LOG-OPTIMALITY AND PRICING WITH A SMALL STREAM

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Abstract. In an incomplete financial model we consider a traded asset with stochastic exponential price driven by a general Itô process and an investor with log-utility preferences who, in addition to an initial capital, receives units of a non-traded endowment at maturity. Using duality techniques we derive a "localized" fourth-order expansion of the (primal) value function for a small number of units held in the non-traded endowment, ϵ, and identify the corresponding nearly optimal "wealth" process that allows for matching the value function up to that order. Using the above we also provide a result in the context of utility-based pricing by expanding the already well-established quadratic approximation of the utility-based certainty equivalent w.r.t. ϵ, up to fourth order for the case of log-utility. In addition we examine the behavior of both the value function expansions (quadratic and fourth order) and the respective nearly optimal wealth processes as the time horizon T tends to infinity. Particularly we show that the former remain within the correct order, but not for an arbitrary choice of bounding constants, all the way as $T \to \infty$. In turn the asymptotic behavior described above for the infinite horizon setting is inherited by the log-utility certainty equivalent, extending the previous results for arbitrarily large maturities.

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Introduction

Discussion. A central problem in financial economics involves an investor allocating initial wealth across a financial market with the goal of maximizing the expected utility of terminal wealth. This optimal investment problem in continuous-time settings was analyzed by Merton [Mer69; Mer71], who used dynamic programming techniques to derive a non-linear partial differential equation characterizing the value function. For various utility functions, Merton obtained explicit closed-form solutions.

A major conceptual advancement came with the development of the theory of equivalent martingale measures by Ross [Ros76], Harrison and Kreps [HK79], and Harrison and Pliska [HP81], which

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enabled the application of martingale and duality methods to such optimization problems. Under the assumption of market completeness—where the set of equivalent martingale measures reduces to a singleton—this duality approach was further developed by Pliska [Pli86]; Karatzas, Lehoczky, and Shreve [KLS87]; and Cox and Huang [CH89; CH91]. The more intricate case of incomplete markets was addressed in foundational works by He and Pearson [HP91a; HP91b], and by Karatzas, Lehoczky, Shreve, and Xu [Kar+91]. Building on these contributions, Kramkov and Schachermayer [KS99; KS03] established minimal conditions on both the utility function and the financial market under which the core results of the theory remain valid.

In the context of incomplete markets, a natural extension of the problem involves maximizing expected utility when the investor receives additional exogenous random endowments. Typical examples are the pension funds whose endowment is the difference between the contributions and the liabilities and the institutional investors with existing non-tradeble placements. In complete markets, endowments (processes or simple random payoffs at a certain point in time) can be perfectly replicated using traded assets, effectively reducing the problem to one with augmented initial wealth and no random endowment. However, as noted among others in [HH07], real-world markets are typically incomplete, with perfect replication impeded by frictions such as transaction costs, non-traded assets, and portfolio constraints. In such settings, assets are associated with a range of arbitrage-free prices, and the risk of holding them cannot be fully hedged through market trading alone.

Consequently, in incomplete markets, transforming the problem with random endowments into an equivalent problem without them is generally a challenging one. Analyzing the value function and deriving closed-form solutions is undeniably significantly more difficult. Notable contributions addressing this challenge include Cvitanić, Schachermayer, and Wang [CSW01], who characterized the optimal terminal wealth in a general semimartingale model via a dual formulation. [KŽ03] extends this framework to account for intertemporal consumption. Hugonnier and Kramkov [HK04] treated both the initial capital and the quantity of random endowments as optimization variables, allowing for unbounded endowments. Also, [OŽ09] studies the case of unbounded random endowments and utility functions defined over the entire real line, providing necessary and sufficient conditions for the absence of utility-based arbitrage and the existence of a solution to the primal problem.

Modelling optimal investment under exogenously given endowment processes encounters also another technique challenge. Typically, investment horizon of institutional investors and pension funds do not have a priori time horizon and similarly the endowment process does not have a certain time at which it vanishes. This means that the optimization problem of such an investor should not be constrained by a (deterministic) finite horizon. Rather, the investor's optimization objective should guarantee time consistency for each future horizon taking also into account the effect of the endowment process. This problem is generally hard to tackle, although it is a more appropriate setting for many cases of institutional investors.

Contributions. In an incomplete financial market, we consider a traded asset whose price follows a stochastic exponential driven by a general Itô process and an investor with log-utility preferences who, in addition to an initial capital, receives units of a non-traded endowment process. Closed-form solutions to the associated utility maximization problem are generally unavailable due to the highly non-linear nature of the corresponding stochastic control PDE. To address this, we assume that the payoff from the non-traded endowment is small relative to the investor's total wealth. This approximation approach is usually encountered in the related strand of literature of utility-based pricing a non-traded contingent claim. Therein, an economically appealing choice is the utility-indifference approach put forward by Hodges and Neuberger [HN89] as well as Davis, Panas and Zariphopoulou [DPZ93]. For a given utility function, this method determines a "fair" price by equating the investor's maximal expected utility with and without the non-traded claim. A comprehensive overview of this approach is provided in [HH07].

Within this framework, our contributions to the literature are as follows:

(1) *Fourth-order expansion and nearly optimal strategies:*

Using duality techniques, we derive a "localized" fourth-order expansion of the primal value function with respect to a small number of units ϵ in the non-traded endowment. We then identify the corresponding nearly optimal wealth process in the spirit of Henderson [Hen02], allowing the value function to be matched up to this order. Leveraging the semimartingale characteristics of the return process, we explicitly construct the associated nearly optimal strategy. To the best of our knowledge this is the first result in this direction, extending similar established approximations of quadratic nature to fourth order. Interestingly, the nearly optimal wealth process underpinning this expansion can also be characterized via a Kunita-Watanabe projection, mirroring the case of second-order expansions (cf. [KS06]). This also leads to a novel result in the context of utility-based pricing, since we extend the well-known quadratic approximation of the utility-based certainty equivalent in ϵ to fourth order in the log-utility case.

(2) *Long-horizon asymptotics:*

We also study the robustness of our results as the investment horizon T tends to infinity. Specifically, we examine the behavior of both the quadratic and fourth-order expansions of the value function and the associated nearly optimal wealth processes. We show that the expansions remain within the correct order, but not for an arbitrary choice of bounding constants, all the way as $T \to \infty$. The long-term behavior of the certainty equivalent is then derived as a consequence, generalizing previous results to arbitrarily large maturities.

These results have three key implications. First they allow for a better understanding of log-optimal behavior in the incomplete setting, under the presence of a non-traded endowment. This stems from the fact that the nearly optimal strategy producing the quadratic approximation of the value function is optimal; assuming market completeness. This enables valuable insights on how market incompleteness is expressed in this context via comparison to the complete setting. Second they allow for a more accurate pricing of the investor's position on the non-traded asset. Particularly the already established quadratic expansion of the certainty equivalent provides valuable information in relation to the complete market. However, it fails to capture the effect of the nearly optimal process associated with the fourth-order expansion of the value function, also referred as the "second-order" nearly optimal process in contrast to the one that produces the quadratic expansion of the value function. Also, considering the case of arbitrarily large time horizons, i.e. "long-term setting" comes with its own merits. Namely it enables analyzing assets which do not have a certain pre-specified maturity. A prominent example of such a situation arises, for example, in the context of a pension fund's liabilities.

Extending the previously established results on the expansion of the value function w.r.t. a small ϵ up to fourth order is no trivial task. A key challenge lies in establishing rigorous lower and upper bounds. Unlike the second-order case, where discrepancies in bounding constants affect only the second-order term, the fourth-order case sees such discrepancies propagate through all terms. This is due to the non-regularity of the underlying processes. A crucial step to resolve this is to consider a range of investment horizons instead of a fixed terminal time T, allowing for a localized analysis that yields the desired fourth-order expansion. However, this approach becomes more delicate as $T \to \infty$, since the bounding constants also depend on T and thus influence the order of the approximation.

Related literature. The existing literature on optimal investment is vast in order to give a meaningful overview. Instead, we focus on the specific area of utility-based hedging and pricing, which is central to our work. Although this field has produced a wealth of results, explicit computations of utility-based prices and hedging strategies are typically infeasible or require restrictive model assumptions. This difficulty arises from the nonlinearity of the Hamilton-Jacobi-Bellman (HJB) partial differential equation associated with the value function, which generally precludes closed-form solutions.

One notable exception is the exponential utility function, which often permits analytical tractability. This is due to its property of separating the value function into components associated with wealth and trading, simplifying the analysis considerably. Prominent works in this context include [Hen02], [MZ04], [GH07], and [LL12]. These studies leverage a linearization technique—commonly referred to as the Cole-Hopf transformation or distortion power—first introduced in claim valuation by [Zar01], which reduces the nonlinear HJB PDE to a linear form solvable via standard methods. Further generalizations by [FS08] and [FS10] showed that, even in models with general asset dynamics, the exponential utility-based price admits a closed-form expression, although these formulas are often implicit and less interpretable. Complementary to these results, Davis [Dav06] used duality techniques to derive an explicit form for the optimal hedging strategy, with related developments in [Mon13]. Another line of research, such as [AIR10], adopts a more stochastic perspective: by applying the martingale optimality principle, the utility maximization problem is reformulated in terms of a forward-backward stochastic differential equation (FBSDE) with quadratic nonlinearity, yielding a characterization of both price and hedging strategy.

Even within the relatively tractable exponential utility framework, closed-form expressions are not always obtainable. For example, in models where the claim depends on the traded asset, Sircar and Zariphopoulou [SZ05] derive asymptotic expansions for the utility-indifference price in the regime of fast mean-reverting volatility. Henderson and Liang [HL16] consider a multidimensional non-traded asset model subject to intertemporal default risk, and develop a semigroup approximation using splitting techniques.

Given the scarcity of explicit results and their reliance on exponential utility, various asymptotic approaches have been proposed for pricing and investment strategy in incomplete markets. Monoyios [Mon04; Mon07], for example, works in a Black-Scholes framework with basis risk and approximates the hedging strategy in powers of $1-\rho^2$, where ρ denotes the correlation between traded and non-traded assets. In [Hen02] and [HH02] the authors consider power utility preferences and derive second-order expansions of the investor's value function with respect to a small position in the contingent claim, thereby approximating both the hedging strategy and reservation price.

These early results were significantly extended in [KS06; KS07], where the authors operate in a general semimartingale framework with broad utility functions defined on $\mathbb{R}_{>0}$. In [KS06], they use second-order expansions of both the primal and dual value functions to derive first-order approximations for marginal (utility-based) indifference prices and study their qualitative features. Although more general in scope, their analysis is confined to fixed time horizon and focuses solely on quadratic expansions, in contrast to our own work. A related analysis by Kallsen [Kal02] studies first-order marginal price approximations under local utility maximization.

In [KS07], using techniques developed in their earlier work, authors also provide first-order approximations of the utility-based hedging strategy and demonstrate its relation to quadratic hedging. Similar asymptotic results are found in [Mon10], which considers valuation and hedging in the presence of parameter uncertainty under exponential utility and partial information. There, the indifference price is approximated to linear order in the risk aversion parameter via PDE methods.

In the same spirit, [KR11] analyzes utility-based pricing and hedging under exponential utility in the limit of vanishing risk aversion or small claim quantities. First-order expansions are derived for both price and hedging, extending earlier results in [MS05], [Bec06] and [KS06; KS07]. Building on this line of research, [KMKV14] presents alternative representations of the results in [KS06; KS07] for power utility functions that avoid the need for a change of numéraire. Their approach leverages semimartingale characteristics and applies to exponential Lévy models.

Within the setting of exponential Lévy processes, [MT16] derives a novel non-asymptotic approximation for the exponential utility-based indifference price. The approach therein extends the earlier small risk-aversion asymptotics and yields a closed-form approximation by treating the Lévy model as a perturbation of the classical Black-Scholes framework.

Structure of the paper. Section 1 introduces the general setup of the market as well as the investor's optimal investment problem. Section 2 extends already known results relevant to the quadratic expansion of the value function for arbitrarily large horizons. Section 3 is dedicated to the (localized) fourth order expansion of the value function and the respective nearly optimal wealth process. Finally, in Section 4 we study the concept of log-based pricing in the context of its quadratic and fourth-order expansions as well as their behavior as the time horizon goes to infinity.

1. Setup

Fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and consider a set of time horizons $\mathcal{A}$, where either $\mathcal{A} = [0, U]$, $U > 0$ ("short-term setting") or $\mathcal{A} = \mathbb{R}_{\geq 0}$ ("long-term setting") s.t. each $T \in \mathcal{A}$ is connected with a collection of assets on $[0, T]$ equipped with a filtration, which in this work will be generated by a standard two-dimensional Brownian motion $W = (W^1, W^2)$ (satisfying the usual hypotheses), i.e. $\mathbb{F}_T^W := (\mathcal{F}(t))_{t \in [0,T]}$. Note that a natural common reference point for all the above planning horizons is the "unified" market on $\mathcal{A}$ equipped with $\mathbb{F}^W := (\mathcal{F}(t))_{t \in \mathcal{A}}$. In this unified setting, consider an agent that trades in two assets; a savings account (with a constant rate of return, r) and a risky asset. We work in discounted terms, i.e. we have that the price of the bond is constant, and denote by S the discounted price process of the risky asset with the following Itô return and stochastic exponential price dynamics on $\mathcal{A}$:

$$
\begin{aligned}
&dR(t) = \mu(t)dt + \sigma(t)dW^1(t), \qquad R(0) = 0, \\
&dS(t) = S(t)dR(t), \qquad S(0) > 0,
\end{aligned} \tag{1.1}
$$

for predictable processes μ, σ s.t. $\sigma > 0$; denoting the local mean and volatility of the asset respectively. In this context, a (self-financing) strategy invested in the market is defined as a pair (x, θ) for a constant $x \in \mathbb{R}_{\geq 0}$ representing the initial capital, and a predictable (locally) S-integrable process θ, specifying the amount (in units) invested in the risky asset. More precisely, we have that the wealth process generated by any (x, θ) on $\mathcal{A}$ is given by:

$$
\widetilde{X} := x + \int_0^{\cdot} \theta(t)dS(t). \tag{1.2}
$$

The family of positive wealth processes is given as:

$$
\widetilde{\mathfrak{X}}^+(x) := \{\widetilde{X} : \text{for a given } x > 0 \text{ we have } \widetilde{X} > 0\}.
$$

For $T \in \mathcal{A}$, a probability measure $\mathbb{Q}^T \sim \mathbb{P}$ is an equivalent local martingale measure for that horizon if S (equivalently $\widetilde{X}$, where $\widetilde{X} \in \widetilde{\mathfrak{X}}^+(1)$ in this context) is a local martingale on $[0, T]$ under $\mathbb{Q}^T$. Denote the class of those $\mathbb{Q}^T$ for each time horizon by $\mathcal{Q}^T$ and assume that:

$$
\mathcal{Q}^T \neq \emptyset, \qquad \forall T \in \mathcal{A} \setminus \{0\}. \tag{A1}
$$

This condition is intimately connected with the absence of arbitrage opportunities (refer to [DS94]). Furthermore, define $u^T(x) := \sup_{\widetilde{X} \in \widetilde{\mathfrak{X}}^+(x)} \mathbb{E}[\ln(\widetilde{X}(T))]$ and to exclude the trivial case, assume:

$$
\forall T \in \mathcal{A} \setminus \{0\} \text{ we have } u^T(x) < \infty, \qquad \text{for some } x > 0. \tag{A2}
$$

Then under (A1), (A2); following [KS99], $\forall T \in \mathcal{A}$ we have $u^T(x) < \infty$ for all $x > 0$ and there exists a unique solution to $u^T(x)$, referred as the log-optimal numeraire, for all such planning horizons; given by $xX_{\pi^\star}$ where $X_{\pi^\star} := \mathcal{E}(R_{\pi^\star})$, $R_{\pi^\star} := \int_0^{\cdot} \mu(t)/(\sigma(t))^2 dR(t)$ (in fact local R-integrability of $\pi^\star := \mu/\sigma^2$ should be connected with (A1); see further in [KK21, Theorem 2.31]). Note that $X_{\pi^\star} = 1/S_0^{\pi^\star}$ where

$S_0^{\pi^\star} := \mathscr{E}(R_0^{\pi^\star}),\ R_0^{\pi^\star} := -\int_0^\cdot \mu(t)dW^1(t)/\sigma(t)$ can be used as a new numeraire for the market, under which all the positive wealth processes become local martingales.

Moving forward, besides trading in the financial market we assume that the agent holds $\epsilon \in \mathbb{R}$ units of an exogenous non-traded endowment Λ (for example the cumulative surplus of a pension fund) that is locally absolutely continuous on $\mathscr{A}$ [1], given (in discounted terms) by:

$$\Lambda = \int_0^\cdot e^{-rt}\lambda(t)dt,$$

for a predictable process λ.

In fact, working under the assumption:

$$\text{(A3)} \qquad \text{We have } -\epsilon\Lambda(T) > 0,\ \forall T \in \mathscr{A} \setminus \{0\},$$

we can consider the following value function:

$$u^T(x,\epsilon) := \sup_{\widetilde{X} \in \widetilde{\mathfrak{X}}^+(x)} \mathbb{E}[\ln(\widetilde{X}(T) - \epsilon\Lambda(T))],$$

and show that $u^T(x,\epsilon) < \infty$, $\forall T \in \mathscr{A}$. More precisely, the fact that for the positive parts we have $\ln^+(\widetilde{X}(T)-\epsilon\Lambda(T)) \le \ln^+(X_{\pi^\star}(T))+\ln^+(\widetilde{X}(T)S_0^{\pi^\star}(T)-\epsilon S_0^{\pi^\star}(T)\Lambda(T))$, $\widetilde{X} \in \widetilde{\mathfrak{X}}^+(x)$ along with $u^T(x) < \infty$, $\ln(x) < x$ for all $x > 0$, $\widetilde{X}S_0^{\pi^\star}$ being a supermartingale and $\mathbb{E}[|S_0^{\pi^\star}(T)\Lambda(T)|] < \infty$ should imply, $\ln^+(\widetilde{X}(T) - \epsilon\Lambda(T)) \in \mathscr{L}^1(\mathbb{P})$. In turn we have for any positive wealth process $\widetilde{X}$:

$$\begin{aligned}\mathbb{E}[\ln(\widetilde{X}(T) - \epsilon\Lambda(T))] &\le \underbrace{\mathbb{E}[\widetilde{U}(S_0^{\pi^\star}(T))]}_{<\infty} + \mathbb{E}[S_0^{\pi^\star}(T)(\widetilde{X}(T) - \epsilon\Lambda(T))], \quad \textit{for } \widetilde{U}(y) := \sup_{x>0}[\ln(x) - yx],\ y>0. \\ &\le x + \mathbb{E}[\widetilde{U}(S_0^{\pi^\star}(T))] - \epsilon\mathbb{E}[S_0^{\pi^\star}(T)\Lambda(T))] < \infty,\end{aligned}$$

yielding $u^T(x,\epsilon) < \infty$.

Changing numeraire, expressing everything in terms of $xX_{\pi^\star}$; denoting by $\mathfrak{X}^+_{\pi^\star}$ the class of processes s.t. $\widetilde{X}/xX_{\pi^\star} - 1,\ \widetilde{X} \in \widetilde{\mathfrak{X}}^+(x)$ (which are in fact integrals w.r.t. W^1) and setting $L := \Lambda S_0^{\pi^\star}$ we have:

$$u^T(x,\epsilon) = \mathbb{E}[\ln(X_{\pi^\star}(T))] + u^T_{\pi^\star}(x,\epsilon),$$

where $u^T_{\pi^\star}(x,\epsilon) := \sup_{X \in \mathfrak{X}^+_{\pi^\star}} \mathbb{E}[\ln(1 + X(T) - \epsilon L(T)/x)]$ [2].

2. Second order asymptotics for the value function

For $p \ge 1$ denote the class of martingales on $\mathscr{A}$ s.t. for each M we have $\mathbb{E}[(\sup_{t\in[0,T]}|M(t)|)^p] < \infty$, $\forall T \in \mathscr{A}$ by $\mathscr{H}^p$ and assume:

$$\text{(A4)} \qquad \mathbb{E}[(L(T))^2] < \infty, \qquad \forall T \in \mathscr{A} \setminus \{0\}.$$

[1] Note that the term "locally" here refers to $K \subseteq \mathscr{A}$ that are compact.

[2] Following similar steps as the ones for $u^T(x,\epsilon)$ we can see that $\ln^+(1 + X(T) - \epsilon L(T)/x) \in \mathscr{L}^1(\mathbb{P}),\ X \in \mathfrak{X}^+_{\pi^\star}$ and $u^T_{\pi^\star}(x,\epsilon) < \infty$.

Now, defining the class $\mathscr{M}^p := \{M \in \mathscr{H}^p : M = \int_0^{\cdot} \theta(t) dW^1(t)\}$ we produce a second order asymptotic of $u^T_{\pi^\star}(1,\epsilon)$ using the following unique solution:

$$D_1^T := \underset{X \in \mathscr{M}^2}{\operatorname{argmin}} \mathbb{E}[(X(T) - L(T))^2], \qquad \forall T \in \mathscr{A}, \tag{FO}$$

where for any time horizon $T \in \mathscr{A}$, by (FO) D_1^T is characterized uniquely on $[0,T]$ and considered as being equal to its value at T for all times after, i.e. D_1^T equals $D_1^{T,T} := D_1^T(\cdot \wedge T)$ (this is usually omitted for notational simplicity). Likewise for any other processes throughout this text that are characterized in a similar manner. In particular D_1^T is given through the Kunita-Watanabe decomposition of the martingale in $\mathscr{H}^2$ with terminal value $L(T)$ w.r.t. W^1, i.e. $M^T(t) = \mathbb{E}[L(T)|\mathscr{F}(t)] = D_1^T(t) + N^T(t)$; where $N^T \in \mathscr{H}^2$ is strongly orthogonal to W^1 s.t. $N^T(0) = M^T(0)$.

In the long-term setting, if we additionally assume that $L(\infty) := \lim_{T\to\infty} L(T)$ is well defined (a.s.) and $\mathbb{E}[(L(\infty))^2] < \infty$ we can also denote $M^\infty(t) = \mathbb{E}[L(\infty)|\mathscr{F}(t)]$ and project it on the space of Brownian integrals (w.r.t. W^1) that are square integrable on $\mathbb{R}_{\geq 0}$ (through Kunita-Watanabe decomposition), i.e. on $\mathscr{M}^2_\infty := \{M \in \mathscr{H}^2_\infty : M = \int_0^{\cdot} \theta(t) dW^1(t)\}$ for $\mathscr{H}^2_\infty$ being the class of martingales on $\mathbb{R}_{\geq 0}$ s.t. $\mathbb{E}[(\sup_{t\in\mathbb{R}_{\geq 0}} |M(t)|)^2] < \infty$. In turn we get $M^\infty = D_1^\infty + N^\infty$, similarly to the short-term setting. To this end, consider:

$$\mathbb{E}[(L(T))^2] < \infty,\ \forall T > 0;\ L(\infty) \text{ exists (a.s.) and } \mathbb{E}[(L(\infty))^2] < \infty. \tag{A5}$$

In turn, we get:

Proposition 2.1. Consider the short term setting. Assuming (A1), (A2), (A3) and (A4), then for each such horizon we have:

$$u^T_{\pi^\star}(1,\epsilon) + \epsilon\mathbb{E}[L(T)] + (\epsilon^2/2)\mathbb{E}[(D_1^T(T) - L(T))^2] = o(\epsilon^2). \tag{2.1}$$

Consider the long-term setting. Extending the first three assumptions on $\mathbb{R}_{\geq 0}$ and strengthening (A4) to (A5), we additionally have $\lim_{T\to\infty} D_1^T = D_1^\infty$ (locally uniformly) in probability and:

$$\overline{\lim_{T\to\infty}} \left| u^T_{\pi^\star}(1,\epsilon) + \epsilon\mathbb{E}[L(T)] + (\epsilon^2/2)\mathbb{E}[(D_1^T(T) - L(T))^2] \right| = \mathcal{O}(\epsilon^2). \tag{2.2}$$

Note that a respective form of (2.1), but for a more general class of utility functions on the positive real line, was derived in [KS06].

Proof. The idea is to use D_1^T, N^T in order to produce an upper and a lower bound for $u_{\pi^\star}(\epsilon)$ that agree up to order ϵ^2. Beginning with the former, consider the localizing (sub)sequence $(\widetilde{\tau}_{m(k)})_{k\in\mathbb{N}_{>0}}$ s.t. $\widetilde{\tau}_{m(k)} = \inf\{t : |N^T(t)| + [N^T](t) \geq m(k)\}$ (where $m(k)$ is sufficiently big s.t. it bounds $|N^T(0)|$ from above). Now, define $Y := \exp(\epsilon\mathbb{E}[L(T)])S_0^{\pi^\star}(T)\mathscr{E}(\epsilon N^T)(T \wedge \widetilde{\tau}_m)$ and note that we have $N^{T,m}$, $[N^{T,m}]$, $\mathscr{E}(\epsilon N^T)(T \wedge \widetilde{\tau}_m)$ are uniformly bounded (omitting k in $m(k)$ for notational simplicity) [3]. In turn, for any $\widetilde{X} \in \widetilde{\mathfrak{X}}^+(1)$ we have:

$$\mathbb{E}[\ln(\widetilde{X}(T) - \epsilon\Lambda(T))] \leq \mathbb{E}[\widetilde{U}(Y)] + \mathbb{E}[Y(\widetilde{X}(T) - \epsilon\Lambda(T))]$$

[3]Note that the notation $N^{T,m}$ refers to the stopped process $N^{T,\widetilde{\tau}_m}$. Likewise for relevant cases throughout this text.

$$
\begin{aligned}
&= -1 - \mathbb{E}[\ln(Y)] + \mathbb{E}[Y(\widetilde{X}(T) - \epsilon\Lambda(T))] \\
&\leq (e^{\epsilon\mathbb{E}[L(T)]} - 1) + \mathbb{E}[\ln(X_{\pi^\star}(T))] - \mathbb{E}[\ln(e^{\epsilon\mathbb{E}[L(T)]}\mathscr{E}(\epsilon N^T)(T \wedge \widetilde{\tau}_m))] \\
&- \epsilon\mathbb{E}[S_0^{\pi^\star}(T)e^{\epsilon\mathbb{E}[L(T)]}\mathscr{E}(\epsilon N^T)(T \wedge \widetilde{\tau}_m)\Lambda(T)] \\
&= (e^{\epsilon\mathbb{E}[L(T)]} - 1 - \epsilon\mathbb{E}[L(T)] - \frac{1}{2}(\epsilon\mathbb{E}[L(T)])^2) + \mathbb{E}[\ln(X_{\pi^\star}(T))] + \frac{\epsilon^2}{2}\mathbb{E}[(N^{T,m}(T))^2] \\
&- \epsilon\mathbb{E}[\underbrace{e^{\epsilon\mathbb{E}[L(T)]}\mathscr{E}(\epsilon N^T)(T \wedge \widetilde{\tau}_m)M^T(T)}_{\substack{\text{apply} \\ \text{Itô}}}] \\
&= (e^{\epsilon\mathbb{E}[L(T)]} - 1 - \epsilon\mathbb{E}[L(T)] - \frac{1}{2}(\epsilon\mathbb{E}[L(T)])^2) + \mathbb{E}[\ln(X_{\pi^\star}(T))] + \frac{\epsilon^2}{2}\mathbb{E}[(N^{T,m}(T))^2] \\
&- \epsilon\mathbb{E}\bigg[\Big(e^{\epsilon\mathbb{E}[L(T)]}\mathbb{E}[L(T)] + \epsilon\int_0^{T\wedge\widetilde{\tau}_m} e^{\epsilon\mathbb{E}[L(T)]}\mathscr{E}(\epsilon N^T)(t)d[N^T](t)\Big)\bigg] \\
&\leq o(\epsilon^2) + \mathbb{E}[\ln(X_{\pi^\star}(T))] + \frac{\epsilon^2}{2}\mathbb{E}[(N^{T,m}(T))^2] - \epsilon(1 + \epsilon\mathbb{E}[L(T)] + o(\epsilon))\mathbb{E}[L(T)] \\
&- \epsilon^2\mathbb{E}\bigg[\int_0^{T\wedge\widetilde{\tau}_m}(1 + \epsilon N^T(t) - \frac{\epsilon^2}{2}[N^T](t))d[N^T](t)\bigg], \qquad e^h \geq 1 + h,\ h \in \mathbb{R}, \\
&= \mathbb{E}[\ln(X_{\pi^\star}(T))] - \epsilon\mathbb{E}[L(T)] - \frac{\epsilon^2}{2}\mathbb{E}[((N^{T,m}(T))^2] + o(\epsilon^2) \\
&= \mathbb{E}[\ln(X_{\pi^\star}(T))] - \epsilon\mathbb{E}[L(T)] - \frac{\epsilon^2}{2}\mathbb{E}[(D_1^T(T) - L(T))^2] \\
&+ \frac{\epsilon^2}{2}\mathbb{E}[[N^T](T) - [N^{T,m}](T)] + o(\epsilon^2).
\end{aligned}
$$

Recalling that $u^T(1,\epsilon) = \mathbb{E}[\ln(X_{\pi^\star}(T))] + u^T_{\pi^\star}(1,\epsilon)$ we get:

$$
u^T_{\pi^\star}(1,\epsilon) \leq -\epsilon\mathbb{E}[L(T)] - \frac{\epsilon^2}{2}\mathbb{E}[(D_1^T(T) - L(T))^2] + \frac{\epsilon^2}{2}\mathbb{E}[[N^T](T) - [N^{T,m}](T)] + o(\epsilon^2). \tag{2.3}
$$

Let's now move forward to the lower bound. Consider the localizing sequence $(\tau_n)_{n\in\mathbb{N}_{>0}}$ s.t. $\tau_n = \inf\{t : |D_1^T(t)| \geq n\}$ and suppose $\epsilon \in (-1/2n, 0)$ or $\epsilon \in (0, 1/2n)$ depending on the sign choice in (A3); then $X_{\pi^\star}(1 + \epsilon D_1^{T,n})$ is a positive wealth process. Below we consider the case where $\epsilon \in (0, 1/2n)$, the other one follows similarly. We have:

$$
\begin{aligned}
u^T(1,\epsilon) &\geq \mathbb{E}[\ln(X_{\pi^\star}(T)(1 + \epsilon D_1^{T,n}(T)) - \epsilon\Lambda(T))] \\
&= \mathbb{E}[\ln(X_{\pi^\star}(T))] + \mathbb{E}[\ln(1 + \epsilon(D_1^{T,n}(T) - L(T)))],
\end{aligned}
$$

where $1 + \epsilon(D_1^{T,n}(T) - L(T)) \geq 1 + \epsilon D_1^{T,n}(T) \geq 1 - \epsilon n > 0$. Furthermore, denoting $U(x) := \ln(x)$, Taylor's theorem yields: $U(x+h) = U(x) + U^{'}(x)h + (1/2)U^{''}(x+ih)h^2$ for $i \in [0,1]$. In turn since $1 + i\epsilon(D_1^{T,n}(T) - L(T)) = (1-i) + i(1 + \epsilon D_1^{T,n}(T) - \epsilon L(T)) \geq (1 - \epsilon n)$ and $U^{''}$ is increasing, we have $U^{''}(1 + i\epsilon(D_1^{T,n}(T) - L(T))) \geq U^{''}(1 - \epsilon n)$. Hence:

$$
u^T(1,\epsilon) \geq \mathbb{E}[\ln(X_{\pi^\star}(T))] - \epsilon\mathbb{E}[L(T)] - \frac{\epsilon^2}{2}\mathbb{E}[(D_1^{T,n}(T) - L(T))^2]\frac{1}{(1-\epsilon n)^2}
$$

$$
\begin{aligned}
&= \mathbb{E}[\ln(X_{\pi^\star}(T))] - \epsilon\mathbb{E}[L(T)] - \frac{\epsilon^2}{2}\mathbb{E}[(D_1^{T,n}(T) - L(T))^2](1 + 2\epsilon n + o(\epsilon)) \\
&= \mathbb{E}[\ln(X_{\pi^\star}(T))] - \epsilon\mathbb{E}[L(T)] - \frac{\epsilon^2}{2}\Big((M^T(0))^2 + \mathbb{E}\Big[[M^T - D_1^{T,n}](T) + [M^{T,n}](T) - [M^{T,n}](T)\Big]\Big) + o(\epsilon^2) \\
&\geq \mathbb{E}[\ln(X_{\pi^\star}(T))] - \epsilon\mathbb{E}[L(T)] - \frac{\epsilon^2}{2}\Big((M^T(0))^2 + \mathbb{E}\Big[[M^T - D_1^{T}](T) + [M^{T}](T) - [M^{T,n}](T)\Big]\Big) + o(\epsilon^2) \\
&= \mathbb{E}[\ln(X_{\pi^\star}(T))] - \epsilon\mathbb{E}[L(T)] - \frac{\epsilon^2}{2}\mathbb{E}[(D_1^{T}(T) - L(T))^2] - \frac{\epsilon^2}{2}\mathbb{E}[[M^T](T) - [M^{T,n}](T)] + o(\epsilon^2).
\end{aligned}
$$

Recalling (2.3) we get:

$$
\begin{aligned}
-\frac{1}{2}\mathbb{E}[[M^T](T) - [M^{T,n}](T)] &\leq \lim_{\epsilon\to 0} \frac{u^T_{\pi^\star}(1,\epsilon) + \epsilon\mathbb{E}[L(T)] + (\epsilon^2/2)\mathbb{E}[(D_1^T(T) - L(T))^2]}{\epsilon^2} \\
&\leq \frac{1}{2}\mathbb{E}[[N^T](T) - [N^{T,m}](T)].
\end{aligned} \tag{2.4}
$$

Letting $n \to \infty,\ m \to \infty$, using the dominated convergence theorem concludes the first part of the proof.

For the second part initially note that $T \mapsto L(T)$ is (a.s.) continuous on $\mathbb{R}_{\geq 0}$ and the limit at infinity exists (with $L(\infty)$ being its value). Hence $T \mapsto L(T)$ is (a.s.) bounded by a square integrable random variable. To see this note that due to $\lim_{T\to\infty} L(T) = L(\infty)$ there exists a $N > 0$ s.t. for $T > N$, $L(T) \in (L(\infty) - 1, L(\infty) + 1)$. Finally, as $L(T)$ is continuous on $[0, N]$ there exist $T^{'}, T^{''} \in [0, N]$ s.t. $L(T^{'}) \leq L(T) \leq L(T^{''})$, $\forall T \in [0, N]$. Thus, $|L(T)| \leq C^L$ on $[0,\infty)$, where $C^L := \max\{|L(\infty) - 1|, |L(\infty)+1|, |L(T^{'})|, |L(T^{''})|\}$. Now, $\forall T$, D_1^T is the closed martingale derived from $D_1^T(T)$, which in turn is the (orthogonal) projection of $L(T)$. Hence by the aforementioned and Doob's inequality we have: $\mathbb{P}(\sup_{t\in[0,T]} |D_1^T(t) - D_1^\infty(t)| \geq K) \leq K^{-1}\|D_1^T(T) - D_1^\infty(\infty)\|_{\mathscr{L}^1(\mathbb{P})} \leq K^{-1}\|L(T) - L(\infty)\|_{\mathscr{L}^2(\mathbb{P})}$, $K > 0$; which should converge to zero as $T \to \infty$ by the dominated convergence theorem. Similar results hold for the convergence of N^T as $T \to \infty$.

Beginning with the lower bound of the value function, we note that $\mathbb{E}[[M^T](T) - [M^{T,n}](T)] = \mathbb{E}[(M^T(T))^2 - (M^{T,n}(T))^2] \leq \mathbb{E}[(M^T(T))^2] \leq \mathbb{E}[(C^L)^2] < \infty$. Moreover, for $\epsilon > 0$ (similar arguments work for $\epsilon < 0$) we have:

$$
\begin{aligned}
-\frac{\epsilon^2}{2}\mathbb{E}[(D_1^{T,n}(T) - L(T))^2](2\epsilon n + o(\epsilon)) &\geq \inf_{T\in\mathbb{R}_{\geq 0}} \Big(-\frac{\epsilon^2}{2}\mathbb{E}[(D_1^{T,n}(T) - L(T))^2]\frac{1 - (1-\epsilon n)^2}{(1-\epsilon n)^2}\Big) \\
&= -\frac{\epsilon^2(1 - (1-\epsilon n)^2)}{2(1-\epsilon n)^2} \sup_{T\in\mathbb{R}_{\geq 0}} \mathbb{E}[(D_1^{T,n}(T) - L(T))^2],
\end{aligned} \tag{2.5}
$$

where the infimum/supremum above are finite by the uniform boundedness of $D_1^{T,n}$ and the fact that $|L(T)| \leq C^L$; for square integrable r.v. C^L. In turn, we get:

$$
\begin{aligned}
u^T_{\pi^\star}(1,\epsilon) + \epsilon\mathbb{E}[L(T)] + (\epsilon^2/2)\mathbb{E}[(D_1^T(T) - L(T))^2] \geq& -\frac{\epsilon^2}{2}\mathbb{E}[(C^L)^2] \\
&- \frac{\epsilon^2(1 - (1-\epsilon n)^2)}{2(1-\epsilon n)^2} \sup_{T\in\mathbb{R}_{\geq 0}} \mathbb{E}[(D_1^{T,n}(T) - L(T))^2].
\end{aligned}
$$

For the upper bound of the value function we use similar arguments as above, along with the fact that $N^{T,m}$, $[N^{T,m}]$ are uniformly bounded, $\mathbb{E}[[N^T](T) - [N^{T,m}](T)] \leq \mathbb{E}[(N^T(T))^2] \leq C\mathbb{E}[(L(T))^2] \leq C\mathbb{E}[(C^L)^2] < \infty,\ C > 0$ (we also choose m in $\widetilde{\tau}_m$ s.t. it bounds $|N^T(0)|$ uniformly in T). As well as that the remainder of the nth order Taylor polynomial of e^h is bounded by $\max(1, e^h)|h|^{n+1}/(n+1)!$, for $h \in \mathbb{R}$. Hence $|u^T_{\pi^\star}(1,\epsilon) + \epsilon\mathbb{E}[L(T)] + (\epsilon^2/2)\mathbb{E}[(D^T_1(T) - L(T))^2]|$ has an upper bound that does not depend on the horizon and is $\mathcal{O}(\epsilon^2)$. ☐

Remark 2.1. Note that in our model, under the new numeraire $X_{\pi^\star}$, the assets become $S^{\pi^\star}_0 := \mathcal{E}(R^{\pi^\star}_0) = 1/X_{\pi^\star},\ R^{\pi^\star}_0 := -\int_0^\cdot \mu(t)dW^1(t)/\sigma(t)$ and $S^{\pi^\star} := S(0)\mathcal{E}(R^{\pi^\star}) = S/X_{\pi^\star},\ R^{\pi^\star} := \int_0^\cdot(\sigma(t) - \mu(t)/\sigma(t))dW^1(t)$. In turn, for each $\widetilde{X} \in \widetilde{\mathfrak{X}}^+(1)$ we have:

$$\widetilde{X}/X_{\pi^\star} = 1 + \underbrace{\int_0^\cdot \frac{\overbrace{\left(\frac{\widetilde{X}(t)}{X_{\pi^\star}(t)} - \theta(t)S^{\pi^\star}(t)\right)}^{\theta_0(t)}}{S^{\pi^\star}_0(t)}\, dS^{\pi^\star}_0(t) + \int_0^\cdot \theta(t)dS^{\pi^\star}(t)}_{X\in\mathfrak{X}^+_{\pi^\star}}.$$

In turn, using the above and the Kunita-Watanabe characterization of D^T_1 we can directly get the nearly optimal strategy associated with $1 + \epsilon D^T_1$, in terms of its wealth process.

Example 2.1 (Short-term setting)**.** Let's consider simplified example where the exogenous endowment is generated through a continuum of European call options driven by a correlated to the market factor process. More precisely, we impose the following dynamics for the factor process, the tradeable asset and the endowment:

$$\begin{aligned}
dZ(t) &= Z(t)(adt + bdB(t)), \qquad a, b \in \mathbb{R};\ B := \rho W^1 + \sqrt{1-\rho^2}W^2,\ |\rho| \leq 1,\\
dS(t) &= S(t)(\mu dt + \sigma dW^1(t)), \qquad \mu \in \mathbb{R},\ \sigma > 0,\\
\Lambda(t) &= \int_0^t e^{-rs}\lambda(Z(s))ds, \qquad \lambda(z) = (z-K)^+;\ K > 0,\ r = 0.
\end{aligned}$$

We may consider this type of endowment as an already bought hedging position against the low values of the factor process. This is suitable for example in the cases where the correlation of the tradeable asset and the factor is negative and the investor has already buys small number of call options in order to hedge the upward movement of the factor. Under another perspective, such form of Λ could be linked to the bonus payments that is linked to a factor: when a factor (say the profitability of a firm) is higher than a specified level, the manager, who is also an investor, gets a bonus.

Note that it is readily verified that assumptions (A2), (A3) (for $\epsilon \leq 0$) and (A4) of Proposition 2.1 hold. Now, note that $\mathbb{E}[L(T)|\mathcal{F}(t)] = \int_0^t \Lambda(s)dS^{\pi^\star}_0(s) + \mathbb{E}[\int_0^T S^{\pi^\star}_0(s)\lambda(Z(s))ds|\mathcal{F}(t)] = \int_0^t \Lambda(s)dS^{\pi^\star}_0(s) + \int_0^t S^{\pi^\star}_0(s)\lambda(Z(s))ds + S^{\pi^\star}_0(t)\psi(Z(t))$; for $\psi(z) := \mathbb{E}^{\mathbb{Q}^{\pi^\star}}_z[\int_0^T \lambda(Z(s))ds]$ and $\mathbb{Q}^{\pi^\star}$, where $d\mathbb{Q}^{\pi^\star}|_{\mathcal{F}(T)} = S^{\pi^\star}_0(T)d\mathbb{P}|_{\mathcal{F}(T)}$.

Hence, D^T_1 is given as $\int_0^\cdot \Lambda(t)dS^{\pi^\star}_0(t)$ plus the Kunita-Watanabe projection of $S^{\pi^\star}_0(t)\psi(Z(t))$ w.r.t. B on $[0,T]$. In fact, as we know that the dynamics of Z under $\mathbb{Q}^{\pi^\star}$ are $dZ(t) = Z(t)(\widetilde{a}dt + bdB^{\mathbb{Q}^{\pi^\star}}(t)),\ \widetilde{a} :=$

$a - \mu\rho b/\sigma$ we have that:

$$\begin{aligned}\mathbb{E}_z^{\mathbb{Q}^{\pi^\star}}[(Z(s)-K)^+] &= e^{\widetilde{a}s} z\Phi\big(m_+(s,z)\big) - K\Phi\big(m_-(s,z)\big) =: \chi(s,z),\\ \Phi &:= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\cdot} e^{-x^2/2}dx,\\ m_\pm(s,z) &:= \frac{1}{b\sqrt{s}}\big[\ln(z/K) + (\widetilde{a} \pm b^2/2)s\big],\\ \partial_z \chi(s,z) &= e^{\widetilde{a}s}\Phi\big(m_+(s,z)\big) =: \Delta(s,z).\end{aligned}$$

Thus, D_1^T is given as $\int_0^{\cdot} \Lambda(t) dS_0^{\pi^\star}(t)$ plus the Kunita-Watanabe projection of $S_0^{\pi^\star}(t)\int_0^T \chi(s,Z(t))ds$. Now, using Itô's lemma we should have that the part of $S_0^{\pi^\star}(t)\int_0^T \chi(s,Z(t))ds$ which contributes to the K-W decomposition (i.e. it is not of finite variation) is of the form $(\int_0^T \chi(s,Z(t))ds)dS_0^{\pi^\star}(t) + (S_0^{\pi^\star}(t)\int_0^T \Delta(s,z)ds)dZ(t)$. In turn, $1+\epsilon D_1^T$ should be given as:

$$1+\epsilon D_1^T(t) = 1 + \epsilon\int_0^t \Lambda(s)dS_0^{\pi^\star}(s) - \epsilon\int_0^t S_0^{\pi^\star}(u)\bigg(\int_0^T \frac{\mu\chi(s,Z(u)) - \rho b\sigma\Delta(s,Z(u))}{\sigma}ds\bigg)dW^1(u).$$

Hence, we get an analytic form of the main ingredient of the second order approximation (2.1).

Example 2.2 (Long-term setting)**.** Consider the following model of an exogenous endowment that is generated by a linear payoff:

$$\begin{aligned}dZ(t) &= Z(t)(adt + bdB(t)), \qquad a,b\in\mathbb{R};\ B := \rho W^1 + \sqrt{1-\rho^2}W^2,\ |\rho|\le 1,\\ dS(t) &= S(t)(\mu dt + \sigma(1+\kappa t)dW^1(t)), \qquad \mu\in\mathbb{R},\ \sigma,\kappa\in\mathbb{R}_{>0},\\ dS_0^{\pi^\star}(t) &= -S_0^{\pi^\star}(t)\frac{\mu}{\sigma(1+\kappa t)}dW^1(t),\\ \Lambda(t) &= \int_0^t e^{-rs}\lambda(Z(s))ds, \qquad \lambda(z) = z;\ r>0.\end{aligned}$$

The pressence of the weight function $w(t) := 1+\kappa t$ gives a type of (hyperbolic) discounting in $S_0^{\pi^\star}$ which in turn ensures $\mathbb{E}[[S_0^{\pi^\star}](\infty)] < \infty$ and in general $\mathbb{E}[(S_0^{\pi^\star}(\infty))^n] < \infty$, $n\in\mathbb{N}_{>0}$. In fact we should have $\mathbb{E}[(L(\infty))^2]^2 \le \mathbb{E}[(S_0^{\pi^\star}(\infty))^4]\mathbb{E}[(\int_0^\infty e^{-rt}Z(t)dt)^4] < \infty$, for sufficiently big r. This, however, comes at the expense of Z not being time homogeneous (Markov process) under $\mathbb{Q}^{\pi^\star}$ (for $\mathbb{Q}^{\pi^\star}$ as in Example 2.1). Again, we readily check that the rest of the assumptions of Proposition 2.1 hold.

This example could model a pension fund which collectively invests in a tradeable asset (stock market), whereas the process Λ stands for the accumulated contribution-liability surplus. Herein, we also model the (most likely) positive correlation between the return of the stock market and the contributions.

In turn, we should have:

$$\begin{aligned}\mathbb{E}[L(T)|\mathscr{F}(t)] &= \int_0^t \Lambda(s)dS_0^{\pi^\star}(s) + \mathbb{E}\bigg[\int_0^T S_0^{\pi^\star}(s)Z(s)ds\bigg|\mathscr{F}(t)\bigg]\\ &= \int_0^t \Lambda(s)dS_0^{\pi^\star}(s) + \int_0^t S_0^{\pi^\star}(s)Z(s)ds + \mathbb{E}\bigg[\int_t^T e^{-rs}S_0^{\pi^\star}(s)Z(s)ds\bigg|\mathscr{F}(t)\bigg]\end{aligned}$$

$$= \int_0^t \Lambda(s) dS_0^{\pi^\star}(s) + \int_0^t S_0^{\pi^\star}(s) Z(s) ds + S_0^{\pi^\star}(t) \int_t^T e^{-rs} \mathbb{E}^{\mathbb{Q}^{\pi^\star}} [Z(s)|\mathscr{F}(t)] ds$$

$$= \int_0^t \Lambda(s) dS_0^{\pi^\star}(s) + \int_0^t S_0^{\pi^\star}(s) Z(s) ds + e^{-rt} C^T(t) S_0^{\pi^\star}(t) Z(t),$$

$$\text{for } C^T(t) := \int_t^T e^{-(r-a)(s-t)} \Big(\frac{w(t)}{w(s)}\Big)^{\frac{\mu\rho b}{\sigma\kappa}} ds,$$

Thus $1 + \epsilon D_1^T$ should be given as:

$$1 + \epsilon D_1^T(t) = 1 + \epsilon \int_0^t \Lambda(s) dS_0^{\pi^\star}(s) + \epsilon \int_0^t e^{-rs} C^T(s) S_0^{\pi^\star}(s) Z(s) \Big(b\rho - \frac{\mu}{\sigma w(s)}\Big) dW^1(s).$$

In fact following a similar procedure as above for $\mathbb{E}[L(\infty)|\mathscr{F}(t)] = \int_0^t \Lambda(s) dS_0^{\pi^\star}(s) + \int_0^t S_0^{\pi^\star}(s) Z(s) ds + e^{-rt} C(t) S_0^{\pi^\star}(t) Z(t)$, $C(t) := \int_t^\infty e^{-(r-a)(s-t)} (w(t)/w(s))^{\frac{\mu\rho b}{\sigma\kappa}} ds$ (which is finite for sufficiently big r) we can check the convergence result in Proposition 2.1 for D_1^T. To this end note that we have $\lim_{T\to\infty} e^{-rt} C^T(t) S_0^{\pi^\star}(t) Z(t)(b\rho - \mu/\sigma w(t)) = e^{-rt} C(t) S_0^{\pi^\star}(t) Z(t)(b\rho - \mu/\sigma w(t))$ (pointwise) and furthermore $e^{-rt} C^T(t) S_0^{\pi^\star}(s) Z(t)(b\rho - \mu/\sigma w(t))$ can be bounded in T by a W^1-integrable process. Therefore, by the (stochastic) dominated convergence we have $\lim_{T\to\infty} D_1^T = D_1^\infty$ (locally uniformly in probability). Hence, we manage to arrive to an analytic expression of the second order approximation even when time horizon goes to infinity.

3. Fourth order asymptotics for the value function

The fourth order asymptotic can be also derived by considering an appropriate Kunita-Watanabe decomposition. Particularly let Q^T denote the continuous version of the uniformly integrable martingale $\mathbb{E}[(N^T(T))^2|\mathscr{F}(t)]$ (by (A4)). In turn, by the continuity of its paths, Q^T is locally square integrable and hence it admits a Kunita-Watanabe decomposition w.r.t. W^1, similarly to the second order asymptotic, i.e. $Q^T = D_2^T + P^T$; where P^T is strongly orthogonal to W^1 s.t. $P^T(0) = Q^T(0)$. In the case of the long-term setting we also denote the martingale $\mathbb{E}[(N^\infty(\infty))^2|\mathscr{F}(t)]$, which is well-defined by (A5), by Q^∞. In turn as it is locally square integrable, by its continuity, we decompose it (through Kunita-Watanabe w.r.t. W^1) as $Q^\infty = D_2^\infty + P^\infty$; where P^∞ is strongly orthogonal to W^1 s.t. $P^\infty(0) = Q^\infty(0)$.

Remark 3.1. Note that while in Proposition 2.1 we could work with an arbitrary time horizon, the situation here seems more complicated. The main thing to consider being the correct order to take the limits on the upper and lower bounds of the value function. More precisely, if we follow similar steps as in the second order asymptotic there is a discrepancy between the upper and lower bounds of the value function when using localized versions of our processes. This discrepancy affects terms of all orders (w.r.t. to ϵ), which is in contrast with the second order asymptotic where this procedure only affected the second order term. This complication is compounded by the fact that in Proposition 2.1, the positivity of $1 + \epsilon D_1^T$ is tied with ϵ. One way to somewhat disentangle this is to modify the time horizon that the fourth order asymptotic holds, as we shall see in the following result.

Before presenting the next result, we introduce the following relevant class. For $I \subseteq \mathbb{R}_{\geq 0}$ let $\mathscr{K}(I)$ denote the class of stochastic processes $(X(T))_{T\in I}$ where there exist constants $\beta, \gamma, C > 0$ s.t.:

$$\mathbb{E}[|X(T) - X(S)|^{\beta}] \leq C|T-S|^{1+\gamma}, \qquad \forall T, S \in I.$$

Note that by Kolmogorov's continuity theorem, each $X \in \mathscr{K}(I)$ admits a continuous version on I.

Theorem 3.1. Consider the short-term setting. Assuming (A1), (A2), (A3), as well as (A4) and $\mathbb{E}[([M^{T,T} - M^{S,S}](T \vee S))^{\beta/2}] \leq C|T-S|^{1+\gamma}$, $\forall T, S \in [0, U]$ and some $\beta, \gamma, C > 0$; we have that for each such time horizon there exists a sequence of stopping times $\mathcal{T}_T^m \in [0, T]$ with $\lim_{m\to\infty} \mathcal{T}_T^m = T$ and $\lim_{m\to\infty} D_1^{\mathcal{T}_T^m} = D_1^T$, $\lim_{m\to\infty} D_2^{\mathcal{T}_T^m} = D_2^T$ (uniformly in probability) s.t.:

$$u_{\pi^\star}^{\mathcal{T}_T^m}(1,\epsilon) - g(\epsilon(D_1^{\mathcal{T}_T^m}(\mathcal{T}_T^m) - L(\mathcal{T}_T^m))) + \epsilon^4 \mathbb{E}[(D_2^{\mathcal{T}_T^m}(\mathcal{T}_T^m))^2/2 - (N^{\mathcal{T}_T^m}(\mathcal{T}_T^m))^2 D_2^{\mathcal{T}_T^m}(\mathcal{T}_T^m)] = o(\epsilon^4), \tag{3.1}$$

where $g(\Psi) := \mathbb{E}[\Psi - \Psi^2/2 + \Psi^3/3 - \Psi^4/4]$.

Consider the long-term setting. Extending the first three assumptions on $\mathbb{R}_{\geq 0}$, strengthening (A4) to (A5) and assuming $\mathbb{E}[([M^{S_1,S_1} - M^{S_2,S_2}](S_1 \vee S_2))^{\beta/2}] \leq C|S_1 - S_2|^{1+\gamma}$, $\forall S_1, S_2 \in [0,T]$ and every $T \in \mathbb{R}_{\geq 0}$ [4], we additionally have $\lim_{T\to\infty} D_1^T = D_1^\infty$, $\lim_{T\to\infty} D_2^T = D_2^\infty$ (locally uniformly in probability) and:

$$\overline{\lim_{T\to\infty}} |u_{\pi^\star}^{\mathcal{T}_T^m}(1,\epsilon) - g(\epsilon(D_1^{\mathcal{T}_T^m}(\mathcal{T}_T^m) - L(\mathcal{T}_T^m))) + \epsilon^4 \mathbb{E}[(D_2^{\mathcal{T}_T^m}(\mathcal{T}_T^m))^2/2 - (N^{\mathcal{T}_T^m}(\mathcal{T}_T^m))^2 D_2^{\mathcal{T}_T^m}(\mathcal{T}_T^m)]| = \mathcal{O}_m(\epsilon^4) \text{ [5]}. \tag{3.2}$$

Proof. We begin by constructing the horizon which we will be working with. To this end we establish the continuity of the adapted process $(D_1^T(T))_{T\in[0,U]}$. Note that $\forall T, S \in [0, U]$ we have by the Burkholder-Davis-Gundy (BDG) inequality:

$$\begin{aligned}
\mathbb{E}[|D^T(T) - D_1^S(S)|^{\beta}] &= \mathbb{E}[|\mathbb{E}[D_1^T(T) - D_1^S(S)|\mathscr{F}(T \vee S)]|^{\beta}] \\
&\leq \mathbb{E}[(\sup_{t\in[0,T\vee S]} |\mathbb{E}[D_1^T(T) - D_1^S(S)|\mathscr{F}(t)]|)^{\beta}] \\
&\leq C_\beta \mathbb{E}[([D_1^{T,T} - D_1^{S,S}](T \vee S))^{\beta/2}] \\
&\leq C_\beta \mathbb{E}[([M^{T,T} - M^{S,S}](T \vee S))^{\beta/2}] \text{ [6]},
\end{aligned}$$

which ensures the continuity of the process by Kolmogorv's continuity theorem. As $(D_1^T(T))_{T\in[0,U]}$ [7] is continuous, its restriction on $\Omega \times [0,T]$ denoted by $D_1^S(S)|_{[0,T]}$, $T \in [0, U]$ also is. By the continuity of $D_1^S(S)|_{[0,T]}$ on the compact interval $[0, T]$ we have $M', M \in [0, T]$ s.t. $D_1^{M'}(M')|_{[0,T]} \leq D_1^S(S)|_{[0,T]} \leq D_1^M(M)|_{[0,T]}$, $\forall S \in [0, T]$. In turn, let $\breve{\xi}^{D_1}$ denote the continuous version of the closed martingale $\mathbb{E}[(|D_1^{M'}(M')|_{[0,T]}| + |D_1^M(M)|)|_{[0,T]}|\mathscr{F}(t)]$ on $[0, U]$ (ensured by the fact that we are working with the Brownian filtration). Following a similar process for $(N^T(T))_{T\in[0,U]}$ we get $\breve{\xi}^N$ and $\breve{\xi} := \breve{\xi}^{D_1} + \breve{\xi}^N$,

[4]Note that here C may also depend on T.
[5]Note that the notation $\mathcal{O}_m(\epsilon^4)$ implies that the upper bound w.r.t. ϵ also depends on m.
[6]Recall that $D_1^{S,S}$, $M^{S,S}$ denote the (continuous) martingales D_1^S, M^S stopped at time S.
[7]This also implies the continuity of $(N^T(T))_{T\in[0,U]}$ by the continuity of L.

where the above conditional expectations are well defined by (A4). As $\breve{\xi}$ is continuous and adapted, $\tau_{m(k)} = \inf\{t : \breve{\xi}(t) \geq m(k)\}$, for sufficiently big $m(k) \in \mathbb{N}_{>0}$ s.t. in bounds $\breve{\xi}(0)$ from above, is a localizing (sub)sequence that makes the stopped process $\breve{\xi}^m$ uniformly bounded (omitting k in $m(k)$ for notational simplicity). Now consider the following sequence of bounded stopping times: $\mathcal{T}_T^m := \tau_m \wedge T$ and note that for each m, $L(\mathcal{T}_T^m) \in \mathcal{L}^2(\mathbb{P}, \mathcal{F}(\mathcal{T}_T^m))$. Hence it admits the following orthogonal decomposition (which also gives rise to its Kunita-Watanabe projection): $L(\mathcal{T}_T^m) = D_1^{\mathcal{T}_T^m}(\mathcal{T}_T^m) + N^{\mathcal{T}_T^m}(\mathcal{T}_T^m)$. In turn the above and the fact that for each m, $\mathcal{T}_T^m \in [0,T]$ should imply the uniform boundedness of the processes $D_1^{\mathcal{T}_T^m,m}$, $N^{\mathcal{T}_T^m,m}$. To see this note $|D_1^{\mathcal{T}_T^m}(\mathcal{T}_T^m)| \leq \sup_{S\in[0,T]} |D_1^S(S)| = \sup_{S\in[0,T]} |D_1^S(S)|_{[0,T]}| \leq |D_1^{M'}(M')|_{[0,T]}| + |D_1^M(M)|_{[0,T]}|$. Hence taking conditional expectations on $\mathcal{F}(t\wedge\tau_m)$, using Jensen's inequality and optional sampling we get the desired result for $D_1^{\mathcal{T}_T^m,m}$ (likewise for $N^{\mathcal{T}_T^m,m}$). As $N^{\mathcal{T}_T^m,m}$ is uniformly bounded, we have that $(N^{\mathcal{T}_T^m}(\mathcal{T}_T^m))^2 \in \mathcal{L}^2(\mathbb{P}, \mathcal{F}(\mathcal{T}_T^m))$. In turn it admits the following orthogonal decomposition (which once more gives rise to its Kunita-Watanabe projection) $(N^{\mathcal{T}_T^m}(\mathcal{T}_T^m))^2 = D_2^{\mathcal{T}_T^m}(\mathcal{T}_T^m) + P^{\mathcal{T}_T^m}(\mathcal{T}_T^m)$. Moreover, by Doob's inequality we have for $K > 0$:

$$
\begin{aligned}
\mathbb{P}(\sup_{t\in[0,U]} |Q^{\mathcal{T}_T^m,\mathcal{T}_T^m}(t) - Q^{T,T}(t)| \geq K) &\leq K^{-1}\|(N^{\mathcal{T}_T^m}(\mathcal{T}_T^m))^2 - (N^T(T))^2\|_{\mathcal{L}^1(\mathbb{P})} \\
&\leq K^{-1}(\|N^{\mathcal{T}_T^m}(\mathcal{T}_T^m) - N^T(T)\|^2_{\mathcal{L}^2(\mathbb{P})} + 2\|N^T(T)(N^{\mathcal{T}_T^m}(\mathcal{T}_T^m) - N^T(T))\|_{\mathcal{L}^1(\mathbb{P})}) \\
&\leq 4K^{-1}(\|L(\mathcal{T}_T^m) - L(T)\|^2_{\mathcal{L}^2(\mathbb{P})} + 2\|L(T)\|_{\mathcal{L}^2(\mathbb{P})}\|L(\mathcal{T}_T^m) - L(T)\|_{\mathcal{L}^2(\mathbb{P})}),
\end{aligned}
$$

which should go to zero as $m \to \infty$ by the dominated convergence theorem. Now for $Q^{\mathcal{T}_T^m} - Q^{\mathcal{T}_T^m}(0) =: \widetilde{Q}^{\mathcal{T}_T^m} = D_2^{\mathcal{T}_T^m} + \widetilde{P}^{\mathcal{T}_T^m}$, $\widetilde{P}^{\mathcal{T}_T^m} := P^{\mathcal{T}_T^m} - P^{\mathcal{T}_T^m}(0)$ we have $\sup_{t\in[0,U]} |\widetilde{Q}^{\mathcal{T}_T^m,\mathcal{T}_T^m}(t) - \widetilde{Q}^{T,T}(t)| \to 0 \Leftrightarrow [\widetilde{Q}^{\mathcal{T}_T^m,\mathcal{T}_T^m} - \widetilde{Q}^{T,T}](U) \to 0$ in probability, as $m \to \infty$ and noting that $[\widetilde{Q}^{\mathcal{T}_T^m,\mathcal{T}_T^m} - \widetilde{Q}^{T,T}](U) = [D_2^{\mathcal{T}_T^m,\mathcal{T}_T^m} - D_2^{T,T}](U) + [\widetilde{P}^{\mathcal{T}_T^m,\mathcal{T}_T^m} - \widetilde{P}^{T,T}](U)$ we get $\lim_{m\to\infty} D_2^{\mathcal{T}_T^m} = D_2^T$ (uniformly in probability). Similarly we have $\lim_{m\to\infty} D_1^{\mathcal{T}_T^m} = D_1^T$, since $\mathbb{P}(\sup_{t\in[0,U]} |D_1^{\mathcal{T}_T^m,\mathcal{T}_T^m}(t) - D_1^{T,T}(t)| \geq K) \leq K^{-1}\|L(\mathcal{T}_T^m) - L(T)\|_{\mathcal{L}^2(\mathbb{P})}$.

For each fixed m consider $\tau'_{i(j)} = \inf\{t : |D_2^{\mathcal{T}_T^m}(t)| + |P^{\mathcal{T}_T^m}(t)| \geq i(j)\}$, for sufficiently big $i(j) \in \mathbb{N}_{>0}$ s.t. it bounds $|P^{\mathcal{T}_T^m}(0)|$ (omitting once more j in $i(j)$ for notational simplicity). Let's now begin with the upper bound. Unline the second order asymptotic we shall constrain ϵ for both bounds of the value function. Focusing on the case where $\epsilon > 0$, constraining $\epsilon \in (0, 1/2c)$ for $c := m + i$ (similar arguments work for $\epsilon < 0$) we have that $1 + \epsilon N^{\mathcal{T}_T^m}(t \wedge \tau_m) + \epsilon^2 P^{\mathcal{T}_T^m,i}(t \wedge \tau_m) \geq 1 - c\epsilon > 0$. Define $\widetilde{Y} := S_0^{\pi^\star}(\mathcal{T}_T^m)(1 + \epsilon N^{\mathcal{T}_T^m} + \epsilon^2 P^{\mathcal{T}_T^m,i})$. In turn, following similar steps as in Proposition 2.1, we have for any $\widetilde{X} \in \widetilde{\mathfrak{X}}^+(1)$ [8]:

$$
\begin{aligned}
\mathbb{E}[\ln(\widetilde{X}(\mathcal{T}_T^m) - \epsilon\Lambda(\mathcal{T}_T^m))] \leq &-\epsilon^3\mathbb{E}[N^{\mathcal{T}_T^m}(\mathcal{T}_T^m)P^{\mathcal{T}_T^m,i}(\mathcal{T}_T^m)] + \mathbb{E}[\ln(X_{\pi^\star}(\mathcal{T}_T^m))] \\
&- \mathbb{E}[\ln(1 + \epsilon N^{\mathcal{T}_T^m}(\mathcal{T}_T^m) + \epsilon^2 P^{\mathcal{T}_T^m,i}(\mathcal{T}_T^m))].
\end{aligned}
$$

Following similar steps as in the lower bound for the second order asymptotic, we have for $U(x) := \ln(x)$: $U(x+h) = U(x) + U'(x)h + (1/2)U''(x)h^2 + (1/6)U'''(x)h^3 + (1/24)U''''(x+jh)h^4$ for some $j \in [0,1]$. In turn since $1 + j(\epsilon N^{\mathcal{T}_T^m}(\mathcal{T}_T^m) + \epsilon^2 P^{\mathcal{T}_T^m,i}(\mathcal{T}_T^m)) = (1-j) + j(1 + \epsilon N^{\mathcal{T}_T^m}(\mathcal{T}_T^m) + \epsilon^2 P^{\mathcal{T}_T^m,i}(\mathcal{T}_T^m)) \geq (1-j) + j(1-$

[8]Note that $\mathbb{E}[\ln(\widetilde{X}(\mathcal{T}_T^m) - \epsilon\Lambda(\mathcal{T}_T^m))]$ is well-defined by the continuity of $\widetilde{X}$, Λ and the fact that $\ln^+(\widetilde{X}(T) - \epsilon\Lambda(T)) \in \mathcal{L}^1(\mathbb{P})$, $\forall T \in [0,U]$. Similarly for $\mathbb{E}[\ln(X_{\pi^\star}(\mathcal{T}_T^m))]$, $u_{\pi^\star}^{\mathcal{T}_T^m}(1,\epsilon)$ and the continuous supermartingale $S_0^{\pi^\star}$.

$c\epsilon) \geq 1-c\epsilon > 0$ and $U^{''''}$ is increasing, we have $-U^{''''}(1+j(\epsilon N^{\mathfrak{T}_T^m}(\mathfrak{T}_T^m)+\epsilon^2 P^{\mathfrak{T}_T^m,i}(\mathfrak{T}_T^m))) \leq -U^{''''}(1-c\epsilon)$. Hence:

$$
\begin{aligned}
u_{\pi^\star}^{\mathfrak{T}_T^m}(1,\epsilon) &\leq -\epsilon\mathbb{E}[L(\mathfrak{T}_T^m)] - \frac{\epsilon^2}{2}\mathbb{E}[(N^{\mathfrak{T}_T^m}(\mathfrak{T}_T^m))^2] - \frac{\epsilon^3}{3}\mathbb{E}[(N^{\mathfrak{T}_T^m}(\mathfrak{T}_T^m))^3] + \frac{\epsilon^4}{2}\mathbb{E}[(P^{\mathfrak{T}_T^m,i}(\mathfrak{T}_T^m))^2] \\
&\quad - \epsilon^4\mathbb{E}[(N^{\mathfrak{T}_T^m}(\mathfrak{T}_T^m))^2 P^{\mathfrak{T}_T^m,i}(\mathfrak{T}_T^m)] + \frac{1}{4}\mathbb{E}[(\epsilon N^{\mathfrak{T}_T^m}(\mathfrak{T}_T^m) + \epsilon^2 P^{\mathfrak{T}_T^m,i}(\mathfrak{T}_T^m))^4]\frac{1}{(1-c\epsilon)^4} + o(\epsilon^4) \\
&= -\epsilon\mathbb{E}[L(\mathfrak{T}_T^m)] - \frac{\epsilon^2}{2}\mathbb{E}[(N^{\mathfrak{T}_T^m}(\mathfrak{T}_T^m))^2] - \frac{\epsilon^3}{3}\mathbb{E}[(N^{\mathfrak{T}_T^m}(\mathfrak{T}_T^m))^3] + \frac{\epsilon^4}{2}\mathbb{E}[(P^{\mathfrak{T}_T^m,i}(\mathfrak{T}_T^m))^2] \\
&\quad - \epsilon^4\mathbb{E}[(N^{\mathfrak{T}_T^m}(\mathfrak{T}_T^m))^2 P^{\mathfrak{T}_T^m,i}(\mathfrak{T}_T^m)] + \frac{1}{4}\mathbb{E}[(\epsilon N^{\mathfrak{T}_T^m}(\mathfrak{T}_T^m) + \epsilon^2 P^{\mathfrak{T}_T^m,i}(\mathfrak{T}_T^m))^4](1+4c\epsilon+o(\epsilon)) + o(\epsilon^4).
\end{aligned}
$$

Let's now move forward to the lower bound. To this end recall τ_m, $\tau_i^{'}$, introduced at the beginning of this proof. Focusing once more on the case where $\epsilon > 0$ and constraining as above $\epsilon \in (0, 1/2c)$ (similar arguments work for $\epsilon < 0$), we have that $X_{\pi^\star}(t)(1+\epsilon D_1^{\mathfrak{T}_T^m}(t\wedge\tau_m)+\epsilon^2 D_2^{\mathfrak{T}_T^m,i}(t\wedge\tau_m))$ is a positive wealth process since $1+\epsilon D_1^{\mathfrak{T}_T^m}(t\wedge\tau_m)+\epsilon^2 D_2^{\mathfrak{T}_T^m,i}(t\wedge\tau_m) \geq 1-c\epsilon > 0$. Following similar as for the upper bound above, we have $U(x+h) = U(x) + U^{'}(x)h + (1/2)U^{''}(x)h^2 + (1/6)U^{'''}(x)h^3 + (1/24)U^{''''}(x+jh)h^4$ for some $j \in [0,1]$. In turn since $1+j(\epsilon D_1^{\mathfrak{T}_T^m}(\mathfrak{T}_T^m)+\epsilon^2 D_2^{\mathfrak{T}_T^m,i}(\mathfrak{T}_T^m)-\epsilon L(\mathfrak{T}_T^m)) = (1-j)+j(1+\epsilon D_1^{\mathfrak{T}_T^m}(\mathfrak{T}_T^m)+\epsilon^2 D_2^{\mathfrak{T}_T^m,i}(\mathfrak{T}_T^m)-\epsilon L(\mathfrak{T}_T^m)) \geq (1-j)+j(1+\epsilon D_1^{\mathfrak{T}_T^m}(\mathfrak{T}_T^m)+\epsilon^2 D_2^{\mathfrak{T}_T^m,i}(\mathfrak{T}_T^m)) \geq (1-j)+j(1-c\epsilon) \geq 1-c\epsilon > 0$ and $U^{''''}$ is increasing, we have $U^{''''}(1+j(\epsilon D_1^{\mathfrak{T}_T^m}(\mathfrak{T}_T^m)+\epsilon^2 D_2^{\mathfrak{T}_T^m,i}(\mathfrak{T}_T^m)-\epsilon L(\mathfrak{T}_T^m))) \geq U^{''''}(1-c\epsilon)$. Hence:

$$
\begin{aligned}
u_{\pi^\star}^{\mathfrak{T}_T^m}(1,\epsilon) &\geq \mathbb{E}[\epsilon D_1^{\mathfrak{T}_T^m}(\mathfrak{T}_T^m) + \epsilon^2 D_2^{\mathfrak{T}_T^m,i}(\mathfrak{T}_T^m) - \epsilon L(\mathfrak{T}_T^m)] - \frac{1}{2}\mathbb{E}[(\epsilon D_1^{\mathfrak{T}_T^m}(\mathfrak{T}_T^m) + \epsilon^2 D_2^{\mathfrak{T}_T^m,i}(\mathfrak{T}_T^m) - \epsilon L(\mathfrak{T}_T^m))^2] \\
&\quad + \frac{1}{3}\mathbb{E}[(\epsilon D_1^{\mathfrak{T}_T^m}(\mathfrak{T}_T^m) + \epsilon^2 D_2^{\mathfrak{T}_T^m,i}(\mathfrak{T}_T^m) - \epsilon L(\mathfrak{T}_T^m))^3] \\
&\quad - \frac{1}{4}\mathbb{E}[(\epsilon D_1^{\mathfrak{T}_T^m}(\mathfrak{T}_T^m) + \epsilon^2 D_2^{\mathfrak{T}_T^m,i}(\mathfrak{T}_T^m) - \epsilon L(\mathfrak{T}_T^m))^4]\frac{1}{(1-c\epsilon)^4} \\
&= g(\epsilon(D_1^{\mathfrak{T}_T^m}(\mathfrak{T}_T^m) - L(\mathfrak{T}_T^m))) - \epsilon^4\mathbb{E}[(D_2^{\mathfrak{T}_T^m,i}(\mathfrak{T}_T^m))^2/2 - (D_1^{\mathfrak{T}_T^m}(\mathfrak{T}_T^m) - L(\mathfrak{T}_T^m))^2 D_2^{\mathfrak{T}_T^m,i}(\mathfrak{T}_T^m)] + o(\epsilon^4).
\end{aligned}
$$

Combining the above with the previously established upper bound for the fourth order asymptotic, letting first $\epsilon \to 0$ and then $i \to \infty$ using Doob's maximal inequality and applying dominated convergence, we get the desired result.

The steps for the second part are very closely related to the ones of the short-term setting. To this end note that since $D_1^S(S)|_{[0,T]} \in \mathcal{K}([0,T])$, $T \in \mathbb{R}_{\geq 0}$, by the BDG inequality, we once more have $M^{'}, M \in [0,T]$ (depending on T) s.t. $|D_1^S(S)|_{[0,T]}| \leq |D_1^{M^{'}}(M^{'})|_{[0,T]}| \vee |D_1^M(M)|_{[0,T]}|$, $\forall S \in [0,T]$. In turn, letting τ_m be any stopping time, we get the decomposition $L(\tau_m \wedge T) = D_1^{\tau_m\wedge T}(\tau_m \wedge T) + N^{\tau_m\wedge T}(\tau_m \wedge T)$ by (A4) and $|D_1^{\tau_m\wedge T}(\tau_m \wedge T)| \leq \sup_{S\in[0,T]}|D_1^S(S)| = \sup_{S\in[0,T]}|D_1^S(S)|_{[0,T]}| \leq |D_1^{M^{'}}(M^{'})|_{[0,T]}| + |D_1^M(M)|_{[0,T]}|$. Hence we have that $|D_1^{\tau_m\wedge T,m}| \leq \breve{\xi}^{D_1,m}$, where $\breve{\xi}^{D_1}$ is the continuous version of the martingale $\mathbb{E}[(|D_1^{M^{'}}(M^{'})|_{[0,T]}| + |D_1^M(M)|_{[0,T]}|)|\mathcal{F}(t)]$ on $\mathbb{R}_{\geq 0}$. Similar steps lead to $\breve{\xi}^N$ and $\breve{\xi} := \breve{\xi}^{D_1} + \breve{\xi}^N$. Using $\tau_{m(k)} = \inf\{t : \breve{\xi}(t) \geq m(k)\}$, $k \in \mathbb{N}_{\geq 0}$ we have that $\breve{\xi}^m$ is bounded by $|\breve{\xi}(0)| \vee m$. Since $\breve{\xi}(0)$ depends on T we use (A5) and the fact that the orthogonal projection is Lipschitz to choose k (and hence m) sufficiently big s.t. it bounds $\breve{\xi}(0)$ uniformly in T. Thus we once more have the uniform boundedness of $D_1^{\mathfrak{T}_T^m,m}$, $N^{\mathfrak{T}_T^m,m}$. Similarly for $\tau_{i(j)}^{'} = \inf\{t : |D_2^{\mathfrak{T}_T^m}(t)| + |P^{\mathfrak{T}_T^m}(t)| \geq i(j)\}$

and sufficiently big $i(j) \in \mathbb{N}_{>0}$ s.t. it bounds $|P^{\mathcal{T}_T^m}(0)|$ uniformly in T (using once more (A5) and the fact that the orthogonal projection is Lipschitz). Recalling how we derived the fourth order bounds of the value function at $\mathcal{T}_T^m$ above, working as for (2.5) and using the boundedness of $D_1^{\mathcal{T}_T^m,m}$, $N^{\mathcal{T}_T^m,m}$, $D_2^{\mathcal{T}_T^m,i}(t \wedge \tau_m)$, $P^{\mathcal{T}_T^m,i}(t \wedge \tau_m)$ as well as the fact that the orthogonal projection of $(N^{\mathcal{T}_T^m}(\mathcal{T}_T^m))^2$ is Lipschitz we get (3.2). For the case of $\lim_{T\to\infty} D_1^T = D_1^\infty$ locally uniformly in probability, note that for $K > 0$ we have:

$$\mathbb{P}(\sup_{t\in[0,T]} |Q^T(t) - Q^\infty(t)| \geq K) \leq 4K^{-1}(\|L(T) - L(\infty)\|^2_{\mathscr{L}^2(\mathbb{P})} + 2\|L(\infty)\|_{\mathscr{L}^2(\mathbb{P})}\|L(T) - L(\infty)\|_{\mathscr{L}^2(\mathbb{P})}).$$

Using (A5) and arguing as in the case of the short-term setting, we get the desired result. □

Before moving forward to some examples note that the discussion of Remark 2.1 can be naturally extended for the case of $1 + \epsilon D_1^T + \epsilon^2 D_2^T$. Now, as $D_1^{\mathcal{T}_T^m}$, $D_2^{\mathcal{T}_T^m}$ converge to D_1^T, D_2^T, we focus on the latter for the examples below.

Example 3.1 (Short-term setting)**.** We continue with Example 2.2 but for $r, \kappa = 0$ (as we are working in a finite horizon setting), which satisfies the assumptions of Theorem 3.1 on $[0, U]$. More precisely, following similar calculations as in Example 2.2 we have:

$$M^T(t) = \int_0^t \Lambda(s) dS_0^{\pi^\star}(s) + \int_0^t S_0^{\pi^\star}(s) Z(s) ds + C^T(t) S_0^{\pi^\star}(t) Z(t),$$
$$\text{for } C^T(t) = \int_t^T e^{\widetilde{a}(s-t)} ds, \text{ where } \widetilde{a} := a - \mu\rho b/\sigma,$$

and

$$1 + \epsilon D_1^T(t) = 1 + \epsilon \int_0^t \Lambda(s) dS_0^{\pi^\star}(s) + \epsilon \int_0^t C^T(s) S_0^{\pi^\star}(s) Z(s)(b\rho - \mu/\sigma) dW^1(s).$$

Now note that since $(D_1^T(T) - L(T))^2 = (D_1^T(T) - M^T(T))^2 = (N^T(T))^2$ and $(N^T(T))^2 = (N^T(0))^2 + 2\int_0^T N^T(t) dN^T(t) + [N^T](T)$, the W^1-projection (with zero initial value) in the Kunita-Watanabe decomposition of the martingale with terminal value $(N^T(T))^2$ will coincide with the projection of the martingale with terminal value $[N^T](T)$. Hence:

$$\begin{aligned}\mathbb{E}[[N^T](T)|\mathcal{F}(t)] &= \mathbb{E}\Big[\Big[\int_0^\cdot C^T(s) S_0^{\pi^\star}(s) Z(s) b\sqrt{1-\rho^2} dW^2(s)\Big](T)\Big|\mathcal{F}(t)\Big] \\ &= \int_0^t (C^T(s) S_0^{\pi^\star}(s) Z(s) b\sqrt{1-\rho^2})^2 ds \\ &\quad + \mathbb{E}\Big[\int_t^T e^{(\mu^2/\sigma^2)s}(C^T(s))^2 \mathcal{E}(2R_0^{\pi^\star})(s)(Z(s))^2 b^2(1-\rho^2) ds\Big|\mathcal{F}(t)\Big].\end{aligned}$$

Letting $\mathbb{Q}^{2\pi^\star}$ denote the measure induced by $\mathcal{E}(2R_0^{\pi^\star})$, the above becomes:

$$\begin{aligned}\mathbb{E}[[N^T](T)|\mathcal{F}(t)] &= \int_0^t (C^T(s) S_0^{\pi^\star}(s) Z(s) b\sqrt{1-\rho^2})^2 ds \\ &\quad + \mathcal{E}(2R_0^{\pi^\star})(t) \int_t^T e^{(\mu^2/\sigma^2)s}(C^T(s))^2 \mathbb{E}^{\mathbb{Q}^{2\pi^\star}}[(Z(s))^2|\mathcal{F}(t)] b^2(1-\rho^2) ds\end{aligned}$$

$$= \int_0^t (C^T(s)S_0^{\pi^\star}(s)Z(s)b\sqrt{1-\rho^2})^2 ds + \widetilde{C}^T(t)\mathscr{E}(2R_0^{\pi^\star})(t)(Z(t))^2,$$

$$\text{for } \widetilde{C}^T(t) := e^{(\mu^2/\sigma^2)t}\int_t^T e^{((\mu^2/\sigma)^2+2\widetilde{a}+b^2-2\mu\rho b/\sigma)(s-t)}(C^T(s))^2 b^2(1-\rho^2)ds.$$

Thus, we have that:

$$\begin{aligned} 1+\epsilon D_1^T+\epsilon^2 D_2^T =& 1+\epsilon\Big(\int_0^t \Lambda(s)dS_0^{\pi^\star}(s) + \int_0^t C^T(s)S_0^{\pi^\star}(s)Z(s)(b\rho-\mu/\sigma)dW^1(s)\Big) \\ &+\epsilon^2\int_0^t 2\widetilde{C}^T(s)\mathscr{E}(2R_0^{\pi^\star})(s)(Z(s))^2(b\rho-\mu/\sigma)dW^1(s). \end{aligned}$$

Lastly note that for $\rho=\pm 1$, D_2^T vanishes as $\widetilde{C}^T=0$.

Example (Example 2.2 continued, D_2^T)**.** Note that since $(D_1^T(T)-L(T))^2 = (D_1^T(T)-M^T(T))^2 = (N^T(T))^2$ and $(N^T(T))^2 = (N^T(0))^2 + 2\int_0^T N^T(t)dN^T(t) + [N^T](T)$, the W^1-projection (with zero initial value) in the Kunita-Watanabe decomposition of the martingale with terminal value $(N^T(T))^2$ will coincide with the projection of the martingale with terminal value $[N^T](T)$. Hence:

$$\begin{aligned} \mathbb{E}[[N^T](T)|\mathscr{F}(t)] &= \mathbb{E}\Big[\Big[\int_0^\cdot e^{-rs}C^T(s)S_0^{\pi^\star}(s)Z(s)b\sqrt{1-\rho^2}dW^2(s)\Big](T)\Big|\mathscr{F}(t)\Big] \\ &= \int_0^t (e^{-rs}C^T(s)S_0^{\pi^\star}(s)Z(s)b\sqrt{1-\rho^2})^2 ds \\ &+ \mathbb{E}\Big[\int_t^T e^{-(2r-\mu^2/(\sigma w(s))^2)s}(C^T(s))^2\mathscr{E}(2R_0^{\pi^\star})(s)(Z(s))^2 b^2(1-\rho^2)ds\Big|\mathscr{F}(t)\Big]. \end{aligned}$$

Letting $\mathbb{Q}^{2\pi^\star}$ denote the measure induced by $\mathscr{E}(2R_0^{\pi^\star})$, the above becomes:

$$\begin{aligned} \mathbb{E}[[N^T](T)|\mathscr{F}(t)] =& \int_0^t (e^{-rs}C^T(s)S_0^{\pi^\star}(s)Z(s)b\sqrt{1-\rho^2})^2 ds \\ &+\mathscr{E}(2R_0^{\pi^\star})(t)\int_t^T e^{-(2r-\mu^2/(\sigma w(s))^2)s}(C^T(s))^2\mathbb{E}^{\mathbb{Q}^{2\pi^\star}}[(Z(s))^2|\mathscr{F}(t)]b^2(1-\rho^2)ds \\ =& \int_0^t (e^{-rs}C^T(s)S_0^{\pi^\star}(s)Z(s)b\sqrt{1-\rho^2})^2 ds + \widetilde{C}^T(t)\mathscr{E}(2R_0^{\pi^\star})(t)(Z(t))^2, \\ \widetilde{C}^T(t) :=& e^{-(2a+b^2)t}\int_t^T e^{-(2r-\mu^2/(\sigma w(s))^2-2a-b^2)s}(C^T(s))^2(w(t)/w(s))^{4\mu\rho b/\sigma\kappa}b^2(1-\rho^2)ds. \end{aligned}$$

Thus $\epsilon^2 D_2^T$ should be given as:

$$\epsilon^2 D_2^T(t) = \epsilon^2\int_0^t 2\widetilde{C}^T(s)\mathscr{E}(2R_0^{\pi^\star})(s)(Z(s))^2(b\rho-\mu/\sigma w(s))dW^1(s).$$

In fact, working similarly as in Example 2.2, we have $\lim_{T\to\infty} D_2^T = D_2^\infty$ (locally uniformly in probability) for sufficiently big r, where D_2^∞ is the W^1-projection of the martingale with terminal value $[N^\infty](\infty)$. Lastly note that for $\rho=\pm 1$, D_2^T vanishes as $\widetilde{C}^T=0$.

4. Utility-based pricing

The utility-based approach can also be used for the sake of pricing non-traded contingent claims. To this end we introduce the notion of certainty equivalent value $c^T(\epsilon) > -1$ of the position $(1, \epsilon)$, which is defined as the solution of the equation:

$$u^T(1 + c^T(\epsilon)) = u^T(1, \epsilon), \qquad \forall T \in \mathcal{A}. \tag{4.1}$$

In words, for each time horizon the investor is indifferent between having the position $(1, \epsilon)$ and the cash amount $c^T(\epsilon)$. Note that as $x \mapsto u(x)$ is continuous and strictly increasing (and hence bijective), (4.1) has a unique solution.

The valuation method (4.1) is well-known and widely-used in mathematical economics and finance (see the related discussion in the introductory section). In particular, the indifference price yields a subjective valuation of a non-replicable payoff in a incomplete market, which takes into account both the investor's risk preferences but also the part of the payoff that could be replicable through investing in tradeable asset. In our setting, the application of the indifference price could provide a way to measure the value of the endowment process even when there is no specific time horizon at which the endowment vanishes. Under this perspective, the indifference price can be seen a subjective way to value a promised pension plan that is designed by a pension fund manager. If the market were complete, this value would be coincide with the unique non-arbitrage price and hence the pension plan could be fully hedged. However, markets are incomplete (for example the contributions are only correlated to the market) and hence there should be another way to value the pension plan, that takes into account the presence of unhedgeable risk. The use of the indifference price is on that direction.

Based on the analysis of the previous sections, we get the following approximation of the indifference price.

Proposition 4.1. Consider the short-term setting. Assuming (A1), (A2), (A3) and (A4), then for each such horizon we have:

$$\begin{aligned} &c^T(\epsilon) - \widehat{c}_2^T(\epsilon) = o(\epsilon^2), \\ &\widehat{c}_2^T(\epsilon) := -\epsilon \mathbb{E}[L(T)] - \frac{\epsilon^2}{2}\Big(\mathbb{E}[(D_1^T(T) - L(T))^2] - \mathbb{E}[L(T)]^2\Big). \end{aligned} \tag{4.2}$$

Consider the long-term setting. Extending the first three assumptions on $\mathbb{R}_{\geq 0}$ and strengthening (A4) to (A5) we additionally have:

$$\overline{\lim_{T\to\infty}} |c^T(\epsilon) - \widehat{c}_2^T(\epsilon)| = \mathcal{O}(\epsilon^2). \tag{4.3}$$

Note that a respective form of (4.2), but for a more general class of utility functions on the positive real line, was derived in [KS06].

Proof. We make the following ansatz of $c^T(\epsilon)$'s second order asymptotic:

$$\widehat{c}_2^T(\epsilon) := -\epsilon \mathbb{E}[L(T)] - \frac{\epsilon^2}{2}\Big(\mathbb{E}[(D_1^T(T) - L(T))^2] - \mathbb{E}[L(T)]^2\Big).$$

In turn, direct computations, as also discussed in [KS06], show that for sufficiently small ϵ (s.t. $1+\widehat{c}_2^T(\epsilon)$ is bounded below by a positive constant) we have:

$$u^T(1+\widehat{c}_2^T(\epsilon)) - u^T(1,\epsilon) = o(\epsilon^2).$$

Now applying the Mean Value Theorem (MVT) on $x \mapsto u^T(x)$ and noting that $\partial_x u^T(x) > 0$, we get for $\xi^T(\epsilon) \leq \max(1+c^T(\epsilon), 1+\widehat{c}_2^T(\epsilon))$:

$$\frac{|u^T(1+\widehat{c}_2^T(\epsilon)) - u^T(1+c^T(\epsilon))|}{|\widehat{c}_2^T(\epsilon) - c^T(\epsilon)|} = \partial_x u^T(\xi^T(\epsilon)) \geq \partial_x u^T(\max(1+c^T(\epsilon), 1+\widehat{c}_2^T(\epsilon))),$$

where the inequality follows by the concavity of $x \mapsto u^T(x)$. In fact we can get a lower bound on the above that does not depend on ϵ. To this end note that:

$$\begin{aligned} 1+c^T(\epsilon) &= (u^T)^{-1}(u^T(1,\epsilon)) \\ &= (u^T)^{-1}\Big(\mathbb{E}[\ln(X_{\pi^\star}(T))] + u^T_{\pi^\star}(1,\epsilon)\Big). \end{aligned}$$

Now using the fact that $(u^T)^{-1}$ is increasing we get:

$$1+c^T(\epsilon) \leq \underbrace{(u^T)^{-1}\Big(\mathbb{E}[\ln(X_{\pi^\star}(T))] + \sup_{X\in\mathfrak{X}^+_{\pi^\star}} \mathbb{E}[\ln(1+X(T)-L(T))]\Big)}_{K_1(T)},$$

for $|\epsilon| \leq 1$. We can similarly bound $1+\widehat{c}_2^T(\epsilon)$ from above by some constant $K_2(T)$ (through appropriately constraining ϵ). Hence, for $K(T) := \max(K_1(T), K_2(T)) > 0$ we have:

$$|\widehat{c}_2^T(\epsilon) - c^T(\epsilon)| \leq \frac{|u^T(1+\widehat{c}_2^T(\epsilon)) - u^T(1+c^T(\epsilon))|}{K(T)} = \frac{|u^T(1+\widehat{c}_2^T(\epsilon)) - u^T(1,\epsilon)|}{K(T)}.$$

By the above along with (2.1) we get (4.2).

For the second part, firstly we refine the choice of $K(T)$ by choosing some $K^{'}$ s.t. it no longer depends on either ϵ or the horizon. In particular, using $\ln(x) \leq x-1$ for $x>0$, $1+X,\ X\in\mathfrak{X}^+_{\pi^\star}$ being a positive local martingale (hence a supermartingale) and $\mathbb{E}[|L(T)|] \leq \mathbb{E}[C_L]$, $\forall T\in\mathbb{R}_{\geq 0}$ we have:

$$1+c^T(\epsilon) \leq (u^T)^{-1}\Big(\mathbb{E}[\ln(X_{\pi^\star}(T))] + \mathbb{E}[C_L]\Big), \tag{4.4}$$

for $|\epsilon| \leq 1$. Now recall that e^x can be defined as the unique real function that maps zero to one and has derivative equals to its value. In turn noting that by the inverse function theorem we have $((u^T)^{-1})^{'} = (u^T)^{-1}$ implies that $1+c^T(\epsilon) \leq (u^T)^{-1}(0)\exp(\mathbb{E}[\ln(X_{\pi^\star}(T))] + \mathbb{E}[C_L]) = \exp(\mathbb{E}[C_L]) =: K_1^{'}$. Similarly for $K_2^{'}$ by using $L(T) \leq C_L$, the fact that $D_1^T(T)$ is the orthogonal projection of $L(T)$. Hence, for $K^{'} := \max(K_1^{'}, K_2^{'}) > 0$ we have for sufficiently small ϵ (that does not depend on the time horizon) s.t. $1+\widehat{c}_2^T(\epsilon)$ is bounded below by a positive constant):

$$|\widehat{c}_2^T(\epsilon) - c^T(\epsilon)| \leq \frac{|u^T(1+\widehat{c}_2^T(\epsilon)) - u^T(1,\epsilon)|}{K^{'}}.$$

Now note that $u^T(1+\widehat{c}_2^T(\epsilon))$ can be written as $\mathbb{E}[\ln((1+\widehat{c}_2^T(\epsilon))X_{\pi^\star}(T))]$. Taylor expanding $\ln((1+z)X_{\pi^\star}(T))$ for the deterministic $z>-1$, at $z=0$ yields:

$$\ln((1+z)X_{\pi^\star}(T)) = \ln(X_{\pi^\star}(T)) + z - \frac{z^2}{2} + \frac{z^3}{3}\frac{1}{(1+y)^3}, \qquad \text{for } y \text{ between } 0 \text{ and } z,$$

where the absolute value of the remainder is bounded above by $\max(|z|^3, |z|^3/(1+z))/3$. Hence we get:

(4.5)

$$\begin{aligned}|u^T(1,\epsilon) - u^T(1+\widehat{c}_2^T(\epsilon))| \leq \left|u^T(1,\epsilon) + \epsilon\mathbb{E}[L(T)] + \frac{\epsilon^2}{2}\mathbb{E}[(D_1^T - L(T))^2]\right| + \left|\frac{1}{2}(\epsilon^2\mathbb{E}[L(T)]^2 - (\widehat{c}_2^T(\epsilon))^2)\right| \\ + \max(|\widehat{c}_2^T(\epsilon)|^3, |\widehat{c}_2^T(\epsilon)|^3/(1+\widehat{c}_2^T(\epsilon)))/3.\end{aligned}$$

As shown for (2.2), the first term on the right-hand side of the above has an upper bound that does not depend on T and is $\mathcal{O}(\epsilon^2)$. Likewise, the second term can be shown to have an upper bound (that does not depend on T) and is $o(\epsilon^2)$ by the fact that $D_1^T(T)$ is the orthogonal projection of $L(T)$ and $|L(T)| \leq C_L$. Similarly for the last term, also using the fact that $1+\widehat{c}_2^T(\epsilon)$ is bounded below by a positive constant (for sufficiently small ϵ that does not depend on T). In other words, $\max(|\widehat{c}_2^T(\epsilon)|^3, |\widehat{c}_2^T(\epsilon)|^3/(1+\widehat{c}_2^T(\epsilon)))/3$ has an upper bound (that does not depend on T) and is $o(\epsilon^2)$. This concludes the proof. □

Example (Example 2.2 continued, $\widehat{c}_2^T$)**.** For $\mathbb{Q}^{\pi^\star}$ as in Example 2.1 we have:

$$\begin{aligned}\widehat{c}_2^T(\epsilon) &= -\epsilon\mathbb{E}^{\mathbb{Q}^{\pi^\star}}\left[\int_0^T e^{-rt}Z(t)dt\right] - \frac{\epsilon^2}{2}\mathbb{E}\left[\left[\int_0^\cdot e^{-rt}C^T(t)S_0^{\pi^\star}(t)Z(t)b\sqrt{1-\rho^2}dW^2(t)\right](T)\right] \\ &= -\epsilon\int_0^T e^{-(r-a)t}(1/w(t))^{\mu\rho b/\sigma\kappa}dt \\ &\quad - \frac{\epsilon^2}{2}\mathbb{E}\left[\int_0^T e^{-(2r-\mu^2/(\sigma w(t))^2)t}(C^T(t))^2\mathcal{E}(2R_0^{\pi^\star})(t)(Z(t))^2b^2(1-\rho^2)dt\right].\end{aligned}$$

Letting $\mathbb{Q}^{2\pi^\star}$ denote the measure induced by $\mathcal{E}(2R_0^{\pi^\star})$, the above becomes:

$$\begin{aligned}\widehat{c}_2^T(\epsilon) &= -\epsilon\int_0^T e^{-(r-a)t}(1/w(t))^{\mu\rho b/\sigma\kappa}dt - \frac{\epsilon^2}{2}\int_0^T e^{-(2r-\mu^2/(\sigma w(t))^2)t}(C^T(t))^2\mathbb{E}^{\mathbb{Q}^{2\pi^\star}}[(Z(t))^2]b^2(1-\rho^2)dt \\ &= -\epsilon\int_0^T e^{-(r-a)t}(1/w(t))^{\mu\rho b/\sigma\kappa}dt - \frac{\epsilon^2}{2}\int_0^T e^{-(2r-\mu^2/\sigma^2 w(t)-2a-b^2)t}(C^T(t))^2(1/w(t))^{4\mu\rho b/\sigma\kappa}b^2(1-\rho^2)dt.\end{aligned}$$

In fact for a sufficiently big r we should have:

$$\begin{aligned}\lim_{T\to\infty}\widehat{c}_2^T(\epsilon) &= -\epsilon\int_0^\infty e^{-(r-a)t}(1/w(t))^{\mu\rho b/\sigma\kappa}dt \\ &\quad - \frac{\epsilon^2}{2}\int_0^\infty e^{-(2r-\mu^2/\sigma^2 w(t)-2a-b^2)t}(C(t))^2(1/w(t))^{4\mu\rho b/\sigma\kappa}b^2(1-\rho^2)dt \\ &= \underbrace{-\epsilon\mathbb{E}[L(\infty)] - \frac{\epsilon^2}{2}\Big(\mathbb{E}[(D_1^\infty(\infty) - L(\infty))^2] - \mathbb{E}[L(\infty)]^2\Big)}_{\widehat{c}_2^\infty(\epsilon)}.\end{aligned}$$

Lastly note that for $\rho=\pm 1$ we get: $\widehat{c}_2^T(\epsilon) = -\epsilon\mathbb{E}[L(T)] = -\epsilon\mathbb{E}^{\mathbb{Q}^{\pi^*}}[\Lambda(T)] = -\epsilon\int_0^T e^{-(r-a)t}(1/w(t))^{\mu\rho b/\sigma\kappa}dt$. In turn, $\lim_{T\to\infty}\widehat{c}_2^T(\epsilon) = -\epsilon\mathbb{E}[L(\infty)] = -\epsilon\int_0^\infty e^{-(r-a)t}(1/w(t))^{\mu\rho b/\sigma\kappa}dt$.

4.1. **Utility-based pricing: fourth order asymptotics.** We are now ready to move forward to the (localized) fourth order expansion of the log-based certainty equivalent, using results of the previous section.

Proposition 4.2. Consider the short-term setting. Under the assumptions for (3.1) in Theorem 3.1 we have that for each time horizon $T\in[0,U]$ there exists a sequence of stopping times $\mathcal{T}_T^m\in[0,T]$ with $\lim_{m\to\infty}\mathcal{T}_T^m = T$, s.t.:

$$c^{\mathcal{T}_T^m}(\epsilon) - \widehat{c}_4^{\mathcal{T}_T^m}(\epsilon) = o(\epsilon^4), \tag{4.6}$$

where:

$$\begin{aligned}\widehat{c}_4^{\mathcal{T}_T^m}(\epsilon) &:= -\epsilon\mathbb{E}[L(\mathcal{T}_T^m)] - \frac{\epsilon^2}{2}\Big(\mathbb{E}[(D_1^{\mathcal{T}_T^m}(\mathcal{T}_T^m) - L(\mathcal{T}_T^m))^2] - \mathbb{E}[L(\mathcal{T}_T^m)]^2\Big)\\ &+\frac{\epsilon^3}{3}\Big(\mathbb{E}[(D_1^{\mathcal{T}_T^m}(\mathcal{T}_T^m) - L(\mathcal{T}_T^m))^3] + \mathbb{E}[L(\mathcal{T}_T^m)]^3\Big) - \frac{\epsilon^4}{4}\Big(\mathbb{E}[(D_1^{\mathcal{T}_T^m}(\mathcal{T}_T^m) - L(\mathcal{T}_T^m))^4] - \mathbb{E}[L(\mathcal{T}_T^m)]^4\Big)\\ &-\epsilon^4\mathbb{E}[(D_2^{\mathcal{T}_T^m}(\mathcal{T}_T^m))^2/2 - (N^{\mathcal{T}_T^m}(\mathcal{T}_T^m))^2 D_2^{\mathcal{T}_T^m}(\mathcal{T}_T^m)] + \frac{\epsilon^3}{2}\mathbb{E}[L(\mathcal{T}_T^m)]\mathbb{E}[(D_1^{\mathcal{T}_T^m}(\mathcal{T}_T^m) - L(\mathcal{T}_T^m))^2]\\ &+\frac{\epsilon^4}{8}\mathbb{E}[L(\mathcal{T}_T^m)]^4 + \frac{\epsilon^4}{8}\mathbb{E}[(N^{\mathcal{T}_T^m}(\mathcal{T}_T^m))^2]^2 + \frac{\epsilon^4}{3}\mathbb{E}[L(\mathcal{T}_T^m)]\mathbb{E}[(N^{\mathcal{T}_T^m}(\mathcal{T}_T^m))^3] + \frac{\epsilon^4}{2}\mathbb{E}[L(\mathcal{T}_T^m)]^2\mathbb{E}[(N^{\mathcal{T}_T^m}(\mathcal{T}_T^m))^2].\end{aligned}$$

Consider the long-term setting. Under the assumptions for (3.2) in Theorem 3.1, we additionally have:

$$\overline{\lim_{T\to\infty}}|c^{\mathcal{T}_T^m}(\epsilon) - \widehat{c}_4^{\mathcal{T}_T^m}(\epsilon)| = \mathcal{O}_m(\epsilon^4). \tag{4.7}$$

Proof. Recalling how $\mathcal{T}_T^m$ was constructed in Theorem 3.1, denote the solution of (4.1) for $\mathcal{T}_T^m$ by $c^{\mathcal{T}_T^m}(\epsilon)$ and note that it is well-defined since $u^{\mathcal{T}_T^m}(x)<\infty,\ x>0,\ u^{\mathcal{T}_T^m}(1,\epsilon)<\infty$ as previously noted. We now make the following ansatz for $c^{\mathcal{T}_T^m}(\epsilon)$'s fourth order asymptotic:

$$\begin{aligned}\widehat{c}_4^{\mathcal{T}_T^m}(\epsilon) &:= -\epsilon\mathbb{E}[L(\mathcal{T}_T^m)] - \frac{\epsilon^2}{2}\Big(\mathbb{E}[(D_1^{\mathcal{T}_T^m}(\mathcal{T}_T^m) - L(\mathcal{T}_T^m))^2] - \mathbb{E}[L(\mathcal{T}_T^m)]^2\Big)\\ &+\frac{\epsilon^3}{3}\Big(\mathbb{E}[(D_1^{\mathcal{T}_T^m}(\mathcal{T}_T^m) - L(\mathcal{T}_T^m))^3] + \mathbb{E}[L(\mathcal{T}_T^m)]^3\Big) - \frac{\epsilon^4}{4}\Big(\mathbb{E}[(D_1^{\mathcal{T}_T^m}(\mathcal{T}_T^m) - L(\mathcal{T}_T^m))^4] - \mathbb{E}[L(\mathcal{T}_T^m)]^4\Big)\\ &-\epsilon^4\mathbb{E}[(D_2^{\mathcal{T}_T^m}(\mathcal{T}_T^m))^2/2 - (N^{\mathcal{T}_T^m}(\mathcal{T}_T^m))^2 D_2^{\mathcal{T}_T^m}(\mathcal{T}_T^m)] + \frac{\epsilon^3}{2}\mathbb{E}[L(\mathcal{T}_T^m)]\mathbb{E}[(D_1^{\mathcal{T}_T^m}(\mathcal{T}_T^m) - L(\mathcal{T}_T^m))^2]\\ &+\frac{\epsilon^4}{8}\mathbb{E}[L(\mathcal{T}_T^m)]^4 + \frac{\epsilon^4}{8}\mathbb{E}[(N^{\mathcal{T}_T^m}(\mathcal{T}_T^m))^2]^2 + \frac{\epsilon^4}{3}\mathbb{E}[L(\mathcal{T}_T^m)]\mathbb{E}[(N^{\mathcal{T}_T^m}(\mathcal{T}_T^m))^3] + \frac{\epsilon^4}{2}\mathbb{E}[L(\mathcal{T}_T^m)]^2\mathbb{E}[(N^{\mathcal{T}_T^m}(\mathcal{T}_T^m))^2].\end{aligned}$$

Following similar steps as for (4.2) in Proposition 4.1 we get for sufficiently small ϵ s.t. $1+\widehat{c}_4^{\mathcal{T}_T^m}(\epsilon)$ is bounded below by a positive constant (note that here ϵ may depend on m, T):

$$|\widehat{c}_4^{\mathcal{T}_T^m}(\epsilon) - c^{\mathcal{T}_T^m}(\epsilon)| \le \frac{|u^{\mathcal{T}_T^m}(1+\widehat{c}_4^{\mathcal{T}_T^m}(\epsilon)) - u^{\mathcal{T}_T^m}(1+c^{\mathcal{T}_T^m}(\epsilon))|}{K(\mathcal{T}_T^m)} = \frac{|u^{\mathcal{T}_T^m}(1+\widehat{c}_4^{\mathcal{T}_T^m}(\epsilon)) - u^{\mathcal{T}_T^m}(1,\epsilon)|}{K(\mathcal{T}_T^m)}. \tag{4.8}$$

Note that $u^{\mathcal{T}_T^m}(1+\widehat{c}_4^{\mathcal{T}_T^m}(\epsilon))$ can be written as $\mathbb{E}[\ln((1+\widehat{c}_4^{\mathcal{T}_T^m}(\epsilon))X_{\pi^\star}(\mathcal{T}_T^m))]$ (the log-optimal numeraire maximizes simultaneously all intertemporal conditional expected growth rates; see further in [Bec01]). Taylor expanding $\ln((1+z)X_{\pi^\star}(\mathcal{T}_T^m))$ for the deterministic $z>-1$, at $z=0$ yields (also recalling the bounds of the Taylor remainder for logarithm in Proposition 4.1):

$$\ln((1+z)X_{\pi^\star}(\mathcal{T}_T^m)) = \ln(X_{\pi^\star}(\mathcal{T}_T^m)) + z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \frac{z^5}{5}\frac{1}{(1+y)^5}, \qquad \text{for } y \text{ between } 0 \text{ and } z,$$

where the absolute value of the remainder is bounded above by $\max(|z|^5, |z|^5/(1+z))/5$. Applying this to (4.8) and after some algebra we get:

$$K(\mathcal{T}_T^m)|\widehat{c}_4^{\mathcal{T}_T^m}(\epsilon) - c^{\mathcal{T}_T^m}(\epsilon)| \le \Big| g(\epsilon(D_1^{\mathcal{T}_T^m}(\mathcal{T}_T^m) - L(\mathcal{T}_T^m))) - \epsilon^4\mathbb{E}[(D_2^{\mathcal{T}_T^m}(\mathcal{T}_T^m))^2/2 - (N^{\mathcal{T}_T^m}(\mathcal{T}_T^m))^2 D_2^{\mathcal{T}_T^m}(\mathcal{T}_T^m)] - u_{\pi^\star}^{\mathcal{T}_T^m}(1,\epsilon)\Big| + o(\epsilon^4).$$

In turn, using Theorem 3.1 we get (4.6).

For the second part we work similarly to (4.3). The goal now is to derive bounding constants s.t. they don't depend on T (although they may depend on m). In fact using (A5) we get, as in (4.4), $1+c^{\mathcal{T}_T^m}(\epsilon) \le \exp(\mathbb{E}[C_L])$. For the case of $1+\widehat{c}_4^{\mathcal{T}_T^m}(\epsilon)$ we also use the boundedness of $N^{\mathcal{T}_T^m}(\mathcal{T}_T^m)$ as well as the fact that $D_2^{\mathcal{T}_T^m}(\mathcal{T}_T^m)$ is the orthogonal projection of its square. Hence, for sufficiently small ϵ (that does not depend on T, but may depend on m) s.t. $1+\widehat{c}_4^{\mathcal{T}_T^m}(\epsilon)$ is bounded below by a positive constant, using (3.2) and arguing as in (4.5) we get (4.7). □

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