sigma\pi - \frac{X}{Y}\delta\right)Z\phi dt. \end{aligned}$$

For the deterministic term of the above equation, using (2.18), (2.19) and (2.16) we get that

$$\frac{1}{2}v_{xx}Z\left(\frac{1}{Y}\sigma\pi - \left(\frac{X}{Y} - R\right)\delta + R\sigma(\lambda+\phi)\right)^2 dt + \left(v_t - \frac{1}{2}v_{xx}\left(\frac{v_x}{v_{xx}}\right)^2\right)(\sigma(\lambda+\phi)-\delta)^2\, dt$$
$$= \frac{1}{2}v_{xx}Z\left(\frac{1}{Y}\sigma\pi - \left((\frac{X}{Y} - R)\delta + R\sigma(\lambda+\phi)\right)\right)^2 dt.$$

Now we observe that the process U is a supermartingale for each π and it is a martingale for π^* such that

$$\frac{1}{Y}\sigma\pi - \left((\frac{X}{Y} - R)\delta + R\sigma(\lambda+\phi)\right) = 0,$$

and hence the optimal portfolio in risky assets is

$$\frac{1}{Y}\pi^*_f = \frac{\left((\frac{X^*}{Y} - R^*)\delta + R^*(\lambda+\phi)\right)}{\sigma} \tag{2.21}$$

where X^* representing the wealth process corresponding to π^* and R^* as in (2.20). □

Remark 2.8. Consider now a specific case where $\phi, \delta = 0$. Then, from equations (2.17), we get that $Y, Z = 1$ and $A = (\mu - r)^2 t$. Hence, the forward utility process is

$$U(x,t) = v\left(\frac{x}{Y}, A\right)Z = v(x, (\mu-r)^2 t).$$

At that case, the optimal portfolio is

$$\pi^*_f = \frac{R^*\lambda}{\sigma} = -\frac{(\mu-r)}{\sigma^2}\frac{v_x(X^*,A)}{v_{xx}(X^*,A)} = -\frac{(\mu-r)}{\sigma^2}\frac{U_x(X^*,t)}{U_{xx}(X^*,t)}.$$

Therefore, when the forward utility U is the initial input v as function of wealth x and time $(\mu-r)^2 t$, then the optimal (forward) portfolio (2.21), coincides with Merton's optimal (2.5) as in the backward case but now we allow the coefficients of risky asset to be stochastic.

Remark 2.9. Assume now that $\delta = 0$ and $\phi \neq 0$. Hence the optimal portfolio is

$$\frac{1}{Y}\pi^*_f = \frac{R^*(\lambda+\phi)}{\sigma} = \frac{r\left(\frac{X^*}{Y}, A\right)(\lambda+\phi)}{\sigma}.$$

It is also that $U(X,t) = v(\frac{x}{Y}, A)Z_t$. For $\lambda = 0$ it is that $Y = 1$. Therefore $v(x^*, A) = U(x^*,t)/Z_t$, and the optimal portfolio

$$\pi^*_f = -\frac{(\lambda+\phi)}{\sigma}\frac{U_x(x^*,t)}{U_{xx}(x^*,t)}\frac{1}{Z_t}.$$

Alternatively,

$$\pi^*_f = -\left(\frac{\mu-r}{\sigma^2}\frac{U_x(x^*,t)}{U_{xx}(x^*,t)} + \frac{\phi}{\sigma}\frac{U_x(x^*,t)}{U_{xx}(x^*,t)}\right)\frac{1}{Z_t}.$$

2.2.2. *Qualitative comparison of backward and forward utility framework.*

(1) Trading Horizon

As we have already mentioned, at the backward case there is a pre-specified horizon (terminal date) as well an a priori specification of the utility at this terminal date, while at the case of the forward in time process there is not a pre-specified terminal date. On the contrary, there is the forward normalization point $t = 0$ and the today's specification of $U(x, 0)$.

(2) Construction of utility function
In a backward framework, utility function is exogenously specified at date T, and working backwardly in time using a value function that satisfies the time consistency and the dynamic programming principle, we reach at time $t = 0$ and get the optimal portfolio strategy. On the other side, in a forward framework utility function is a self-generating process since the investor defines the today's utility and for each $t \geq 0$ she constructs the utility based on the $\mathcal{F}_t$ - filtration. This implies that forward utility process is also indifferent among the subhorizons.

(3) Preferences
At the case of the backward process the investor has not the ability to change his preferences at other times in response to changes in the market environment, while at the forward case she can adjust the market information and hence his preferences into his utility's construction.

(4) Assessment of strategies
In a backward framework the investor cannot evaluate her trading strategy at times later than the specified terminal date, while in a forward framework she has a tool to consistently assess her trading strategy without specific time limit.

3. Forward Utility Process for CDC Pension Schemes

In contrast to a single generation or member's optimization problem, a pension fund does not have a specific terminal (retirement) time, since new members enter and old ones retire through time. This means that placing optimization goals at any future time needs attention in order to guarantee time-consistency. While the notion of forward performance criterion has studied by several authors and directions, models that are dedicated to pension funds have been rare. Forward utility seems an appropriate approach, since it does not require a pre-determined time-horizon, and secondly it imposes stochastic nature of the pension fund's investment goals.

Collective Defined Contribution (CDC) pension schemes are very common in practice. In a pure DC pension plan, retirement payment is defined as the aggregate investment outcome of all the contributions that a participant will have given to the fund until the day of her retirement. In other words, each participant pays contributions to the fund on a regular basis (normally a proportion of the wage), the fund invests these contributions to the risky security market and the retirement benefits are the aggregate investment payoffs. The main addition of a CDC plan, is that fund manager invests the contributions from all participants together on an on-going basis and hence she creates a continuum of investment goals for the existed and the future participants (see among others [BST16], [BB09], [CKZ21], [CDJP09], [Don17], [Gol08], [Kur18], [Pot+07]). CDC pension funds become reasonably more and more popular in the pension fund industry because of their advantages. Some of them are the following:

- Retirement in a single package: Members of CDC scheme can both build up a pension (accumulation) and receive a pension (decumulation) in the same scheme. This is similar to defined benefit schemes, although the income is not guaranteed.
- An income without a risk premium: As CDC schemes do not guarantee an income, there is not an additional cost to savers or employers of securing that guarantee.
- Possible longevity risk sharing: In DC schemes, members manage their own pension pots. Because they cannot accurately predict how long they will live, there is a risk that people under-spend (dying with unused funds) or overspend (running out of money). CDC schemes can potentially manage this risk collectively by paying pensions based on average life expectancy across the scheme.
- Investment strategy: It is argued that CDC schemes can take a longer - term investment strategy than single DC schemes, because they have a mix of members in accumulation and decumulation.
- Employers: CDC schemes could be attractive to employers by allowing them to offer employees a pension scheme, with an income in retirement, but without the ongoing risk that the employer will need to continue to fund the fund if it cannot afford to pay that income.

3.1. **The Pension Fund Model.** We start by stating a main standing assumption: Each generation will remain in the fund for a specific period of time (thus a specific "target date" or "target period" for each generation is imposed). That period is considered to be a known at the beginning of the pension plan. Particularly, having target period equal to N (years) means that generations who enter to the pension fund at the beginning of year n, will retire (i.e., will leave the fund) at the beginning of year $n+N$.

Therefore, we have to manage different generations that enter and retire from the fund, while the contributions are invested collectively. This collective investment for all generations implies that there is no specific terminal horizon to place the investment goal.

We consider a sequence of discrete points in time $\mathcal{T} := \{T_0 = 0, T_1, T_2, ...\}$ and the wealth process of the pension fund depends on the investment in risky asset, as well as on the contributions that the fund receives as a percentage of the workers' wages. For each period $[T_n, T_{n+1})$ the pension fund updates the coefficients of the dynamics of both the risky asset and the contribution process. In other words, for each period $[T_n, T_{n+1})$, the price of the risky asset is given as

$$S_t = S_{T_n} + \mu_n \int_{T_n}^{t} S_s ds + \sigma_n \int_{T_n}^{t} S_s dW_s, \qquad \text{for } t \in [T_n, T_{n+1}], \tag{3.1}$$

where μ_n and σ_n are $\mathcal{F}_{T_n}$-measureable random variables for each $n \in \mathbb{N}$.

Let $(\pi_t)_{t\geq 0}(:= l_t X_t)$ stand for the amount of capital invested by the pension fund (for all generations) in the risky asset and $(\tilde{h}_t)_{t\geq 0}(:= \tilde{l}_t X_t)$ the amount of capital that is created by the contributions of all generations. The number of people that enters the fund at each time $T_n \in \mathcal{T}$ is given by a discrete-time stochastic process $(P_n)_{T_n \in \mathcal{T}}$, which is independent on (W, W') and P_n is $\mathcal{F}_{T_n}$-measureable. This means that the wealth process for $t \in [0, T_1]$ is given as

$$X_t = X_0 + \underbrace{\int_0^t \pi_s \frac{dS_s}{S_s}}_{\substack{1^{st} \text{ source of wealth:} \\ \text{investment's return in stocks}}} + \underbrace{\int_0^t \pi'_s \frac{dR_s}{R_s}}_{\substack{2^{nd} \text{ source of wealth:} \\ \text{investment's return in riskless asset}}} + \underbrace{\int_0^t \tilde{h}_s \frac{dy_s}{y_s}}_{\substack{3^{rd} \text{ source of wealth:} \\ \text{money of all the contributions}}},$$

where $(y_t)_{t\geq 0}$ is the *process of random endowment* for the fund, such that

$$dy_t = m_n y_t dt + u_n y_t dW'_t,$$

where m_n and u_n are $\mathcal{F}_{T_n}$-measureable random variables for each $n \in \mathbb{N}$. Note that $(y_t)_{t\geq 0}$ is driven only by W', where W, W' are correlated Brownian Motions with correlation coefficient equal to ρ.

As already mentioned, $N \in \mathbb{N}$ is the number of periods each generation stays in the pension fund. For example, the first generation (generation-0) is going to leave the fund at time T_N. In the sequel, we are going to need the process that stands for the existing population of the funds at time $t \in (T_n, T_{n+1}]$. We denote this process as $(\Pi_n)_{n\in\mathbb{N}}$, where

$$\Pi_n = \sum_{i=0}^{n} P_i, \quad \text{for } n \leq N.$$

$$\Pi_n = \sum_{i=(n+1)-N}^{n+1} P_i, \quad \text{for } n > N.$$

The running liability of the fund to the generation-0 shall be denoted by the process $(Z_t^0)_{t\in[0,T_N]}$. For this, we consider the following reasonable liability structure: Generation-0 will receive at its retirement, the share of the profits/losses from investments and the its compounded contributions. Formally,

(3.2)

$$Z_t^0 = \int_0^t \pi_s \frac{dS_s}{S_s} + \int_0^t \pi'_s \frac{dR_s}{R_s} + \int_0^t \tilde{h}_s \frac{dy_s}{y_s} +, \quad \forall t \in [0, T_1]$$

$$Z_t^0 = Z_{T_n}^0 + \frac{P_0}{\Pi_n}\Big(\int_{T_n}^t \pi_s \frac{dS_s}{S_s} + \int_{T_n}^t \pi'_s \frac{dR_s}{R_s} + \int_{T_n}^t \tilde{h}_s \frac{dy_s}{y_s}\Big), \quad \forall t \in (T_n, T_{n+1}] \text{ and } n = 1, 2, ..., N-1.$$

Similarly, for each generation-k, we define the process

$$Z_t^k = 0, \quad \forall t \in [0, T_k)$$

$$\text{and} \quad \forall n = k, k+1, ..., k+N$$

$$Z_t^k = Z_{T_n}^k + \underbrace{\frac{P_k}{\Pi_n}\Big(\int_{T_n}^t \pi_s \frac{dS_s}{S_s} + \int_0^t \pi'_s \frac{dR_s}{R_s}\Big)}_{\substack{\text{percentage of wealth} \\ \text{created by investments}}} + \underbrace{\frac{P_k}{\Pi_n}\int_{T_n}^t \tilde{h}_s \frac{dy_s}{y_s}}_{\substack{\text{percentage of wealth} \\ \text{created by all the contributions}}}, \quad \forall t \in [T_n, T_{n+1}).$$

Remark 3.1. Note that (as a consequence of the collective scheme) both parts of liability process depend not only on the size of generation-0, but also on population of all the existing generation in the fund at the periods that generation-0 stays in the fund.

We now define the surplus process $(x_t)_{t\geq 0}$ under which the forward utility is going to be defined:

$$x_t := X_t - Z_t, \qquad t \geq 0,$$

where X is given in (4.2) and Z is defined in the following way:

$$Z_t := \begin{cases} 0, & t \leq T_{N-1}, \\ Z_t^0, & T_{N-1} < t \leq T_N \\ Z_t^1, & T_N < t \leq T_{N+1} \\ \vdots & \vdots \\ Z_t^k, & T_{N+k-1} < t \leq T_{N+k}, \end{cases} \tag{3.3}$$

for any $k \in \mathbb{N}$. In other words, for each period, the optimization problem considers the liability at the end of the period (the payments that the fund has to deliver to the generation that retires).

Let us now see closer what happens when the first generation retires: Note that for each $t < T_N$ we define the forward utility at process $x_t = X_t$ (since there is no liability). At $t = T_N$, generation-0 leaves and gets $Z^0_{T_N}$. Hence,

$$\begin{aligned} x_{T_N} :=& X_{T_N} - Z^0_{T_N} \\ =& X_{T_{N-1}} + \int_{T_{N-1}}^{T_N} \pi_s \frac{dS_s}{S_s} + \int_{T_{N-1}}^{T_N} \pi'_s \frac{dR_s}{R_s} + \int_{T_{N-1}}^{T_N} \tilde{h}_s \frac{dy_s}{y_s} \\ &- \left(Z^0_{T_{N-1}} + \frac{P_0}{\Pi_{N-1}} \Big(\int_{T_{N-1}}^{T_N} \pi_s \frac{dS_s}{S_s} + \int_{T_{N-1}}^{T_N} \pi'_s \frac{dR_s}{R_s} + \int_{T_{N-1}}^{T_N} \tilde{h}_s \frac{dy_s}{y_s} \Big) \right) \\ =& x_{T_{N-1}} + \left(\frac{\Pi_{N-1} - P_0}{\Pi_{N-1}} \right) \Big(\int_{T_{N-1}}^{T_N} \pi_s \frac{dS_s}{S_s} + \int_{T_{N-1}}^{T_N} \pi'_s \frac{dR_s}{R_s} + \int_{T_{N-1}}^{T_N} \tilde{h}_s \frac{dy_s}{y_s} \Big). \end{aligned}$$

More general, we have that the surplus process of the fund that we consider for the optimization problem $\forall t \in (T_n, T_{n+1})$ is given as

$$x_t := \begin{cases} x_{T_n} + \int_{T_n}^t \pi_s \frac{dS_s}{S_s} + \int_{T_n}^t \pi'_s \frac{dR_s}{R_s} + \int_{T_n}^t \tilde{h}_s \frac{dy_s}{y_s}, & \text{for } n \leq N-2 \\ x_{T_n} + \frac{\tilde{\Pi}_n}{\Pi_n} \Big(\int_{T_n}^t \pi_s \frac{dS_s}{S_s} + \int_{T_n}^t \pi'_s \frac{dR_s}{R_s} + \int_{T_n}^t \tilde{h}_s \frac{dy_s}{y_s} \Big), & \text{for } n \geq N-1 \end{cases} \tag{3.4}$$

where $x_0 := X_0$ and

$$\tilde{\Pi}_n := \begin{cases} \Pi_n, & \text{for } n \leq N-2 \\ \Pi_n - P_{N-n-1}, & \text{for } n \geq N-1 \end{cases} \tag{3.5}$$

is the reminder members of the fund at the end of period $[T_n, T_{n+1})$. Therefore, the dynamics of the surplus process at the period $[T_n, T_{n+1}]$ is

$$\begin{aligned} dx_t =&\frac{\tilde{\Pi}_n}{\Pi_n}\left(\pi_t \frac{dS_t}{S_t} + \pi_t' \frac{dR_t}{R_t} + \tilde{h}_t \frac{dy_t}{y_t}\right) \\ =&\frac{\tilde{\Pi}_n}{\Pi_n}\left(l_t x_t \frac{dS_t}{S_t} + (1 - l_t - \tilde{l}_t) x_t \frac{dR_t}{R_t} + \tilde{l}_t x_t \frac{dy_t}{y_t}\right). \end{aligned}$$

Equivalently, we may rewrite the above equation as

$$\frac{dx_t}{x_t} = B_n(l_t\sigma_n\lambda_{1,n} + \tilde{l}_n u_n \lambda_{2,n})dt + B_n l_t \sigma_n dW_t + B_n \tilde{l}_n u_n dW_t',$$

(which is similar to the wealth process of Anthropelos et al. [AGZ22]), where

$$\lambda_{1,n} = \frac{\mu_n}{\sigma_n}, \quad \lambda_{2,n} = \frac{m_n}{u_n}, \quad B_n := \frac{\tilde{\Pi}_n}{\Pi_n}.$$

Note that both quantities $\tilde{\Pi}_n/\Pi_n$ and $\tilde{\Pi}_n \in \mathcal{F}_{T_n}$ and hence they can be considered constants in the interval $(T_n, T_{n+1}]$.

Proposition 3.2. *Let $\rho^2 \neq 1$. Consider the random PDE*

$$v_t - \frac{1}{2}\lambda_{1,n}^2 \frac{v_x^2}{v_{xx}} + \frac{1}{2}(1-\rho^2) v_{xx} x_t^2 B_n^2 \tilde{l}_n^2 u_n^2 + (\lambda_{2,n} - \rho\lambda_{1,n}) x_t v_x B_n \tilde{l}_n u_n = 0. \tag{3.6}$$

Then, the process $v(x_t^l, t)_{t\geq 0}$ is a forward utility process, and the investment strategy

$$l_t^* = \frac{\lambda_{1_n}}{B_n\sigma_n} K(x,t) - \rho \tilde{l}_n \frac{u_n}{\sigma_n} \tag{3.7}$$

is the optimal one, where $K(x,t) = -v_x(x,t)/(x v_{xx}(x,t))$.

To provide further insights on the forward utility process, we consider the CRRA initial datum $v_0(x) = x^{1-\gamma}/(1-\gamma)$.

Proposition 3.3. *Let $\rho^2 \neq 1$, $\gamma \neq 1$ and $(\eta_{n,t})_{t\geq 0}$ be given by*

$$\eta_n = \lambda_{1,n}^2 + 2\gamma B_n \tilde{l}_n u_n (\lambda_{2,n} - \rho\lambda_{1,n}) - \gamma^2 B_n^2 \tilde{l}_n^2 u_n^2 (1-\rho^2), \quad \forall n \in \mathbb{N}$$

Then, if $v_0(x) = x^{1-\gamma}/(1-\gamma)$ the process

$$v_n(x,t) = \frac{x^{1-\gamma}}{1-\gamma}\beta_{n-1} e^{-\int_{T_{n-1}}^t \frac{1-\gamma}{2\gamma}\eta_{s,n} ds} \tag{3.8}$$

is a forward utility process and the investment strategy

$$l_{t,n}^* = \frac{\Pi_n}{\tilde{\Pi}_n}\frac{\lambda_{1,n}}{\gamma\sigma_n} - \rho\tilde{l}_n \frac{u_n}{\sigma_n}$$

is the optimal one.

Proof. We consider as a candidate the separable form $v(x) = \frac{x^{1-\gamma}}{1-\gamma}\beta$, where $(\beta_t)_{t\geq 0}$ is an $\mathcal{F}_t$-adapted process, differentiable in t with β_0=1. By replacing the above form of v into (3.6), we readily conclude. □

3.2. **Comparative Statics.** We observe that under CRRA initial datum, the first part of the optimal strategy departs from the Merton's optimal strategy in that it is scaled to the ratio of the population that is going to retire at the end of the period. In particular, higher population of the existing generations implies higher investment in the risky asset (while the opposite occurs for $\tilde{\Pi}_n$). This is again a reasonable feature of the model, since positive market premium means positive investment on the risky asset and if this investment is for more members, the capital invested in the risky asset is naturally more.

More interesting is the dependence of the optimal on the correlation between the contributions and the risky asset. When the correlation is positive (a reasonable assumption), the investment in the risky asset is lower. This reflects a diversification of the risk, in the sense that due to positive correlation, the desirable position in the risky asset is indirectly placed through the contributions. Hence the actual investment on the risky asset is lower. This effect is analogous to the size of the contributions (captured by the process $\tilde{l}$).

Finally, another meaningful feature of the optimal strategy is its dependence to the ratio of u/σ. Therein, higher relative risk of the contribution means lower investment in the risky asset, provided that the correlation is positive.

3.3. **Time-Consistency.** So far, we have guaranteed the time-consistency within each time period thanks to the setting of fund's preference by the forward utility process. It remains to guarantee that the time-consistency holds for all time.

For this, we want to find the appropriate conditions for the forward utility process any time we switch from one period to the other. In particular, we need the terminal value function of a period to be the initial utility for the next one. However, under our formulation, this transmission is rather easy to get. For each $n \in \mathbb{N}$, we need to have the equality

$$v_n(x_{T_n}, T_n) = v_{n+1}(x_{T_n}, T_n)$$

where, $v_n(x, T_{n-1})$ is $\mathcal{F}_{T_{n-1}}$ measureable for each n. Under CRRA initial datum and the representation (4.15), the required condition becomes

$$\frac{x_{T_n}^{1-\gamma}}{1-\gamma}\beta_{n-1}e^{-\int_{T_{n-1}}^{T_n}\frac{1-\gamma}{2\gamma}\eta_{s,n}ds} = \frac{x_{T_n}^{1-\gamma}}{1-\gamma}\beta_n e^{-\int_{T_n}^{T_n}\frac{1-\gamma}{2\gamma}\eta_{s,n}ds},$$

which readily gives a recursive formula for the discrete-time stochastic process $(\beta_n)_{n\in\mathbb{N}}$

$$\beta_n = \beta_{n-1}e^{-\int_{T_{n-1}}^{T_n}\frac{1-\gamma}{2\gamma}\eta_{s,n}ds}.$$

The above formulation completes CRRA-type of forward utility process for a CDC pension fund, according to which the time-consistency is guaranteed, the optimal investment strategy is similar to the classic backward one and the pension fund can update model's parameters at the beginning of each population-renewal period.

4. Forward Utility Process for Collective Pension Funds under CARA Preferences

In this section we slightly change the setting of the previous section in terms of the nature of the liabilities, which allows us to consider both Collective DC and DB pension plans.

As already mentioned, generations enter and retire from the fund, while stochastic (correlated to the market) contributions are invested collectively.

The Model. Assume as in the previous section, that there is a sequence of discrete points in time $\mathcal{T} := \{T_0 = 0, T_1, T_2, ...\}$, which indicates the times where the pension's population changes. For any time-period $[T_n, T_{n+1}]$, the pension fund updates the coefficients of the dynamics of both the risky asset price and the contribution process. For each period $[T_n, T_{n+1})$, the price of the risky asset is given as (3.1), and the contribution process as

$$y_t = \int_{T_n}^t b_n(y_s)ds + \int_{T_n}^t g_n(y_s)dW'_s, \tag{4.1}$$

where W, W' are correlated Brownian Motions with correlation ρ and b_n and g_n are sequence of real smooth functions (in the sense that (4.1) admits a strong solution).

Let $(\pi_t)_{t\geq 0}$ stand for the amount of capital invested by the pension fund (for all generations) in the risky asset. The number of people that enters the fund at each time $T_n \in \mathcal{T}$ is given by a discrete-time stochastic process $(P_n)_{T_n\in\mathcal{T}}$, which is independent on W and W' (P_n is $\mathcal{F}_{T_n}$-measureable). Hence, the wealth process at the initial period $[0, T_1)$ is given as

$$X_t = X_0 + \int_0^t \pi_s \frac{dS_s}{S_s} + P_0 \int_0^t y_s ds, \qquad t \in [0, T_1)$$

and

$$X_t = X_{T_1} + \int_{T_1}^t \pi_s \frac{dS_s}{S_s} + (P_0 + P_1) \int_{T_1}^t y_s ds, \qquad t \in [T_1, T_2).$$

As before, the number of periods each generation stays in the pension fund is denoted by $N \in \mathbb{N}$. Hence, the first generation (generation-0) is going to leave the fund at time T_N. In the sequel, we shall need the process that stands for the existing population of the funds at time $t \in (T_n, T_{n+1}]$. Recall that this process $(\Pi_n)_{n\in\mathbb{N}}$ has been defined in previous section as

$$\Pi_n = \sum_{i=0}^{n} P_i, \quad \text{for } n \leq N.$$

$$\Pi_n = \sum_{i=(n+1)-N}^{n+1} P_i, \quad \text{for } n > N.$$

Ignoring for now the liability payments, the fund's assets grow (for all $n \in \mathbb{N}$) as

$$X_t = X_{T_n} + \int_{T_n}^t \pi_s \frac{dS_s}{S_s} + \Pi_n \int_{T_n}^t y_s ds, \qquad \forall t \in (T_n, T_{n+1}], \tag{4.2}$$

where X_0 is the initial capital of the fund.

4.1. **Collective DC Pension Plans.** Under a standard CDC pension plan, the fund's liabilities to the retiring generation are the contributions that each generation had given to the fund throughout its participation in it and the part of the investments' profits attributed to the retiring generation.

So we introduce the process $(Z_t^0)_{t\in[0,T_N]}$ that stands for the running liability of the fund to the generation-0.

$$Z_t^0 = P_0 \int_0^t y_s ds + \int_0^t \pi_s \frac{dS_s}{S_s}, \quad \forall t \in [0, T_1)$$

$$Z_t^0 = Z_{T_n}^0 + \underbrace{P_0 \int_{T_n}^t y_s ds}_{\text{sum of given contributions}} + \underbrace{\frac{P_0}{\Pi_n} \int_{T_n}^t \pi_s \frac{dS_s}{S_s}}_{\substack{\text{percentage of wealth} \\ \text{created by investments}}}, \quad \forall t \in (T_n, T_{n+1}] \text{ and } n = 1, 2, ..., N-1.$$

Remark 4.1. We readily see that the first part of liabilities depends only on the generation-0 (i.e. the generation that retires first), while the second part depends on the other existing generations too. This highlights the difference between the current section and the previous one, where both the first and second part of liabilities depend on all the existing generations and not only the retiring one (see (3.2)).

Similarly, for each generation-k, we define the processes

$$Z_t^k = 0, \quad \forall t \in [0, T_k)$$

$$Z_t^k = Z_{T_n}^k + P_k \int_{T_n}^t y_s ds + \frac{P_k}{\Pi_n} \int_{T_n}^t \pi_s \frac{dS_s}{S_s}, \quad \forall t \in [T_n, T_{n+1}) \text{ and } n = k, k+1, ..., k+N.$$

As before, the aggregate fund's surplus process $(x_t)_{t\geq 0}$ is equal to

$$x_t := X_t - Z_t, \qquad t \geq 0,$$

where X is given in (4.2) and Z is defined in the following way:

$$Z_t := \begin{cases} 0, & t \leq T_{N-1}, \\ Z_t^0, & T_{N-1} < t \leq T_N \\ Z_t^1, & T_N < t \leq T_{N+1} \\ \vdots & \vdots \\ Z_t^k, & T_{N+k-1} < t \leq T_{N+k} \end{cases} \tag{4.3}$$

for any $k \in \mathbb{N}$. In other words, for each period, the optimization problem considers the liability at the end of the period (the payments that the fund has to deliver to the generation that retires).

Let's see what happens when the first generation retires: Note that for each $t < T_N$ we define the forward performance at process $x_t = X_t$ (since there is no payment). At $t = T_N$, generation-0 leaves and gets $Z^0_{T_N}$

$$\begin{aligned} x_{T_N} :=& X_{T_N} - Z^0_{T_N} \\ =& x_{T_{N-1}} + \left(1 - \frac{P_0}{\Pi_{N-1}}\right) \int_{T_{N-1}}^{T_N} \pi_s \frac{dS_s}{S_s} + (\Pi_{N-1} - P_0) \int_{T_{N-1}}^{T_N} y_s ds \\ =& x_{T_{N-1}} + \left(\frac{\Pi_{N-1} - P_0}{\Pi_{N-1}}\right) \int_{T_{N-1}}^{T_N} \pi_s \frac{dS_s}{S_s} + (\Pi_{N-1} - P_0) \int_{T_{N-1}}^{T_N} y_s ds. \end{aligned}$$

More general, the surplus process of the fund that we consider for the optimization problem $\forall t \in (T_n, T_{n+1}]$ is given as

$$x_t := \begin{cases} x_{T_n} + \int_{T_n}^t \pi_s \frac{dS_s}{S_s} + \Pi_n \int_{T_n}^t y_s ds, & \text{for } n \le N-2 \\ x_{T_n} + \frac{\tilde{\Pi}_n}{\Pi_n} \int_{T_n}^t \pi_s \frac{dS_s}{S_s} + \tilde{\Pi}_n \int_{T_n}^t y_s ds, & \text{for } n \ge N-1 \end{cases} \tag{4.4}$$

where $x_0 := X_0$ and

$$\tilde{\Pi}_n := \begin{cases} \Pi_n, & \text{for } n \le N-2 \\ \Pi_n - P_{N-n-1}, & \text{for } n \ge N-1 \end{cases} \tag{4.5}$$

stands for number of the fund's reminder members at the end of period $[T_n, T_{n+1})$. Therefore, the dynamics of the surplus process at the period $[T_n, T_{n+1})$ is

$$\begin{aligned} dx_t =& \frac{\tilde{\Pi}_n}{\Pi_n} \pi_t \frac{dS_t}{S_t} + \tilde{\Pi}_n y_t dt = \alpha_{t,n} \left(\mu_n dt + \sigma_n dW_t\right) + \tilde{\Pi}_n y_t dt \\ =& (\alpha_{t,n} \mu_n + \tilde{\Pi}_n y_t) dt + \alpha_{t,n} \sigma_n dW_t, \end{aligned}$$

where

$$\alpha_{t,n} := \frac{\tilde{\Pi}_n}{\Pi_n} \pi_t.$$

We also note that both coefficients $\tilde{\Pi}_n / \Pi_n$ and $\tilde{\Pi}_n$ are $\mathcal{F}_{T_n}$-measureable, so in the interval $(T_n, T_{n+1}]$ can be considered as constants.

Remark 4.2. Note that at this setting, since the liability process changes each period, there is a jump in the process x_t at the beginning of each period. In particular, for each period $(T_{n-1}, T_n]$, for $n < N$, there is no liability and hence $x_t = X_t$. However, at the end of period $(T_{N-1}, T_N]$, generation-0 retires, and for this period we consider the process $x_t = X_t - Z^0_t$. This means that at time T_{N-1} process x does a jump:

$$x_{T_{N-1}} = X_{T_{N-1}} \quad \text{and} \quad x_{T^+_{N-1}} = X_{T_{N-1}} - Z^0_{T_{N-1}}.$$

This holds in each period. For example

$$x_{T_N} = X_{T_N} - Z^0_{T_N} \quad \text{and} \quad x_{T^+_N} = X_{T_N} - Z^1_{T_N}.$$

In other words, at each period we change the process $\tilde{\Pi}_n$ in the dynamics of (4.4) and the initial surplus of each period changes. As we shall see in the sequel, this jump could be easily incorporated in the model (especially if the initial datum is exponential).

In the rest of the section we work toward the construction of a forward process defined on the above surplus and represented functionally as

$$U_{T_n}(x,t;\omega) = v_n(x_t, y_t, t), \qquad \text{for each } n \in \mathbb{N}.$$

Remark 4.3. The model that we use in the current section is a more general than the previous one. This implies that we would need a richer family of forward process to build a meaningful example. Hence, herein we consider the utility function with an additional state variable (the contribution) y_t. This will help us capture the interaction between the wealth process and the contributions.

Proposition 4.4. *For each $n \in \mathbb{N}$, consider the random PDE*

$$v_t + \frac{1}{2}g_n^2(y)v_{yy} + v_y b_n(y) + v_x\tilde{\Pi}_n y - \frac{1}{2}\frac{(\lambda_n v_x + \rho g_n(y)v_{xy})^2}{v_{xx}} = 0. \tag{4.6}$$

Then v must solve (4.6) in $t \in (T_n, T_{n+1}]$ and the strategy

$$\alpha^*_{t,n} = -\frac{\lambda v_x}{\sigma_n v_{xx}} - \rho\frac{g_n(y)v_{xy}}{\sigma_n v_{xx}},$$

or equivalently,

$$\pi^*_{t,n} = \frac{\Pi_n}{\tilde{\Pi}_n}\left(-\frac{\lambda_n v_x}{\sigma_n v_{xx}} - \rho\frac{g_n(y)v_{xy}}{\sigma_n v_{xx}}\right), \tag{4.7}$$

where $\lambda_n = \mu_n/\sigma_n$, is optimal $\forall t \in (T_n, T_{n+1}]$, when v_n is set as the forward utility process.

Proof. We apply Ito's formula on $v(x_t, y_t, t)$, that is

$$\begin{aligned} dv(x_t,y_t,t) =& v_t dt + v_x dx + \frac{1}{2}v_{xx}(dx)^2 + v_y dy + \frac{1}{2}v_{yy}(dy)^2 + v_{xy}(dx)(dy) \\ =& \left(v_t + \frac{1}{2}g_n^2(y)v_{yy} + v_y b_n(y) + v_x\tilde{\Pi}_n y + \alpha_{t,n}\mu_n v_x + \frac{1}{2}\alpha_{t,n}^2\sigma_n^2 v_{xx} + \rho\sigma_n g_n(y)\alpha_{t,n}v_{xy}\right)dt \\ &+ v_x\sigma_n\alpha_{t,n}dW_t + v_y g_n(y)dW'_t. \end{aligned}$$

We get that v must solve in $t \in (T_n, T_{n+1}]$ the following PDE

$$v_t + \frac{1}{2}g_n^2(y)v_{yy} + v_y b_n(y) + v_x\tilde{\Pi}_n y + \max_\alpha\left(\alpha\mu_n v_x + \frac{1}{2}\alpha^2\sigma_n^2 v_{xx} + \rho\sigma_n g_n(y)\alpha v_{xy}\right) = 0.$$

Particularly from the above equation we have that the optimal strategy is

$$\alpha^*_{t,n} = -\frac{\lambda v_x}{\sigma_n v_{xx}} - \rho\frac{g_n(y)v_{xy}}{\sigma_n v_{xx}},$$

and by replacing that strategy on the above equation, we get the random PDE (4.6) that v must satisfy. Since, we have unify the surplus process in (4.4) we may work for each period $(T_n, T_{n+1}]$ at the same way. □

Proposition 4.5. ***(Exponential solutions)***
Consider the solutions of the following family of variable-separation

$$v_n(x,y,t) = -e^{-\gamma x}F_n(y,t), \tag{4.8}$$

where F_n being a positive function such that $F(y,t) = w(y,t)^{1/(1-\rho^2)}$. Then the positive function $w(y,t)$ should satisfy the second order PDE:

$$w_t + \frac{1}{2}g_n^2(y)w_{yy} + (b_n(y) - \lambda_n\rho g_n(y))\, w_y - \left(\frac{1}{2}\lambda_n^2 + \gamma\tilde{\Pi}_n y\right)(1-\rho^2)w = 0. \tag{4.9}$$

Proof. Simple calculations yield that positive function F solves the problem

$$F_t + \frac{1}{2}g_n^2(y)F_{yy} - \gamma\tilde{\Pi}_n yF - \frac{1}{2}\frac{(\lambda_n F + \rho g_n(y)F_y)^2}{F} = 0,$$

or equivalently

$$F_t + \frac{1}{2}g_n^2(y)F_{yy} + (b_n(y) - \lambda_n\rho g_n(y))\, F_y - \frac{1}{2}\rho^2 g_n^2(y)\frac{F_y^2}{F} = \left(\frac{1}{2}\lambda_n^2 + \gamma\tilde{\Pi}_n y\right)F.$$

Consider utilities of the following family

$$v(x,y,t) = -e^{-\gamma x}F(y,t) = -e^{-\gamma x}w(y,t)^{1/(1-\rho^2)}.$$

Simple calculations yield that

$v_t = -\frac{e^{-\gamma x}}{1-\rho^2}w^{\frac{\rho^2}{(1-\rho^2)}}w_t$, $\quad v_x = \gamma e^{-\gamma x}w^{\frac{1}{(1-\rho^2)}}$, $\quad v_{xx} = -\gamma^2 e^{-\gamma x}w^{\frac{1}{(1-\rho^2)}}$, $\quad v_y = -\frac{e^{-\gamma x}}{1-\rho^2}w^{\frac{\rho^2}{(1-\rho^2)}}w_y$

$v_{xy} = \frac{\gamma e^{-\gamma x}}{1-\rho^2}w^{\frac{\rho^2}{(1-\rho^2)}}w_y$, $\quad v_{yy} = -\frac{e^{-\gamma x}}{1-\rho^2}w^{\frac{\rho^2}{(1-\rho^2)}}(w_{yy} + \frac{\rho^2}{1-\rho^2}w^{-1}w_y^2)$.

By replacing the above formulas of derivatives on (4.6), we get (4.9). □

Note that the parameters of the above pde are $\mathcal{F}_{T_n}$-measurable. Equation (4.9) belongs in the family of linear parabolic equations. Theorem 3.12 of [NT17] states that the positive solutions of (4.9) should be of the form

$$w(t,y) = \int_{\mathbb{R}} e^{-lt}\psi_n(y;l)\nu_n(dl), \tag{4.10}$$

for some Borel probability measure ν_n on $\mathbb{R}$ and a positive function ψ_n which satisfies

$$\frac{1}{2}g_n^2(y)\psi''(y;l) + (b_n(y) - \lambda_n\rho g_n(y))\,\psi_n'(y;l) - \left(\frac{1}{2}\lambda_n^2 + \gamma\tilde{\Pi}_n y\right)(1-\rho^2)\psi_n(y;l) = l\psi_n(y;l).$$

The above is a regular elliptic ode, whose solutions have been well-studied.

Contributions as piece-wise Geometric Brownian Motion. In order to specify a good example that could lead to intuitive formulas, we make a standard simplification. In particular, we assume that stock price's coefficients are piece-wise constant (a piece-wise Geometric Brownian Motion). We also simplify the contribution process in that we set $b_n(y) = m_n y$ and $g_n(y) = u_n y$ (i.e. y_s is a piece-wise a Geometric Brownian motion too). This means that (4.9) is simplified to

$$w_t + \frac{1}{2}u_n^2 y^2 w_{yy} + (m_n - \lambda_n\rho u_n)\, y w_y - \left(\frac{1}{2}\lambda_n^2 + \gamma\tilde{\Pi}_n y\right)(1-\rho^2)w = 0.$$

The positive solutions of the above pde should satisfy the representation (4.10), where $\psi_n(y;l)$ solves the following ode

$$\frac{1}{2}u_n^2 y^2 \psi''(y;l) + (m_n - \lambda_n \rho u_n)\, y\psi_n'(y;l) - \left(\frac{1}{2}\lambda_n^2 + \gamma\tilde{\Pi}_n y\right)(1-\rho^2)\psi(y;l) = l\psi(y;l)$$

or equivalently

$$y^2\psi''(y;l) + \zeta_n y\psi'(y;l) - k_n y\psi(y;l) - \xi_n(l)\psi(y;l) = 0, \tag{4.11}$$

where $\zeta_n = 2(m_n - \lambda_n\rho u_n)/u_n^2$ and $k_n = 2\gamma\tilde{\Pi}_n(1-\rho^2)/u_n^2$ are $\mathcal{F}_{T_n}$-measurable and $\xi_n(l) = (\lambda_n^2(1-\rho^2)+2l)/u_n^2$. The ode (4.11) is a special case of the *general Bessel equation* and its unique solution is given as

$$\psi_n(y;l) = y^{(1-\zeta_n)/2}\left(\beta_n Z_{p_n(l)}(2\sqrt{yk_n}) + \beta_n' Z_{-p_n(l)}(2\sqrt{yk_n})\right), \tag{4.12}$$

where

$$p_n(l) = 2\sqrt{\left(\frac{1-\zeta_n}{2}\right)^2 + \xi_n(l)}$$

β_n, β_n' are $\mathcal{F}_{T_n}$-measurable and $Z_\zeta(y)$ denotes a Bessel function (depending on the values of ζ).

Therefore, a candidate for the forward process between each period $(T_n, T_{n+1}]$ could be

$$U_{T_{n+1}}(x,y,t;\omega) = v_{T_{n+1}}(x,y,t) = -e^{-\gamma x}\left(\int_{\mathbb{R}} e^{-lt}\psi_n(y;l)\nu_n(dl)\right)^{1/(1-\rho^2)}, \tag{4.13}$$

for all $t \in (T_n, T_{n+1}]$, for some Borel probability measure ν_n, where $\psi_n(y;l)$ solves (4.12).

Now, the terminal value function of this period should be the initial utility for the next one, where the coefficients for the stock price, the contribution process and the fund's population are updated. In other words, for the period $(T_{n+1}, T_{n+2}]$ the forward should be on the same form

$$U_{T_{n+2}}(x,y,t;\omega) = v_{T_{n+2}}(x,y,t) = -e^{-\gamma x}\left(\int_{\mathbb{R}} e^{-lt}\psi_{n+1}(y;l)\nu_{n+1}(dl)\right)^{1/(1-\rho^2)},$$

for some other Borel probability measure ν_{n+1}, where $\psi_{n+1}(y;l)$ solves (4.12), for the updated, $\mathcal{F}_{T_{n+1}}$-measurable parameters. To guarantee the time-consistency we need to have

$$v_{T_n}(x,y,T_n) = v_{T_{n+1}}(x,y,T_n) \tag{4.14}$$

or equivalently

$$\int_{\mathbb{R}} e^{-lT_n}\psi_{n-1}(y;l)\nu_{n-1}(dl) = \int_{\mathbb{R}} e^{-lT_n}\psi_n(y;l)\nu_n(dl),$$

which is generally difficult to hold.

Proposition 4.6. ***(Specific example)***
A special member of the family of solutions is when we choose $l_n = -(1/2)\lambda_n^2(1-\rho^2)$. Within this family, the candidate forward utility process (4.13) takes the form

$$U_{T_{n+1}}(x,y,t;\omega) = -e^{-\gamma x + \frac{1}{2}\lambda_n^2 t}(\psi_n(y))^{1/(1-\rho^2)} \tag{4.15}$$

and the ode (4.11) *is simplified to*

$$y\psi''(y) + \zeta_n \psi'(y) - k_n \psi(y) = 0 \tag{4.16}$$

with the unique solution

$$\psi_n(y) = y^{(1-\zeta_n)/2}\left(\beta_n Z_{\zeta_n - 1}(2\sqrt{yk_n}) + \beta'_n Z_{1-\zeta_n}(2\sqrt{yk_n})\right),$$

for some $\mathcal{F}_{T_n}$*-measurable random variables* β_n *and* β'_n.

Proof. We have that a candidate forward process between each period $(T_n, T_{n+1}]$ could be

$$U_{T_{n+1}}(x,y,t;\omega) = v_{T_{n+1}}(x,y,t) = -e^{-\gamma x}\left(\int_{\mathbb{R}} e^{-lt}\psi_n(y;l)\nu_n(dl)\right)^{1/(1-\rho^2)}.$$

For specific $l_n = -(1/2)\lambda_n^2(1-\rho^2)$, forward process becomes

$$\begin{aligned} v_{T_{n+1}}(x,y,t) &= -e^{-\gamma x}\left(e^{(1/2)\lambda_n^2(1-\rho^2)t}\psi_n(y)\right)^{1/(1-\rho^2)} \\ &= -e^{-\gamma x + (1/2)\lambda_n^2 t}\psi_n(y)^{1/(1-\rho^2)}. \end{aligned}$$

Moreover, ode (4.11) $y^2\psi''(y;l) + \zeta_n y\psi'(y;l) - k_n y\psi(y;l) - \xi_n(l)\psi(y;l) = 0$, for that specific $l_n = -(1/2)\lambda_n^2(1-\rho^2)$ (that implies $\xi_n(l) = 0$), becomes $y^2\psi''(y) + \zeta_n y\psi'(y) - k_n y\psi(y) = 0$ (or equivalently $y\psi''(y) + \zeta_n\psi'(y) - k_n\psi(y) = 0$). ☐

This condition will also hold for the next period but with different parameters ζ_{n+1} and k_{n+1}.

Proposition 4.7. ***(Time consistency)***
Consider the case $\beta'_n = \beta'_{n+1} = 0$ *and choose* $\beta_0 = 1$. *We ensure time-consistency between periods and* (4.14) *holds when*

$$\beta_{n+1} = \beta_n e^{\frac{\lambda_{n+1}^2 - \lambda_n^2}{2}T_{n+1}}\frac{\bar{\psi}_n(y_{T_{n+1}})}{\bar{\psi}_{n+1}(y_{T_{n+1}})}, \qquad \text{for each } n \in \mathbb{N}, \tag{4.17}$$

where $\bar{\psi}_n(y_{T_{n+1}}) = y_{T_{n+1}}^{(1-\zeta_n)/2} Z_{\zeta_n - 1}(2\sqrt{y_n k_n})$.

Proof. Considering the case $\beta'_n = \beta'_{n+1} = 0$, the initial condition (4.14) can be written as

$$\beta_n e^{-\frac{\lambda_n^2}{2}T_{n+1}}\bar{\psi}_n(y_{T_{n+1}}) = \beta_{n+1} e^{-\frac{\lambda_{n+1}^2}{2}T_{n+1}}\bar{\psi}_{n+1}(y_{T_{n+1}})$$

for some positive random variable β_{n+1}, which is $\mathcal{F}_{T_{n+1}}$-measureable β_n and β'_n, where $\bar{\psi}_n(y_{T_{n+1}}) = y_{T_{n+1}}^{\frac{1-\zeta_n}{2}} Z_{\zeta_n - 1}(2\sqrt{y_n k_n})$. Therefore, if we choose $\beta_0 = 1$ and

$$\beta_{n+1} = \beta_n e^{\frac{\lambda_{n+1}^2 - \lambda_n^2}{2}T_{n+1}}\frac{\bar{\psi}_n(y_{T_{n+1}})}{\bar{\psi}_{n+1}(y_{T_{n+1}})}, \qquad \text{for each } n \in \mathbb{N} \tag{4.18}$$

we ensure that (4.14) holds and $\psi_{n+1}(y) = \beta_{n+1} y^{(1-\zeta_{n+1})/2} Z_{1-\zeta_{n+1}}(2\sqrt{yk_{n+1}})$. ☐

We use the parameters of the ode's solution to ensure the consistency between the value functions. Under the above conditions, we get that the unique solution takes the form

$$\psi_n(y) = \beta_n y^{(1-\zeta_n)/2} Z_{\zeta_n - 1}(2\sqrt{y k_n}). \tag{4.19}$$

Remark 4.8. If condition (4.18) holds, then we ensure the martingale property of U in the optimal strategy. Particularly,

$$\begin{aligned} \mathbb{E}(U_{T_{n+2}}(x_t^*, y_t, t)|\mathcal{F}_s) =& \mathbb{E}\Big(\mathbb{E}(U_{T_{n+2}}(x_t^*, y_t, t)|\mathcal{F}_{T_{n+1}})|\mathcal{F}_s\Big) \\ =& \mathbb{E}(U_{T_{n+2}}(x_{T_{n+1}}^*, y_{T_{n+1}}, T_{n+1})|\mathcal{F}_s) \\ =& \mathbb{E}(U_{T_{n+1}}(x_{T_{n+1}}^*, y_{T_{n+1}}, T_{n+1})|\mathcal{F}_s) \\ =& U_{T_{n+1}}(x_s^*, y_s, s). \end{aligned} \tag{4.20}$$

4.2. **Power-Series Solutions.** Working with Bessel functions is not convenient. For practical purposes, however we could consider the power-series approximation of the ode solution. Indeed, we may easily express the solution of the above ode's using the theory of Frobenius method on power-series solutions.[2] Ode (4.16) can alternatively be written as

$$\psi''(y) + p(y)\psi'(y) + q(y)\psi(y) = 0,$$

where the functions $p_n(y) = \frac{\zeta_n}{y}$ and $q_n(y) = -\frac{k_n}{y}$. Following the related terminology, we call the point $y_0 = 0$ *regular-singular point*, and we look for solutions of the form

$$\psi_n(y) = \sum_{i=0}^{\infty} a_{n,i} y^i, \qquad \text{for each } n \in \mathbb{N},$$

around y_0.

Proposition 4.9. *Consider the ode (4.16). Then the function*

$$\psi_n(y) = a_{n,0} \sum_{i=0}^{\infty} \frac{(k_n y)^i}{i!(i + \zeta_n - 1)!} \tag{4.21}$$

is a power-series solution.

Proof. Put the form $\psi_n(y) = \sum_{i=0}^{\infty} a_{n,i} y^i$ back into ode (4.16) and group the coefficients. Expansions for the first derivatives are

$$\psi_n'(y) = \sum_{i=1}^{\infty} a_{n,i} i y^{i-1}$$

and

$$\psi_n''(y) = \sum_{i=2}^{\infty} a_{n,i} i(i-1) y^{i-2}.$$

[2]Frobenius method solves second order linear homogeneous differential equations with variable coefficients. According to this method, if the following ode

$$\psi''(y) + p(y)\psi'(y) + q(y)\psi(y) = 0$$

has a *regular-singular point* y_0 , i.e. the functions $(y - y_0)p$ and $(y - y_0)q$ are analytics at y_0, then its solution can be written as $\psi(y) = \sum_{i=0}^{\infty} a_i y^i$ (see Chapter 3 of [Gab21] and Chapter 4 of [Ear90]).

By replacing the above on (4.16), we get that:

$$\sum_{i=1}^{\infty} y^{i-1}\Big(a_{n,i}i(i-1)+\zeta_n a_{n,i}i - k_n a_{n,i-1}\Big) = 0.$$

Then easily we take that

$$\begin{aligned}
a_{n,1} &= \frac{k_n}{\zeta_n}a_{n,0}, \quad \text{for} \quad i=1\\
a_{n,2} &= \frac{k_n^2}{2\zeta_n(\zeta_n+1)}a_{n,0}, \quad \text{for} \quad i=2\\
a_{n,3} &= \frac{k_n^3}{6\zeta_n(\zeta_n+1)(\zeta_n+2)}a_{n,0}, \quad \text{for} \quad i=3\\
&\vdots\\
a_{n,i} &= \frac{k_n^i}{i!\prod_{j=1}^{i}(i+\zeta_n-1)}a_{n,0}, \quad \forall i\in\mathbb{N} \text{ and for each } n\in\mathbb{N},
\end{aligned}$$

and with a slight abuse of notation,

$$a_{n,i} = \frac{k_n^i}{i!(i+\zeta_n-1)!}a_{n,0}, \quad \forall i\in\mathbb{N} \text{ and for each } n\in\mathbb{N}.$$

□

Remark 4.10. In order to ensure the positivity of function (4.21), we set the sufficient condition: $\rho < \lambda_c/\lambda$, where $\lambda_c := m/u$. In words, we assume that the correlation between the risky asset and the contributions to be lower than the rate of their Sharpe ratios. Under that condition parameter ζ is always positive and hence ψ stays positive too.

Remark 4.11. Using the notion of a *Modified Bessel function of the first kind* [3], we may express the series (4.21) as

$$\psi_n(y) = a_{n,0}(yk_n)^{-(\zeta_n-1)/2} I_{\zeta_n-1}(2\sqrt{yk_n}), \tag{4.22}$$

for each $n\in\mathbb{N}$, where the Bessel function I is by definition always a positive function. The above expression resulting from the power-series solution is similar with the Bessel solution (4.19), if we consider the parameter $\beta_n = a_{n,0}k_n^{\frac{1-\zeta_n}{2}}$ and the Bessel function $Z_{\zeta_n-1}(2\sqrt{yk_n}) = I_{\zeta_n-1}(2\sqrt{yk_n})$.

4.3. **Optimal Strategy.** For the optimal strategy, we recall the following form (as we see in (4.7)),

$$\pi^*_{t,n} = \frac{\Pi_n}{\tilde{\Pi}_n}\left(-\frac{\lambda_n v_x}{\sigma_n v_{xx}} - \rho\frac{g_n(y)v_{xy}}{\sigma_n v_{xx}}\right), \qquad \forall t\in(T_n, T_{n+1}]$$

[3]The Modified Bessel function of the first kind is generally given by the form: $I_p(x) = (\frac{1}{2}x)^p\sum_{i=0}^{\infty}\frac{(\frac{1}{4}x^2)^i}{i!\Gamma(p+i+1)}$ [ME23]. So in our case, it is that

$$I_{\zeta_n-1}(2\sqrt{yk_n}) = (\frac{1}{2}2\sqrt{yk_n})^{\zeta_n-1}\sum_{i=0}^{\infty}\frac{(\frac{1}{4}4yk_n)^i}{i!\Gamma(i+\zeta_n-1+1)} = (yk_n)^{\frac{1-\zeta_n}{2}}\sum_{i=0}^{\infty}\frac{(yk_n)^i}{i!\Gamma(i+\zeta_n)}.$$

which now can be analytically expressed under the simplified assumption of piece-wise geometric Brownian Motion for the risky asset and the contributions.

Remark 4.12. The induced optimal strategy seems more complicated containing the second term $\rho(g_n(y)v_{xy})/(\sigma_n v_{xx})$. Through the term v_{xy}, we capture the interaction between the wealth process and the contributions (see also Remark 4.3). The extra term contains not only the correlation (similarly to the previous section and optimal strategy (3.7)), but also the manager's investment criteria (captured by the utility function).

Corollary 4.13. *For candidate forward (exponential) utility (4.15), optimal strategy takes the form*

$$\pi^*_{t,n} = \frac{\Pi_n}{\tilde{\Pi}_n}\left(\frac{\lambda_n}{\gamma\sigma_n} + \frac{\rho}{\gamma(1-\rho^2)}\frac{u_n}{\sigma_n}y_n\psi_n'(y)\Big(\psi_n(y)\Big)^{-1}\right), \qquad \forall t \in (T_n, T_{n+1}]. \tag{4.23}$$

Proof. We replace forward utility of the form (4.15) and its derivatives in the formula of optimal strategy (4.7). Particularly, we have

$$\begin{aligned} v_x &= \gamma e^{-\gamma x+\frac{1}{2}\lambda_n^2 t}(\psi_n(y))^{1/(1-\rho^2)}, \\ v_{xy} &= \frac{\gamma}{1-\rho^2}e^{-\gamma x+\frac{1}{2}\lambda_n^2 t}(\psi_n(y))^{\rho^2/(1-\rho^2)}\psi'(y), \\ v_{xx} &= -\gamma^2 e^{-\gamma x+\frac{1}{2}\lambda_n^2 t}(\psi_n(y))^{1/(1-\rho^2)}. \end{aligned}$$

By replacing the above formulas on (4.7), we get the formula of optimal strategy (4.23) as a function of ψ. □

Proposition 4.14. *For power-series solution (4.21) and* $\forall t \in (T_n, T_{n+1}]$*, optimal strategy is given by*

$$\pi^*_{t,n} = \frac{\Pi_n}{\tilde{\Pi}_n}\left(\frac{\lambda_n}{\gamma\sigma_n} + \frac{\rho}{\gamma(1-\rho^2)}\frac{u_n}{\sigma_n}y_n\frac{\left(\frac{k_n}{\zeta_n} + \frac{k_n^2}{\zeta_n(\zeta_n+1)}y_n + \frac{k_n^3}{2\zeta_n(\zeta_n+1)(\zeta_n+2)}y_n^2 + ...\right)}{\left(1 + \frac{k_n}{\zeta_n}y_n + \frac{k_n^2}{\zeta_n(\zeta_n+1)}y_n^2 + \frac{k_n^3}{6\zeta_n(\zeta_n+1)(\zeta_n+2)}y_n^3 + ...\right)}\right). \tag{4.24}$$

Proof. We have that

$$\pi^*_{t,n} = \frac{\Pi_n}{\tilde{\Pi}_n}\left(\frac{\lambda_n}{\gamma\sigma_n} + \frac{\rho}{\gamma(1-\rho^2)}\frac{u_n}{\sigma_n}y_n\psi_n'(y)\Big(\psi_n(y)\Big)^{-1}\right),$$

and by replacing the power series solution

$$\psi_n(y) = a_{n,0}\sum_{i=0}^{\infty}\frac{(k_n y)^i}{i!(i+\zeta_n-1)!}$$

with

$$a_{n,i} = \frac{k_n^i}{i!(i+\zeta_n-1)!}a_{n,0},$$

we get the final formula for the optimal strategy (4.24) expressing up to the second order of the power series. □

We observe that the optimal strategy starts from Merton's optimal strategy and adds a second term which is quite interesting to study. This terms consists of function ψ in that it contains $y_n\psi'(y)(\psi(y))^{-1}$

which (as we see in the function (4.24)) is written as

$$y_n \frac{\left(\frac{k_n}{\zeta_n} + \frac{k_n^2}{\zeta_n(\zeta_n+1)} y_n + \frac{k_n^3}{2\zeta_n(\zeta_n+1)(\zeta_n+2)} y_n^2 + ...\right)}{\left(1 + \frac{k_n}{\zeta_n} y_n + \frac{k_n^2}{\zeta_n(\zeta_n+1)} y_n^2 + \frac{k_n^3}{6\zeta_n(\zeta_n+1)(\zeta_n+2)} y_n^3 + ...\right)}.$$

While this terms remains always positive, its actual impact on optimal portfolio is almost negligible. In other words, the fraction of the above term is close to one which makes the extra term on the optimal demand approximately equal to $\rho/\gamma(1-\rho^2) \times (y_n u_n)/\sigma_n$. In particular, when ρ is positive, the manager increases the investment in the risky asset.

Proposition 4.15 (Comparative Statics)**.** *The approximation up to the third order of optimal strategy (4.24) is*
(i) increasing w.r.t. the volatility of contributions u
(ii) decreasing w.r.t. the volatility of risky asset σ
(iii) increasing w.r.t. the Sharpe Ratio of risky asset λ*, (or alternatively increasing w.r.t. the return of risky asset* μ*).*

Proof. We base our approximation up to the third order of optimal strategy and particularly on that term which is a function of all the parameters with respect to which we want to examine the monotonicity.

$$\frac{u}{\sigma} \frac{\left(\frac{k}{\zeta} y + \frac{k^2}{\zeta(\zeta+1)} y^2 + \frac{k^3}{2\zeta(\zeta+1)(\zeta+2)} y^3 + ...\right)}{\left(1 + \frac{k}{\zeta} y + \frac{k^2}{\zeta(\zeta+1)} y^2 + \frac{k^3}{6\zeta(\zeta+1)(\zeta+2)} y^3 + ...\right)} \tag{4.25}$$

We replace formulas $\zeta = 2(m - \lambda\rho u)/u^2$, $k = 2\gamma\tilde{\Pi}(1-\rho^2)/u^2$ and we consider the functions $\tilde{\Gamma} = 2\gamma\tilde{\Pi}(1-\rho^2)y$, which is independent of u, σ, λ, and the function $V(u, \sigma, \lambda) = 2(m - \lambda\rho u)$. Hence, (4.25) becomes

$$\frac{u}{\sigma} \frac{\frac{\tilde{\Gamma}}{V} + \frac{\tilde{\Gamma}^2}{V(V+u^2)} + \frac{\tilde{\Gamma}^3}{2V(V+u^2)(V+2u^2)} + ...}{\left(1 + \frac{\tilde{\Gamma}}{V} + \frac{\tilde{\Gamma}^2}{V(V+u^2)} + \frac{\tilde{\Gamma}^3}{6V(V+u^2)(V+2u^2)} + ...\right)}. \tag{4.26}$$

We also set the function

$$g(u, \sigma, \lambda) = \frac{\tilde{\Gamma}}{V} + \frac{\tilde{\Gamma}^2}{V(V+u^2)} + \frac{\tilde{\Gamma}^3}{2V(V+u^2)(V+2u^2)} + ...$$

(which is a positive function, according to (4.10)), and therefore (4.26) becomes

$$\frac{u}{\sigma} \frac{g(u, \sigma, \lambda)}{\left(1 + g(u, \sigma, \lambda)\right)}. \tag{4.27}$$

(i) The partial derivative of (4.27) in terms of u, is

$$\frac{\sigma(1+g)(g + ug_u) - ug\sigma g_u}{(\sigma(1+g))^2} = \frac{g + g^2 + ug_u}{(1+g)^2} > 0,$$

where $g_u > 0$, with $V_u(u, \sigma, \lambda) = -2\lambda\rho < 0$.

(ii) The partial derivative of (4.27) in terms of σ is

$$\frac{ug_\sigma\sigma(1+g) - ug((1+g)+\sigma g_\sigma)}{(\sigma(1+g))^2} = \frac{u\sigma g_\sigma - ug - ug^2}{(\sigma(1+g))^2} < 0,$$

where,

$$g_\sigma = -\frac{2\mu\rho u}{\sigma^2}\Big(\frac{\tilde{\Gamma}}{V^2} + \frac{\tilde{\Gamma}^2}{\big(V(V+u^2)\big)^2}(2V+u^2) + \frac{\tilde{\Gamma}^3}{\big(2V(V+u^2)(V+2u^2)\big)^2}(6V^2+12Vu^2+4u^4)+...\Big) < 0,$$

since $V_\sigma = \frac{2\mu\rho u}{\sigma^2}$. Therefore, optimal strategy (4.24) is a decreasing function of σ, since

$$\pi^*_\sigma = -\frac{2\mu}{\gamma\sigma^3} + \frac{\rho}{\gamma(1-\rho^2)}\frac{u\sigma g_\sigma - ug - ug^2}{(\sigma(1+g))^2} < 0.$$

(iii) The partial derivative of (4.27) in terms of λ is

$$\frac{u}{\sigma}\frac{g_\lambda(1+g) - g_\lambda g}{(1+g^2)} = \frac{u}{\sigma}\frac{g_\lambda}{(1+g)^2} > 0, \tag{4.28}$$

where,

$$g_\lambda = 2\rho u\Big(\frac{\tilde{\Gamma}}{V^2} + \frac{\tilde{\Gamma}^2}{\big(V(V+u^2)\big)^2}(2V+u^2) + \frac{\tilde{\Gamma}^3}{\big(2V(V+u^2)(V+2u^2)\big)^2}(6V^2+12Vu^2+4u^4)+...\Big) > 0,$$

since $V_\lambda = -2\rho u$. Therefore, optimal strategy (4.24) is an increasing function of λ, since

$$\pi^*_\lambda = \frac{1}{\gamma\sigma} + \frac{\rho}{\gamma(1-\rho^2)}\frac{u}{\sigma}\frac{g_\lambda}{(1+g)^2} > 0.$$

□

4.4. **Comparative Statics.** We finish with the analysis of the DC pension funds under forward utility process by giving some extra discussion on comparative statics.

Firstly, we observe that the optimal investment at each period is analogous to the population ratio $\Pi_n/\tilde{\Pi}_n$, which in particularly implies that when the retiring generation is highly populated, the investment in the risky asset is higher. This is a consequence of the forward looking feature of the optimization problem. Intuitively, consider the period at the end of which the first generation retires. Until the beginning of that period $\Pi_n/\tilde{\Pi}_n = 1$. Now if the number of the new members is higher than the number of the retiring generation, the ratio increases to higher than one.

Note also that when $\rho = 0$, the presence of contribution does not change the optimal strategy. This means that the contribution enters the fund are invested in the risk-free asset and then accumulated are distributed to each retiring generation. The only parameter that changes the standard Merton's portfolio is the population ratio $\Pi_n/\tilde{\Pi}_n$. On the contrary, when correlation is positive, then an increase in volatility of contributions, implies an increase in investment in risky asset (see also Figure 1). Intuitively, when contributions' volatility gets higher the asset manager has to capture this volatility through a higher position in risky asset, since in the DC scheme higher volatility of contribution implies higher volatility of the deliverable payments to the retiring generation.

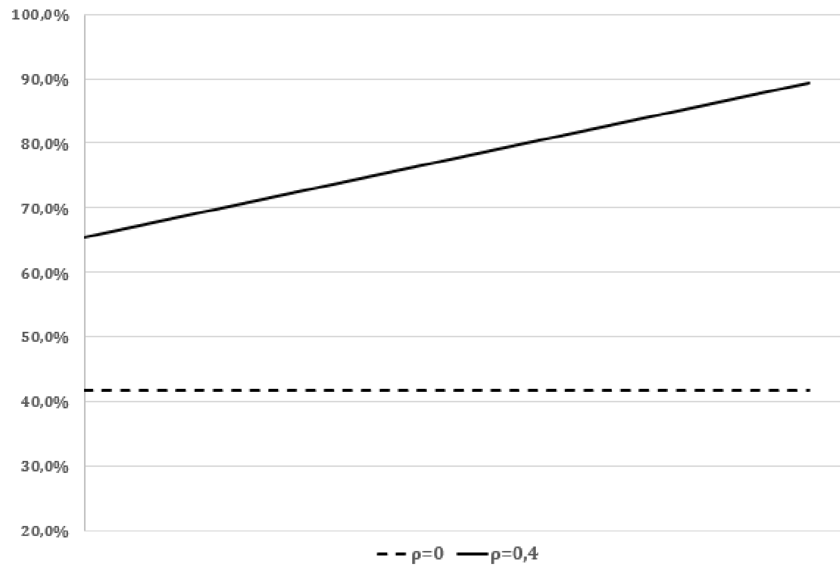

FIGURE 1. Optimal strategy π^* - volatility of contributions u. In this example, $\mu = 0.05$, $\sigma = 0.2$, $m = 0.02$ and $\gamma = 5$.

On the other hand, optimal investment is decreasing with respect to asset's volatility σ, an effect that stems not only from the Merton's part, but also from the contribution's part (when $\rho > 0$). This is of course an expected and reasonable feature of the model (see also Figure 2). Similarly, in terms of the Sharpe ratio λ, we verify that its increase implies higher positions in the risky asset.

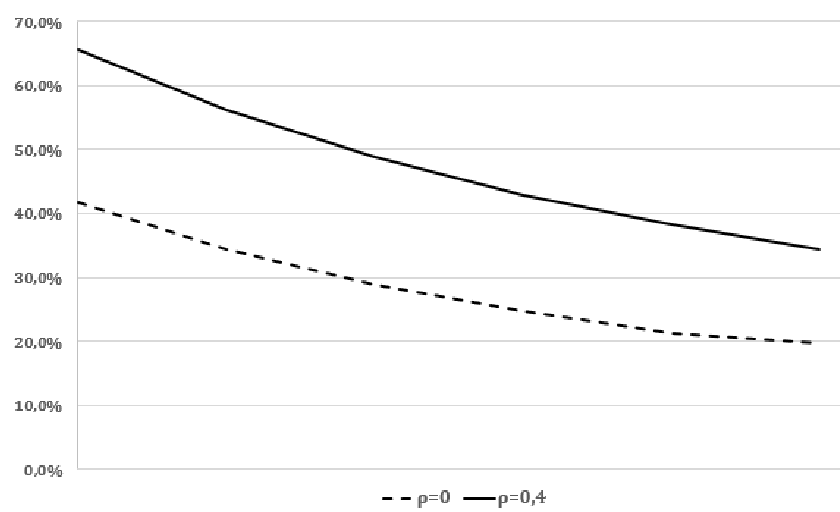

FIGURE 2. Optimal strategy π^* - volatility of risky asset σ. In this example, $\mu = 0.05$, $u = 0.1$, $m = 0.02$ and $\gamma = 5$.

4.5. **Collective DB Pension Plans.** In this subsection, we show that the above setting could be adjusted in order to incorporate the different liability setting of a CDB pension fund. In contrast to CDC pension funds, at the case of CDB the manager invests collectively the contributions of all generations but the promised liabilities to the retiring generations are pre-determined. For this, we consider the running liability of the fund to the generation-i of population P_i to be the *historical average of all the wages* that this generation has received since entering the fund to the retirement year.

In particular, for the generation-0 we have that the running liability is of the form

$$Z_t^0 = P_0 \int_0^t \phi_s C_s ds, \quad \forall t \in [0, T_1]$$

$$Z_t^0 = Z_{T_n}^0 + P_0 \int_{T_n}^t \phi_s C_s ds, \quad \forall t \in (T_n, T_{n+1}] \text{ and } n = 1, 2, ..., N-1,$$

where $\{\phi_t\}_{t\in T}$ denotes the *liability factor*, which are a family of pre-determined positive constants that are usually increasing through time.

Similarly to the CDC pension plans, we define the surplus process $(x_t)_{t\geq 0}$ under which the forward performance is going to be defined:

$$x_t := X_t - Z_t, \qquad t \geq 0,$$

where X is given in (4.2) and Z is defined in the following way:

$$Z_t := \begin{cases} 0, & t \leq T_{N-1}, \\ Z_t^0, & T_{N-1} < t \leq T_N \\ Z_t^1, & T_N < t \leq T_{N+1} \\ \vdots & \vdots \\ Z_t^k, & T_{N+k-1} < t \leq T_{N+k} \end{cases} \tag{4.29}$$

for any $k \in \mathbb{N}$.

Let's consider again what happens when the first generation retires: Note that for each $t < T_N$ we define the forward performance at process $x_t = X_t$ (since there is no payment). At $t = T_N$, generation-0 leaves and gets $Z_{T_N}^0$, hence

$$\begin{aligned} x_{T_N} :=& X_{T_N} - Z_{T_N}^0 \\ =& X_{T_{N-1}} + \int_{T_{N-1}}^{T_N} \pi_s \frac{dS_s}{S_s} + \Pi_{N-1} \int_{T_{N-1}}^{T_N} y_s ds - \left(Z_{T_{N-1}}^0 + P_0 \int_{T_{N-1}}^{T_N} \phi_s C_s ds \right) \\ =& x_{T_{N-1}} + \int_{T_{N-1}}^{T_N} \pi_s \frac{dS_s}{S_s} + \Pi_{N-1} \int_{T_{N-1}}^{T_N} h_s C_s ds - P_0 \int_{T_{N-1}}^{T_N} \phi_s C_s ds \end{aligned}$$

where $\{y_t\}_{t\geq 0}$ is the contribution process which is determined as a known proportion of the wage process $\{C_t\}_{t\geq 0}$, particularly $y_t := h_t C_t$, with $h_t \in (0, 1)$, for each t. Therefore, the wage process is a piece-wise Geometric Brownian Motion process, that is

$$dC_t = b_n(C_t)dt + g_n(C_t)dW'_t$$

where $b_n(C_t) = m_n C_t$, and $g_n(C_t) = u_n C_t$.

In general, we have that the surplus process of the fund that we consider for the optimization problem $\forall t \in (T_n, T_{n+1}]$ is given as

$$x_t := \begin{cases} x_{T_n} + \int_{T_n}^t \pi_s \frac{dS_s}{S_s} + \Pi_n \int_{T_n}^t h_s C_s ds, & \text{for } n \le N-2 \\ x_{T_n} + \int_{T_n}^t \pi_s \frac{dS_s}{S_s} + (\Pi_n h_n - \bar{\Pi}_n \phi_n) \int_{T_n}^t C_s ds, & \text{for } n \ge N-1 \end{cases} \tag{4.30}$$

where $x_0 := X_0$ and

$$\bar{\Pi}_n := P_n, \text{ for } n \ge N-1 \tag{4.31}$$

is the members who leave the fund at the end of period $[T_n, T_{n+1})$. We also note that both coefficients Π_n and $\bar{\Pi}_n \in \mathcal{F}_{T_n}$, so in the interval $(T_n, T_{n+1}]$ can be considered constants. We also consider the process $\{M_n\}_{n\in\mathbb{N}}$ as

$$M_n := \Pi_n h_n - \bar{\Pi}_n \phi_n,$$

which shows the difference between the percentage of capital that enters the fund by the contributions and the percentage of capital that leaves the fund in terms of promised liabilities.

Therefore, the dynamics of the surplus process at the period $(T_n, T_{n+1}]$ is

$$\begin{aligned} dx_t =& \pi_t \frac{dS_t}{S_t} + M_n C_t dt = \pi_t \left(\mu_n dt + \sigma_n dW_t\right) + M_n C_t dt \\ =& (\pi_t \mu_n + M_n C_t)dt + \pi_t \sigma_n dW_t. \end{aligned}$$

Since we have unified the surplus process in (4.30), from Ito's formula we get that v must solve in $t \in (T_n, T_{n+1}]$,

$$v_t + \frac{1}{2} g_n^2(C) v_{CC} + v_y b_n(C) + v_x M_n C + \max_\pi \left(\pi \mu_n v_x + \frac{1}{2}\pi^2 \sigma_n^2 v_{xx} + \rho \sigma_n g_n(C) \pi v_{xC} \right) = 0.$$

The optimal strategy is given by

$$\tilde{\pi}^*_{t,n} = -\frac{\lambda v_x}{\sigma_n v_{xx}} - \rho \frac{g_n(C) v_{xC}}{\sigma_n v_{xx}}, \qquad \forall t \in (T_n, T_{n+1}.] \tag{4.32}$$

As in the case of the CDC pension plan, we are looking for positive solutions of the form

$$v_n(x, C, t) = -e^{-\gamma x} F_n(C, t). \tag{4.33}$$

Following the same line of arguments, we conclude that the positive solution should solve the following ode

$$C^2 \tilde{\psi}''(C; l) + \zeta_n C \tilde{\psi}'(C; l) - \tilde{k}_n C \tilde{\psi}(C; l) - \xi_n(l) \tilde{\psi}(C; l) = 0, \tag{4.34}$$

where $\zeta_n = 2(m_n - \lambda_n \rho u_n)/u_n^2$ and $\tilde{k}_n = 2\gamma M_n (1-\rho^2)/u_n^2$ are $\mathcal{F}_{T_n}$-measurable and $\xi_n(l) = (\lambda_n^2(1 - \rho^2) + 2l)/u_n^2$. The ode (4.34) is the same as (4.11), that is a special case of the general Bessel equation whose unique solution is given as

$$\tilde{\psi}_n(C;l) = C^{(1-\zeta_n)/2}\left(\beta_n Z_{p_n(l)}(2\sqrt{C\tilde{k}_n}) + \beta_n' Z_{-p_n(l)}(2\sqrt{C\tilde{k}_n})\right), \tag{4.35}$$

where

$$p_n(l) = 2\sqrt{\left(\frac{1-\zeta_n}{2}\right)^2 + \xi_n(l)}$$

β_n, β_n' are $\mathcal{F}_{T_n}$-measurable and $Z_{\zeta(C)}$ denotes a Bessel function (depending on the values of ζ).

Therefore, a candidate for the forward process between each period $(T_n, T_{n+1}]$ could be

$$U_{T_{n+1}}(x,C,t;\omega) = v_{T_{n+1}}(x,C,t) = -e^{-\gamma x}\left(\int_{\mathbb{R}} e^{-lt}\tilde{\psi}_n(C;l)\nu_n(dl)\right)^{1/(1-\rho^2)}, \tag{4.36}$$

for all $t \in (T_n, T_{n+1}]$, for some Borel probability measure ν_n, where $\psi_n(C;l)$ solves (4.34).

Similalry, at the case of a special member of this family of solutions when $l_n = -(1/2)\lambda_n^2(1-\rho^2)$, we get that the candidate forward utility (4.36) takes the form

$$U_{T_n}(x,C,t;\omega) = -e^{-\gamma x + \frac{1}{2}\lambda_n^2 t}(\tilde{\psi}_n(C))^{1/(1-\rho^2)}, \tag{4.37}$$

and the ode (4.34) is simplified to

$$C\tilde{\psi}''(C) + \zeta_n\tilde{\psi}'(C) - \tilde{k}_n\tilde{\psi}(C) = 0 \tag{4.38}$$

whose unique solution is

$$\psi_n(C) = C^{(1-\zeta_n)/2}\left(\beta_n Z_{\zeta_n-1}(2\sqrt{C\tilde{k}_n}) + \beta_n' Z_{1-\zeta_n}(2\sqrt{C\tilde{k}_n})\right), \tag{4.39}$$

for some $\mathcal{F}_{T_n}$-measurable random variables β_n and β_n'. Considering the case $\beta_n' = \beta_{n+1}' = 0$, we get that the unique solution at the case of Collective DB pension plans (which is the same as (4.19)), is

$$\tilde{\psi}_n(C) = \beta_n C^{(1-\zeta_n)/2} Z_{\zeta_n-1}(2\sqrt{C\tilde{k}_n}). \tag{4.40}$$

Moreover, as in the previous subsection we can use the Frobeniu's method and we define the power series solution as following

$$\tilde{\psi}_n(C) = \sum_{i=0}^{\infty}\tilde{a}_{n,i}C^i, \quad \text{for each } n \in \mathbb{N},$$

where

$$\tilde{a}_{n,i} = \frac{\tilde{k}_n^i}{i!(i+\zeta_n-1)!}\tilde{a}_{n,0}, \quad \forall i \in \mathbb{N} \text{ and for each } n \in \mathbb{N}.$$

4.5.1. *Optimal Strategy.*

Corollary 4.16. *For candidate forward exponential utility process (4.15), the optimal strategy for a Collective DB pension fund takes the form*

$$\tilde{\pi}_{t,n}^* = \frac{\lambda_n}{\gamma\sigma_n} + \frac{\rho}{\gamma(1-\rho^2)}\frac{u_n}{\sigma_n}C_n\tilde{\psi}_n'(C)\Big(\tilde{\psi}_n(C)\Big)^{-1}, \qquad \forall t \in (T_n, T_{n+1}]. \tag{4.41}$$

Proof. Same calculations as in the proof of Corollary (4.13). □

Proposition 4.17. *For power-series solution (4.21) and $\forall t \in (T_n, T_{n+1}]$, optimal strategy of a Collective DB pension fund is given by*

$$\tilde{\pi}^*_{t,n} = \left(\frac{\lambda_n}{\gamma\sigma_n} + \frac{\rho}{\gamma(1-\rho^2)} \frac{u_n}{\sigma_n} C_n \frac{\left(\frac{\tilde{k}_n}{\zeta_n} + \frac{\tilde{k}_n^2}{\zeta_n(\zeta_n+1)} C_n + \frac{\tilde{k}_n^3}{2\zeta_n(\zeta_n+1)(\zeta_n+2)} C_n^2 + \ldots \right)}{\left(1 + \frac{\tilde{k}_n}{\zeta_n} C_n + \frac{\tilde{k}_n^2}{\zeta_n(\zeta_n+1)} C_n^2 + \frac{\tilde{k}_n^3}{6\zeta_n(\zeta_n+1)(\zeta_n+2)} C_n^3 + \ldots \right)} \right). \tag{4.42}$$

Proof. Similarly as in the case of the previous subsection, at the case of CDC pension plans, we use the power series solutions. We have that

$$\tilde{\pi}^*_{t,n} = \frac{\lambda_n}{\gamma\sigma_n} + \frac{\rho}{\gamma(1-\rho^2)} \frac{u_n}{\sigma_n} C_n \tilde{\psi}'_n(C) \Big(\tilde{\psi}_n(C) \Big)^{-1}, \qquad \forall t \in (T_n, T_{n+1}].$$

and by replacing the power series solution

$$\tilde{\psi}_n(C) = \sum_{i=0}^{\infty} \tilde{a}_{n,i} C^i, \quad \text{for each } n \in \mathbb{N},$$

with

$$\tilde{a}_{n,i} = \frac{\tilde{k}_n^i}{i!(i+\zeta_n-1)!} \tilde{a}_{n,0}, \quad \forall i \in \mathbb{N} \text{ and for each } n \in \mathbb{N},$$

we get the final formula for the optimal strategy (4.42). □

The main difference between the optimal strategy of a Collective DB pension fund (4.42) and the optimal at the case of Collective DC (4.24) is the presence of process M_n that appears in process $\tilde{k}_n$. We see that an increase in M_n, i.e. when h_n increases (or when ϕ_n decreases), implies a very small increase of the optimal investment in risky asset (as in the related discussion of the CDC scheme).

Moreover, we note that the optimal strategy of a Collective DB pension fund departs again from the Merton's portfolio and increases the investment in the risky asset when $\rho > 0$ due to the promised liability. Indeed, the presence of the liability increases the target payment which makes the fund's manager to increase the volatility of the undertaken portfolio. On the other hand, higher volatility ratio u/σ implies higher investment in risky asset too, meaning that higher volatility of the promised liability increase the targeted volatility of the optimal strategy.

5. Conclusion

In this work, we model the investment criteria of both Collective DC and DB pension funds through the notion of the forward performance criterion (forward utility process) targeted on the fund's surplus. Collectiveness means that the investments of the fund are made for all the generations at once. While the investment problem and the risky asset are considered in a continuous-time basis, the proposed model imposes a sequence of discrete times which stand for the times generations retire and new ones enters the fund. At that time-setting, the fund's surplus is defined as the difference between the wealth that is created by investments and contributions minus the running fund's liabilities. Besides the fund's population, at each discrete time, the fund manager could update her model's parameters allowing in that way a dynamic updating of the model's dynamics.

Financial market is incomplete market, in that the risky asset is correlated with contributions that cannot be hedged out).

Under a family of exponential initial utility, we derive the optimal investment strategies for both the CDC and the CDB pension settings. Based on the analytic expression of the optimal strategies we make a several predictions in terms of the models parameters. In short, higher correlation between market and contribution, higher contributions' volatility, higher running population and lower market's volatility result in higher position in the risky asset.

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