CONTRACT PRICING AND UTILITY SHARING

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Abstract. In an incomplete market setting, we consider two financial agents who are willing to trade as counterparties a contract that represents a non-replicable indivisible contingent claim. Market incompleteness allows for an infinity of non-arbitrage prices at which the contract can be traded. Furthermore, indivisibility of the contract does not allow for equilibrium pricing. Assuming that the agents are utility maximizers who will resort into indifference pricing, we suggest a scenario that allows for the establishment of a natural one-to-one correspondence between the agreed transaction price of the contract and the relative bargaining power of the two agents. This amounts to the solution of an optimization problem, seeking to maximize a convex combination of the indirect utilities of the two agents, weighted by their relative bargaining power and under the constraint that the trade takes place. In particular, given the utility functions of the two agents and their relative bargaining power, we obtain an optimal unique price for the contract. Its existence is proved for a large family of utility functions, and a number of its properties are stated and discussed. As an example, we analyze extensively the case where both agents have exponential utility.

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1. Introduction

Realistic financial markets are incomplete, as evidenced by empirical studies and manifested by the difficulty or inability to replicate any contingent claim as a portfolio of traded assets. Seminal studies in the field of incomplete markets (see [35], [16] for overviews in relevant topics) have led to an ever increasing research activity in this area, that by now has allowed for a deeper understanding of the way that financial markets function. It is well known for instance, that in an incomplete market setup there is no longer a unique pricing kernel and this may at best point out a whole band of non-arbitrage prices for a contingent claim.

There is an extensive and very interesting literature focusing on the determination of the upper and lower hedging prices (see e.g., [13], [15], [49] etc.). On the other hand though, further criteria seem to be needed for the determination of the single price, out of the whole band of non-arbitrage prices, at which the contingent claim will eventually be traded. The majority of such criteria, as they have been proposed in the relevant literature, is based on, or related to, the minimization of entropy-like functions (see e.g., [21], [22] etc.) and lead to *subjective* non-linear pricing rules. However a complete theory on the procedure of price selection in incomplete markets is still missing. More recently (see among others [3], [8] and [17]), (partial) equilibrium arguments (i.e., setting supply equal demand) have been proposed for the determination of the price for non-replicable claims, in the case where the agents negotiate not only the price of the claims but also their units they are willing to buy or sell. However, these pricing procedures can not be applied when the claim (or the basket of claims) that is going to be transacted is *indivisible*.

The aim of this paper is to address the question of price selection in incomplete markets and provide new ideas and insights, by introducing a bargaining dimension that allows to elaborate on a "utility sharing" scenario involving two financial agents who negotiate the price of a single indivisible non-replicable contingent claim.

We consider two agents, named "buyer" and "seller", who are risk averse von Neumann-Morgenstern expected utility maximizers and have access to some financial market that is liquid, incomplete and arbitrage free.

At time 0 the two agents are willing to enter as counterparties into a contract that represents a contingent claim, by agreeing on a price P immediately payable by the "buyer" to the "seller" in exchange of a non-replicable contingent payoff B that is due from the seller to the buyer at the end of their common and mutually beforehand agreed investment horizon T. Examples of such a situation range from over-the-counter financial transactions to insurance and reinsurance contracts.

The incompleteness of the market dictates that there is an infinity of prices P that are consistent with the absence of arbitrage. Furthermore, the incompleteness of the market means that the agents cannot fully hedge their positions in the contingent claim B by trading in the market. Therefore, each of our agents resorts to (expected) utility maximization criteria to produce a reference price that makes her indifferent between an optimally invested portfolio that contains the position on the contingent claim and an optimally invested portfolio without the position on the claim. However, the agents are fully aware that their corresponding indifference prices are highly unlikely to coincide (see e.g., [4], [50]). In other words they understand that if they manage to agree on a transaction price for the contingent claim, this price will differ from their reference indifference prices that are dictated by their utility functions, i.e., some “gain” or “loss” of indirect utility should take place. To be more precise, if the indifference price of the seller is lower than the indifference price of the buyer, then by agreeing at any price in between, they both experience some gain of utility. In the case however that the indifference price of the seller is higher than the indifference price of the buyer, then by agreeing at any price in between, they both experience some loss of utility. Although loss of utility would normally forbid the execution of the trade (and thus we could just ignore such a case), one may want to consider instances where this could happen (i.e. instances where the counterparties are forced to execute the trade). Our model is quite general to accommodate such cases as well.

In this sense, the two agents are prepared in advance to embark on some kind of bargaining, in order to determine the price at which they may trade the contract. During the bargaining, they will try to deploy their best bargaining and persuasive skills in order to achieve the best possible price. Within this context we introduce a parameter $\lambda \in (0,1)$ that reflects the relative bargaining (or persuasion) power of the two agents. Then one can address the following question:

“How can we relate the price selected to the bargaining power of the two agents?”

We will suggest a scenario that allows for the establishment of a natural one-to-one correspondence between the agreed transaction price of the contract and the relative bargaining power of the two agents. In particular, given the utility functions of the two agents and their relative bargaining power, we will obtain an optimal unique price for the contract. Conversely, given the utility functions of the two agents and the price at which they agree to trade the contract, our scenario reveals their relative bargaining power.

Our scenario amounts to the solution of an optimization problem, in which we seek to maximize a convex combination of the indirect utility of the two agents, weighted by their relative bargaining power and under the constraint that the transaction is made possible.

In our context, the agents’ utility functions are understood as the basic tools that guide them to informed decision making. However, it should be pointed out that the use of such utilities is not restrictive for the purpose of this paper; as discussed in Section 5 our results can be generalized, using convex or coherent risk measures (see [2] and [20] for the exact definitions of such measures) as decision making rules instead of utility functions.

For the purposes of this paper we adopt the view that utility is cardinal and allows for interpersonal and intrapersonal comparison as is usually done in many branches of economic theory such as social welfare theory (see e.g., [24], [37], [47] but also [30] for an opposite view), the axiomatic theory of bargaining (see e.g., [29], [39]), the theory of coalitional bargaining ([48]) etc. Without entering the long standing debate concerning the ordinal or cardinal nature of utilities and the problem of strength of preferences [6], [7], [19], [40][1], we will not eschew from exploring the implications that such an assumption has in contingent claim pricing theory in incomplete markets.

The utility maximization problem has captured an important part of the mathematical finance literature (see e.g., [11], [33], [41], [46] etc). In these papers, there is a main distinction of utility functions into two types on the basis of their domain (see [46], Chapter 1). The first type refers to those utility functions that are defined on wealth that may take values from the whole real axis, while the utility functions of the second type are defined only for values of wealth on the positive part of the real axis. In this work, we do not impose restrictions on the agents' type of utility function. In fact, the pricing scheme that we suggest below can be applied to both types of utility. However, for presentation purposes and in order to facilitate the reading of the paper, we consider the case of utilities of the first type in the main body of the paper, while the technicalities of utility functions of the second type are treated in the Appendix A. We emphasize though that, under the appropriate assumptions, whenever a utility function is considered in this paper, it can be taken to be of either type.

The structure of the paper is as follows: In Section 2, we fix ideas and notation, we present the general setup of the market and of the agents risk preferences,we introduce the concept of the indirect utility differential and we extend the concept of utility indifference pricing in order to accommodate situations of loss or gain of indirect utility. In Section 3, we deal with our main problem of price selection in incomplete markets by imposing our risk sharing criterion which leads to our main result, a pricing scheme, represented by an optimization problem, which we prove that it offers a unique solution to the price selection problem. In Section 4, we examine in detail the example of agents who report exponential utility, in which case we are able to provide a closed form solution to the price selection problem and discuss further its properties. In Section 5, we summarize and conclude, after discussing possible variations or extensions of our framework (convex risk measures, optimal trading time, other forms of total risk), which hint to some possible directions for future research. Finally in the Appendix A, we treat the technicalities involved with utility functions that are defined only for wealths that take values on the positive real axis.

[1] which to the best of our understanding is still not resolved completely as the literature still addresses the problem of providing conditions under which cardinality of utility functions is feasible see e.g., [5], [9], [10], [18], [32] or even measurable by econometric experiments [34].

2. The Market and the Agents

2.1. **The Market.** We consider an incomplete market setting over a fixed finite time horizon T, based on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, where the filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ satisfies the usual conditions of right continuity and completeness. We assume that the market consists of $d+1$ tradable assets, the discounted price process of which is denoted by $\mathbf{S} = (S_t)_{t\in[0,T]} = \left(S_t^{(0)}, S_t^{(1)}, S_t^{(2)}, ..., S_t^{(d)}\right)_{t\in[0,T]}$. The first of these assets is considered to be riskless (the bond) with discounted price process $S_t^{(0)}$ equal to 1 at any time $t \in [0,T]$, i.e., it plays the rôle of the numéraire. The other assets are risky (the stocks) with discounted price process modeled by a $\mathbb{R}^d$-valued (*locally bounded*) semimartingale $\left(S_t^{(1)}, S_t^{(2)}, ..., S_t^{(d)}\right)_{t\in[0,T]}$ on $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$.

Furthermore, we make the following assumption which ensures that our market is incomplete and free of arbitrage opportunities (see [15] for the related proofs).

Assumption 2.1. *The set* $\mathcal{M}_e := \{\mathbb{Q} \sim \mathbb{P} : \mathbf{S}$ *is local martingale under* $\mathbb{Q}\}$ *is non empty and not a singleton.*

According to Delbaen and Schachermayer, [15], given a contingent claim B, the set of non-arbitrage prices for this claim is the interval $\left(\inf_{\mathbb{Q}\in\mathcal{M}_e}\{\mathbb{E}_{\mathbb{Q}}[B]\}, \sup_{\mathbb{Q}\in\mathcal{M}_e}\{\mathbb{E}_{\mathbb{Q}}[B]\}\right)$. Clearly, if the market were complete, i.e, if $\mathcal{M}_e$ were a singleton, this interval would degenerate to a single point, thus leading to the unique price of the claim.

2.2. **The Utility Functions.** The incompleteness of the market implies the existence of contingent claims that can not be replicated by the market assets $\mathbf{S}$. Thus, an agent who includes such a non-replicable claim in her portfolio will always face *unhedgeable* risk. Therefore, in this market setting, the agent's risk preferences play a crucial rôle regarding her investment decisions. We assume that the risk preferences of an agent are modeled by some utility function on her terminal wealth at time horizon T. Then, the agent faces the problem of maximizing the expected utility of her terminal wealth by employing an appropriate portfolio strategy for trading in the market $\mathbf{S}$ (see problems (4) and (5) below).

So let us suppose that an agent decides on the basis of some utility function $U : \mathbb{R} \to \mathbb{R}$ that is strictly increasing, strictly concave, continuously differentiable and satisfies the *Inada conditions*:

$$\lim_{x\to-\infty} U'(x) := U'(-\infty) = +\infty \text{ and } \lim_{x\to+\infty} U'(x) := U'(+\infty) = 0. \tag{1}$$

Moreover, we assume that U has *reasonable asymptotic elasticity*, i.e.,

$$\liminf_{x\to-\infty} \frac{xU'(x)}{U(x)} > 1 \text{ and } \limsup_{x\to+\infty} \frac{xU'(x)}{U(x)} < 1. \tag{2}$$

As it is proved in [45] (see also [33]) this assumption is required for the well posedness of the utility maximization problem of the agent.

The notation $\lim_{x\to+\infty} U(x) = U(+\infty)$ is also used.

2.3. **The Agents and the Admissible Strategies.** Consider now an agent with initial wealth $x \in \mathbb{R}$ (measured in numéraire units) and risk preferences that are modeled by a utility function U. The agent invests her initial wealth in the market assets by creating *self-financing* portfolios (investment strategies), with the goal to maximize her expected utility of terminal wealth.

A *self-financing* portfolio for this agent is a d-dimensional stochastic process $\vartheta = (\vartheta_t)_{t\in[0,T]}$, that is predictable and $\mathbf{S}$-integrable, specifying the number of units of each asset held in the portfolio at each time $t \in [0, T]$. Then, the wealth process $X = (X_t)_{t\in[0,T]}$ that such a portfolio produces is given as the stochastic integral $X_t = (\vartheta \cdot \mathbf{S})_t = \int_0^t \vartheta_s d\mathbf{S}_s$. The portfolio ϑ is called *admissible* if the wealth process $X_t = (\vartheta \cdot \mathbf{S})_t$ that it produces is uniformly bounded from below by some constant. We denote by $\mathbf{\Theta}$ the set of all admissible portfolios.

In the sequel, we use the following notation: $\mathbb{L}^p = \mathbb{L}^p(\Omega, \mathcal{F}_T, \mathbb{P})$, $1 \le p < \infty$, denotes the set of equivalence classes of $\mathcal{F}_T$-measurable random variables X such that $\mathbb{E}[|X|^p] < \infty$, where $\mathbb{E}[.]$ denotes the expectation under the probability measure $\mathbb{P}$; the case where $p = \infty$ corresponds to the essentially bounded random variables. By $\mathbb{L}^0$ we denote the set of $\mathcal{F}_T$-measurable random variables.

Assume for now that the agent has a (possibly non-replicable) claim $B \in \mathbb{L}^\infty$ that matures at the fixed time horizon T. Then we may follow M. Owen, [41] and define the set $\mathfrak{F}_{U,B}(x) = \{F \in \mathbb{L}^0 : U(F) \in \mathbb{L}^1$ and $F \le x + (\vartheta \cdot \mathbf{S})_T + B$ for some $\vartheta \in \mathbf{\Theta}\}$. According to [41], the set of attainable wealths for this agent which describes the wealths that she can attain at time T by employing some admissible portfolio strategy ϑ it is given as:

$$\mathcal{X}_{U,B}(x) := \{X \in \mathbb{L}^0 : U(x + X + B) \in \overline{\{U(F) : F \in \mathfrak{F}_{U,B}(x)\}}^{\mathbb{L}^1}\}. \tag{3}$$

Then, the problem of *utility maximization* that the agent faces is

$$u(x;B) := \sup_{X\in\mathcal{X}_{U,B}(x)} \mathbb{E}\left\{U\left(X + B\right)\right\} \tag{4}$$

In the special case that $B = 0$ the corresponding problem becomes

$$u(x) := \sup_{X\in\mathcal{X}_{U}(x)} \mathbb{E}\left\{U\left(X\right)\right\} \tag{5}$$

where, by $\mathcal{X}_U(x)$ stands for $\mathcal{X}_{U,0}(x)$.

The functions $u(x)$ and $u(x;B)$ are usually called the “indirect” utilities (the latter under the liability B) or simply the value functions and have been studied by many authors (see e.g., [41], [42], [45] etc.).

The next theorem due to [41] (see also [45]) states some useful properties of the indirect utility.

Theorem 2.1. *(W. Schachermayer, 2001, M. Owen, 2002)*
Assume that:

(i) $B \in \mathbb{L}^{\infty}$.

(ii) *Assumption 2.1 holds.*

(iii) *There exists* $x \in \mathbb{R}$ *such that* $u(x;B) < U(\infty)$.

Then, the value function $u(.;B)$ *is finitely valued, strictly increasing, strictly concave and continuously differentiable on* $\mathbb{R}$ *with*

$$\lim_{x\to+\infty} u'(x;B) = 0 \text{ and } \lim_{x\to-\infty} u'(x;B) = +\infty \tag{6}$$

Furthermore, there exists a solution to problem (4), i.e., the supremum in (4) is attained.

Remark 2.1. Since, $u(x;B)$ is strictly concave, $C^1(\mathbb{R})$ function of x, we obtain that $\frac{du(x;B)}{dx}$ is strictly decreasing and its range is $(0,+\infty)$. We also have that $\lim_{x\to-\infty} u(x;B) = \lim_{x\to-\infty} U(x) = -\infty$ and $\lim_{x\to+\infty} u(x;B) = U(+\infty)$.

2.4. **The Agents and the Indifference Price.** Having defined the indirect utility functions, we recall the notion of the seller's and buyer's *indifference price* for the given claim $B \in \mathbb{L}^{\infty}$. The seller's indifference price, $v^{(s)}(B)$, is given as the unique solution of:

$$u(x) = u\left(x + v^{(s)}(B); -B\right) \tag{7}$$

while the corresponding buyer's indifference price, $v^{(b)}(B)$, solves:

$$u(x) = u\left(x - v^{(b)}(B); B\right) \tag{8}$$

Indifference pricing has been examined by several authors (see e.g., [12], [26], [31], [38], [43] etc.). It can be shown that both indifference prices are non-arbitrage prices, i.e., they belong to the interval $\left(\inf_{\mathbb{Q}\in\mathcal{M}_e}\{\mathbb{E}_{\mathbb{Q}}[B]\}, \sup_{\mathbb{Q}\in\mathcal{M}_e}\{\mathbb{E}_{\mathbb{Q}}[B]\}\right)$ (see among others [42], Proposition 7.2) and when B is replicable, they are both equal to the unique non-arbitrage price $\mathbb{E}_{\mathbb{Q}}[B]$, for some $\mathbb{Q} \in \mathcal{M}_e$.

The indifference pricing rule provides the agents with reference prices of the claim $B \in \mathbb{L}^{\infty}$. An agent who follows this pricing rule is willing to sell the claim B at any price P such that $v^{(s)}(B) \leq P$. In particular, if the agent sells the claim B at a price P strictly greater than $v^{(s)}(B)$, she "gains" some indirect utility. Similarly, the agent is willing to buy the claim B at any price P such that $v^{(b)}(B) \geq P$, while again a strict inequality transaction results a "gain" of indirect utility. On the other hand, selling claim B for a price $P < v^{(s)}(B)$ or buying claim B for a price $P > v^{(b)}(B)$, results a "loss" of indirect utility.

Although a transaction that does not cause any "loss" of indirect utility to any of the transacting agents sounds natural, there may be instances that lead to some "loss" of indirect utility as the following examples illustrate.

Example 2.1. A company C owns two subsidiaries S1 and S2. For some reason (e.g., internal politics, book-keeping, tax purposes) the top management of C is convinced that it would be beneficial for C if S1 and S2 perform a transaction on a structured claim B with S1 acting as the buyer and S2 acting as the seller of the claim. The mother company C is not interested at the exact price that the transaction will take place as long as it is a non-arbitrage price and so it "orders" the management of S1 and S2 to perform the transaction at a convenient to them price. However the managers of the subsidiaries are naturally concerned about the price of the transaction. Let $v^{(b)}(B)$, $v^{(s)}(B)$ the indifference prices of S1 and S2 respectively. If $v^{(b)}(B) < v^{(s)}(B)$ then, since S1 and S2 are forced to perform the transaction, at least one of them will lose some indirect utility (most probably they will both lose some indirect utility as the bargaining procedure may eventually lead to a price P such that $v^{(b)}(B) < P < v^{(s)}(B)$).

Example 2.2. An employer E offers a bonus scheme to an employee M, which is agreed to have a fixed value C at time zero. The bonus consists of two parts: the first one is a contingent non-replicable claim B maturing at time T. The second part of the bonus is just the lump cash amount that remains after subtracting the time zero value of B from the total value C. Therefore, the value C of the bonus is the sum of the time zero value of B and whatever remains in cash; however, the time zero value of B is clearly negotiable. In this sense, the time zero value of B is the price at which the employer "sells" the claim to the employee, who in turn "buys" it from the employer. It is clear that the price of B is important and the two counter-parties may find that it is mutually beneficial to reach an agreement on the price of B even if some loss of indirect utility is needed.

The above examples indicate that in certain cases we need to *extend* the notion of the utility indifference pricing to accommodate situations of loss or gain of indirect utility. Let us fix a contingent claim $\hat{B} \in \mathbb{L}^\infty$ and following [50] consider an agent with utility function U that faces $\hat{B}$ at time T and it is compensated at time 0 by the certain amount $\hat{P} \in \mathbb{R}$. We define her indirect utility differential as the quantity:

$$\varepsilon := u\left(x\right) - u\left(x + \hat{P}; \hat{B}\right). \tag{9}$$

Clearly, the case where $\hat{B} = -B$, $\hat{P} =: P$ corresponds to the "reservation price" stated by the seller of the claim when a decision leading to indirect utility differing by ε from the indifference level is taken. On the other hand, the situation where $\hat{B} = B$, $\hat{P} =: -P$ corresponds to the buyer's side.

Alternatively, one can define the function $P^{(s)}\left(\varepsilon\right)$, which gives the exact price under which the indirect utility differential for the seller is ε. In other words, $P^{(s)}\left(\varepsilon\right)$ is given implicitly as the solution of the equation:

$$\varepsilon = u\left(x\right) - u\left(x + P^{(s)}\left(\varepsilon\right); -B\right) \tag{10}$$

Similarly, for the buyer's side, $P^{(b)}(\varepsilon)$ is given as the solution of the equation:

$$\varepsilon = u(x) - u\left(x - P^{(b)}(\varepsilon); B\right) \tag{11}$$

Notice that $P^{(b)}(0) = v^{(b)}(B)$ and $P^{(s)}(0) = v^{(s)}(B)$ and in general both $P^{(s)}(\varepsilon)$ and $P^{(b)}(\varepsilon)$ depend on the initial wealth x.

2.5. **The Functions $P^{(s)}(.)$ and $P^{(b)}(.)$.** To study the properties of $P^{(s)}(.)$ and $P^{(b)}(.)$, we need to introduce the following notation: For every utility, U, and for every initial wealth, $x \in \mathbb{R}$, we define

$$A_{U,x} := (u(x) - u(+\infty), +\infty) = (u(x) - U(+\infty), +\infty) \tag{12}$$

where $u(x)$ is the corresponding indirect utility given in (5). We also define the function $\varphi(.;B) : (-\infty, U(+\infty)) \to \mathbb{R}$ by

$$\varphi(y; B) := u^{-1}(y; B), \text{ for } y \in (-\infty, U(+\infty)). \tag{13}$$

$\varphi(y; B)$ is well-defined on account of Theorem 2.1.

Using the above definitions, the solutions of equations (10) and (11) are given by

$$P^{(s)}(\varepsilon) = \begin{cases} \varphi(u(x) - \varepsilon; -B) - x, & \text{if } \varepsilon \in A_{U,x}, \\ +\infty, & \text{if } \varepsilon \leq u(x) - U(+\infty) \end{cases} \tag{14}$$

and

$$P^{(b)}(\varepsilon) = \begin{cases} x - \varphi(u(x) - \varepsilon; B), & \text{if } \varepsilon \in A_{U,x}, \\ -\infty, & \text{if } \varepsilon \leq u(x) - U(+\infty) \end{cases} \tag{15}$$

where $\lim_{\varepsilon \to +\infty} P^{(s)}(\varepsilon) = -\infty$ and $\lim_{\varepsilon \to +\infty} P^{(b)}(\varepsilon) = +\infty$.

We should point out that although the indifference prices (i.e., $P^{(s)}(0)$ and $P^{(b)}(0)$) are non-arbitrage prices, the values $P^{(s)}(\varepsilon)$ and $P^{(b)}(\varepsilon)$ may lie outside the interval of non-arbitrage prices $\left(\inf_{\mathbb{Q} \in \mathcal{M}_e} \{\mathbb{E}_{\mathbb{Q}}(B)\}, \sup_{\mathbb{Q} \in \mathcal{M}_e} \{\mathbb{E}_{\mathbb{Q}}(B)\}\right)$ for certain values of ε. In fact, $P^{(s)}(A_{U,x}) = P^{(b)}(A_{U,x}) = \mathbb{R}$ for any pair of initial wealths $x \in \mathbb{R}$ and utility functions U. By (14) and (15), $P^{(s)}(\varepsilon)$ is within the interval of non-arbitrage prices for those ε's that belong to the interval:

$$\left(u(x) - u(\sup_{\mathbb{Q} \in \mathcal{M}_e} \{\mathbb{E}_{\mathbb{Q}}(B)\} + x; -B), u(x) - u(\inf_{\mathbb{Q} \in \mathcal{M}_e} \{\mathbb{E}_{\mathbb{Q}}(B)\} + x; -B)\right) \tag{16}$$

and $P^{(b)}(\varepsilon)$ is within the interval of non-arbitrage prices for those ε's that belong to the interval:

$$\left(u(x) - u(x - \inf_{\mathbb{Q} \in \mathcal{M}_e} \{\mathbb{E}_{\mathbb{Q}}(B)\}; B), u(x) - u(x - \sup_{\mathbb{Q} \in \mathcal{M}_e} \{\mathbb{E}_{\mathbb{Q}}(B)\}; B)\right) \tag{17}$$

The following proposition establishes some useful properties of the functions $P^{(s)}$ and $P^{(b)}$.

Proposition 2.1. *For a given claim $B \in \mathbb{L}^\infty$ and initial wealth $x \in \mathbb{R}$, $P^{(s)}$ ($P^{(b)}$) is a continuously differentiable, strictly decreasing (increasing) and strictly convex (concave) function of ε on $A_{U,x}$.*

Proof. We restrict our attention to $P^{(s)}$, since the proof for $P^{(b)}$ follows along the same lines. The fact that $P^{(s)} \in C^1(A_{U,x})$ follows directly from the definition of $P^{(s)}$ and Theorem 2.1. Also for ε_1,$\varepsilon_2 \in A_{U,x}$ such that $\varepsilon_1 < \varepsilon_2$

$$\varepsilon_1 \quad = \quad u(x) - u\left(x + P^{(s)}(\varepsilon_1); - B\right) < \varepsilon_2 = u(x) - u\left(x + P^{(s)}(\varepsilon_2); - B\right)$$

and $P^{(s)}(\varepsilon_1) > P^{(s)}(\varepsilon_2)$ follows since $u(x; - B)$ is strictly increasing for $x \in \mathbb{R}$.

For the convexity, let us take any $\lambda \in (0,1)$ and some ε_1,$\varepsilon_2 \in A_{U,x}$, with $\varepsilon_1 \neq \varepsilon_2$. Then,

$$\lambda\varepsilon_1 + (1-\lambda)\,\varepsilon_2 = u(x) - u\left(x + P^{(s)}(\lambda\varepsilon_1) + (1-\lambda\varepsilon_2); - B\right),$$

but also

$$\lambda\varepsilon_1 + (1-\lambda)\,\varepsilon_2 = u(x) - \lambda u\left(x + P^{(s)}(\varepsilon_1); - B\right) - (1-\lambda)u\left(x + P^{(s)}(\varepsilon_2); - B\right).$$

Therefore,

$$u\left(x + P^{(s)}(\lambda\varepsilon_1 + (1-\lambda)\varepsilon_2); - B\right) = \lambda u\left(x + P^{(s)}(\varepsilon_1); - B\right) + (1-\lambda)u\left(x + P^{(s)}(\varepsilon_2); - B\right)$$

but by the strict convexity of $u(x; - B)$ we have

$$u\left(x + P^{(s)}(\lambda\varepsilon_1 + (1-\lambda)\,\varepsilon_2); - B\right) < u\left(x + \lambda P^{(s)}(\varepsilon_1) + (1-\lambda)\,P^{(s)}(\varepsilon_2); - B\right)$$

which gives that

$$P^{(s)}(\lambda\varepsilon_1 + (1-\lambda)\,\varepsilon_2) < \lambda P^{(s)}(\varepsilon_1) + (1-\lambda)\,P^{(s)}(\varepsilon_2).$$

This completes the proof. □

3. The Utility Sharing Pricing Scheme

In this section, we deal with the main problem of this paper. Consider a fixed claim $B \in \mathbb{L}^\infty$ and two financial agents whose risk preferences are modeled by utility functions that satisfy the properties listed in subsection 2.2 (or the ones in subsection A.1). One of the agents is supposed to be the seller of B, while the other is the buyer. In other words, we suppose that at time T, the seller will face the liability $-B$ payable to the buyer, as a result of receiving some certain amount $P \in \mathbb{R}$ at time 0 by the buyer. On the other hand, the buyer will receive the payoff B at time T, as a result of paying P to the seller at time 0. Given that the transaction takes place,

How should price P be determined?

Before we give our proposed answer, we need some further notation. Let U_s and U_b denote the utility functions of the seller and the buyer respectively, $u^{(s)}(.;-B), u^{(b)}(.;B)$ their indirect utilities, $x_s, x_b \in \mathbb{R}$ their initial wealths and $P^{(s)}(.), P^{(b)}(.)$ their respective pricing functions. Furthermore, we set $A_s := A_{U_s,x_s}$ and $A_b := A_{U_b,x_b}$. Since both agents are considered to be utility maximizers (i.e., they both encounter problems of the form (4) and (5)), it is reasonable to expect that they give value to non-replicable contingent claims using the indifference pricing rule. Then, two cases are possible, namely, $v^{(s)}(B) \leq v^{(b)}(B)$ and $v^{(s)}(B) > v^{(b)}(B)$. In the former case, the indifference pricing rule leads to an *agreement* price (see [4] for the exact definition of agreement and [27] for a relevant discussion), in the sense that there is a price $P \in [v^{(s)}(B), v^{(b)}(B)]$, the transaction at which implies gain in terms of indirect utility for at least one of the agents. However, determination of the exact price P at which the transaction will take place needs an extra criterion. In the latter situation, the indifference pricing rule implies no agreement between the agents, in other words, transaction at any price leads to loss of indirect utility for at least one of the agents. Hence even in this case, provided though that the agents have to proceed to the transaction, a criterion is needed in order to determine the exact price at which the transaction will take place. As mentioned in the introduction, the market clearing principle (supply equals the demand) that leads to a (partial) equilibrium prices is not applicable in this case, since the payoff B is given and indivisible (determination of the partial equilibrium price in similar situations is established in [3]).

A desired criterion can be given through the functions $P^{(s)}(.)$ and $P^{(b)}(.)$. For this, we consider the following minimization problem:

$$\min_{(\varepsilon_s,\varepsilon_b)\in A_s\times A_b} \{\lambda\varepsilon_s + (1-\lambda)\varepsilon_b\} \quad \text{subject to } P^{(s)}(\varepsilon_s) \leq P^{(b)}(\varepsilon_b) \tag{18}$$

where, $\lambda \in (0,1)$ and $\varepsilon_s, \varepsilon_b$ are the indirect utilities differentials of the seller and the buyer respectively. Assuming the existence of a solution $(\varepsilon_s^*, \varepsilon_b^*)$ of (18), we are then able to define a commonly agreed price, $P = P^*$, for the claim B by

$$P^* := P^{(s)}(\varepsilon_s^*) = P^{(b)}(\varepsilon_b^*)$$

In words, according to the above scenario, the agents will proceed to the transaction on claim B when a convex combination (a weighted average) of their indirect utilities differentials is minimized. This convex combination corresponds to an allocation, between the two agents, of the total utility gain (loss) that would result from the completion of the transaction. As Theorem 3.1 states, the uniqueness of the solution provides a unique price P^*, which will be called the *utility sharing price* of the claim B (see Definition 3.1 below) of these two agents.

The following theorem provides the existence and the uniqueness of the solution to the minimization problem (18).

Theorem 3.1. *Under the Assumption 2.1, for any choice of utilities U_s and U_b and for every weight $\lambda \in (0,1)$, the minimization problem (18) has a unique solution $(\varepsilon_s^*, \varepsilon_b^*) \in A_s \times A_b$, which provides a unique price $P^* = P^{(s)}(\varepsilon_s^*) = P^{(b)}(\varepsilon_b^*)$.*

Proof. First, notice that in (18), the inequality in the constraint can be replaced by the equality $P^{(s)}(\varepsilon_s) = P^{(b)}(\varepsilon_b)$. Indeed, if $(\bar{\varepsilon}_s, \bar{\varepsilon}_b) \in A_s \times A_b$ minimizes $\lambda\varepsilon_s + (1-\lambda)\varepsilon_b$ and $P^{(s)}(\bar{\varepsilon}_s) < P^{(b)}(\bar{\varepsilon}_b)$, there exists $\delta > 0$ such that $\bar{\varepsilon}_s - \delta \in A_s$ and $P^{(s)}(\bar{\varepsilon}_s - \delta) = P^{(b)}(\bar{\varepsilon}_b) \in \mathbb{R}$ (since $P^{(s)}$ is strictly decreasing and $\sup\{P^{(s)}(\varepsilon) : \varepsilon \in A_s\} = +\infty$). But then $\lambda(\bar{\varepsilon}_s - \delta) + (1-\lambda)\bar{\varepsilon}_b < \lambda\bar{\varepsilon}_s + (1-\lambda)\bar{\varepsilon}_b$, which is a contradiction.

Now, since $P^{(s)}$ $(P^{(b)})$ is in $C^1(A_s)$ $(C^1(A_b))$, we apply the Lagrange multiplier method in order to solve (18). To this end, we define the Lagrangian $L : A_s \times A_b \times \mathbb{R} \to \mathbb{R}$

$$L(\varepsilon_s, \varepsilon_b, m) = \lambda\varepsilon_s + (1-\lambda)\varepsilon_b + m\left(P^{(s)}(\varepsilon_s) - P^{(b)}(\varepsilon_b)\right)$$

and we are looking for the triples $(\varepsilon_s, \varepsilon_b, m)$ that solve the following system

$$\frac{\partial L}{\partial \varepsilon_s} = \frac{\partial L}{\partial \varepsilon_b} = \frac{\partial L}{\partial m} = 0$$

or equivalently

$$(P^{(s)})'(\varepsilon_s) = -\frac{\lambda}{m} \tag{19}$$

$$(P^{(b)})'(\varepsilon_b) = \frac{(1-\lambda)}{m} \tag{20}$$

$$P^{(s)}(\varepsilon_s) = P^{(b)}(\varepsilon_b) \tag{21}$$

Since $P^{(s)}$ is strictly decreasing and $P^{(b)}$ is strictly increasing, we can restrict ourselves to $m \in \mathbb{R}_+^*$.

Thanks to the representations (14) and (15) of the functions $P^{(s)}$ and $P^{(b)}$ with respect to the indirect utilities, equations (19) and (20) can be written as

$$u_s'\left(x_s + P^{(s)}(\varepsilon_s); -B\right) = \frac{m}{\lambda} \tag{22}$$

$$u_b'\left(x_b - P^{(b)}(\varepsilon_b); B\right) = \frac{m}{(1-\lambda)} \tag{23}$$

since

$$(P^{(s)})'(\varepsilon) = -\varphi_s'(u_s(x_s) - \varepsilon; -B) = -\frac{1}{u_s'\left(x_s + P^{(s)}(\varepsilon); -B\right)} \quad \text{for every } \varepsilon \in A_s$$

and

$$(P^{(b)})'(\varepsilon) = -\frac{1}{u_b'\left(x_b - P^{(b)}(\varepsilon); B\right)} \quad \text{for every } \varepsilon \in A_b.$$

By Theorem 2.1 (see also Remark 2.1), we have that for every $m \in \mathbb{R}_+^*$ and $\lambda \in (0,1)$, there exists a unique $k(m) \in \mathbb{R}$ such that

$$u_s'(k(m); -B) = \frac{m}{\lambda},$$

and since $P^{(s)}(A_s) = \mathbb{R}$, there exists a unique $\varepsilon_s^*(m) \in A_s$ for which (22) holds. Similarly for the buyer, for every $m \in \mathbb{R}_+^*$ and $\lambda \in (0,1)$, there exists a unique $\varepsilon_b^*(m) \in A_b$ that solves equation (23). The Lagrange multiplier m has to be assigned the value m^*, so that the constraint holds, that is

$$P^{(s)}(\varepsilon_s^*(m^*)) = P^{(b)}(\varepsilon_b^*(m^*)) \tag{24}$$

Hence, since the functions $\varepsilon_s^*(m)$ and $\varepsilon_b^*(m)$ are continuous, the solution of (18) follows from the solution m^* of the equation:

Since, $\varepsilon_s^*(m)$ and $\varepsilon_b^*(m)$ are strictly increasing and continuous (because u_s, u_b are so), $P^{(s)}(\varepsilon_s^*(m))$ is strictly decreasing and continuous and $P^{(b)}(\varepsilon_b^*(m))$ is strictly increasing and continuous as functions of m. Now, to establish that (24) has a unique solution, it is enough to ensure that

$$\lim_{m\to 0} P^{(s)}(\varepsilon_s^*(m)) > \lim_{m\to 0} P^{(b)}(\varepsilon_b^*(m)) \tag{25}$$

and

$$\lim_{m\to +\infty} P^{(s)}(\varepsilon_s^*(m)) < \lim_{m\to +\infty} P^{(b)}(\varepsilon_b^*(m)). \tag{26}$$

Indeed, we have

$$\lim_{m\to 0} P^{(s)}(\varepsilon_s^*(m)) = \lim_{m\to +\infty} P^{(b)}(\varepsilon_b^*(m)) = +\infty \tag{27}$$

$$\lim_{m\to 0} P^{(b)}(\varepsilon_b^*(m)) = \lim_{m\to +\infty} P^{(s)}(\varepsilon_s^*(m)) = -\infty \tag{28}$$

therefore, the required inequalities (25) and (26) hold for any choice of utilities and hence there exists a unique $m^* \in \mathbb{R}_+^*$, which solves (24). Given m^*, $\varepsilon_s^* = \varepsilon_s^*(m^*)$ and $\varepsilon_b^* = \varepsilon_b^*(m^*)$ solve (19) and (20) respectively. The fact that $(\varepsilon_s^*, \varepsilon_b^*)$ is then indeed the unique minimizer of (18) is a direct consequence of the strict convexity of $L(\varepsilon_s, \varepsilon_b, m)$ on $(\varepsilon_s, \varepsilon_b)$, for any given m. □

Thanks to the uniqueness of the solution of (18), we are able to define the *utility sharing price* as follows.

Definition 3.1. *The price $P^* = P^{(s)}(\varepsilon_s^*) = P^{(b)}(\varepsilon_b^*)$, where $(\varepsilon_s^*, \varepsilon_b^*)$ is the solution of problem (18) is called the utility sharing price of claim B for the two agents.*

Remark 3.1. Using the notation $X^{(s)}(P^{(s)}(\varepsilon)) := \underset{X\in\mathcal{X}_{U_s,-B}(x_s+P^{(s)}(\varepsilon))}{\text{argmax}} \mathbb{E}[U_s(X-B)]$, equation (10) can be written as

$$\mathbb{E}\left[U_s\left(X^{(s)}(P^{(s)}(\varepsilon)) - B\right)\right] = u_s(x_s) - \varepsilon.$$

Then, we define the seller's *residual risk* of claim B at price P (see also [38]) as

$$R^{(s)}(P) := X^{(s)}(P) - B. \tag{29}$$

Similarly, the buyer's residual risk of claim B at P defined as the difference $X^{(b)}(-P)+B$. Then problem (18) is equivalent to maximization of

$$\lambda \mathbb{E}[U_s(R^{(s)}(P^{(s)})(\varepsilon_s))] + (1-\lambda)\mathbb{E}[U_b(R^{(b)}(P^{(b)})(\varepsilon_b))] \tag{30}$$

Equation (30) describes the total utility in terms of residual risks. The price P^* is the maximizer of (30) and corresponds to the optimal residual risk allocation according to the sharing rule λ.

Remark 3.2. The utility sharing parameter λ decides how the gain or loss of utility is going to be distributed between the two agents and it reflects the relative *bargaining power* between the two agents. It would be interesting to examine even further the nature and consistency of λ, through properly designed experiments but this is clearly beyond the scope of this work.

The following proposition discusses the effect of parameter λ on the utility sharing price.

Proposition 3.1. *The utility sharing price P^* is continuous and strictly increasing function of $\lambda \in (0,1)$.*

Proof. We first show that P^* is strictly increasing with respect to λ. Consider two strictly positive numbers $\lambda_1 < \lambda_2 < 1$. Since $u_s'(.;-B)$ and $P^{(s)}(.)$ are strictly decreasing functions (see Theorem 2.1), equation (22) implies that $\varepsilon^*_{s,\lambda_1}(m) > \varepsilon^*_{s,\lambda_2}(m)$, for every $m \in \mathbb{R}_+$, where $\varepsilon^*_{s,\lambda}(m)$ denotes the solution the equation (22). Similar arguments, implies $\varepsilon^*_{b,\lambda_1}(m) < \varepsilon^*_{b,\lambda_2}(m), \forall m \in \mathbb{R}_+$, where $\varepsilon^*_{b,\lambda}(m)$ denotes the solution of equation (23). Therefore,

$$P^{(s)}(\varepsilon^*_{s,\lambda_1}(m)) < P^{(s)}(\varepsilon^*_{s,\lambda_2}(m)) \quad \text{and} \quad P^{(b)}(\varepsilon^*_{b,\lambda_1}(m)) < P^{(b)}(\varepsilon^*_{b,\lambda_2}(m))$$

for every m, hence by (24), we obtain the desired monotonicity property of P^* with respect to λ.

The continuity with respect to λ is an easy consequence of Berge's maximum theorem; Indeed Berge's theorem (see e.g., [1], Theorem 16.31) guarantees the upper semicontinuity of the correspondence P^*_λ. Since, the solution of the problem is unique, this correspondence is single-valued, hence continuous. □

Since, functions $P^{(s)}$ and $P^{(b)}$ have unbounded ranges (see (14) and (15)), one can observe that for values of λ sufficiently close to 0 or 1, it may be the case that P^* lies outside the interval of non-arbitrage prices for B (see for example the exponential case (40) below, for $\gamma_s = \gamma_b$). This can be seen as an inconsistency, the resolution of which requires that lower and upper bounds are imposed on λ, which intuitively means that "extreme" loss (or gain) of utilities is not allowed. Note however that in some real situations, agents with weak bargaining power have to trade claims, which can well lead to arbitrage opportunities (weak bargaining power is indeed one of the reasons for the existence of arbitrage in real life).

In the light of Proposition 3.1, there exists a continuous function $b(.)$ (which in fact is the inverse of P^* as a function of λ) such that

$$g < P^* < G \;\Leftrightarrow\; b(g) < \lambda < b(G) \tag{31}$$

In particular, the utility sharing price P^* is an non-arbitrage, if and only if the weight of the pricing scheme λ belongs in the interval $(b(\inf_{\mathbb{Q}\in\mathcal{M}_e}\{\mathbb{E}_{\mathbb{Q}}[B]\}), b(\sup_{\mathbb{Q}\in\mathcal{M}_e}\{\mathbb{E}_{\mathbb{Q}}[B]\}))$. Those bounds can be considered as the limits on the bargaining power of agents that keep the prices within the non-arbitrage interval.

Remark 3.3. Notice the formal similarity of problem (18) with Hicks' expenditure minimization problem ([37], Chapter 3). In our setting, risk plays the rôle of expenditure, λ and $1-\lambda$ play the rôle of the prices, while the price functions $P^{(s)}$ and $P^{(b)}$ play the rôle of utilities. Though this similarity, we may clarify the effect of λ on the utility sharing price P^*. Furthermore, one can consider the dual problem, namely the maximization utility, which in our framework corresponds to the maximization of the difference of the reservation prices of the two agents, given the total quantity of utility to be undertaken.

Remark 3.4. The utility sharing pricing scheme as provided by the solution of (18), gives a price for B traded at $t=0$; it can be extended to include the case where the claim can be traded at any time $t\in[0,T]$.

The utility maximization problems (4) and (5) can be naturally expressed in a dynamic way, i.e., for any time $t\in[0,T]$, we consider the problem

$$u(x,t;B) := \sup_{X\in\mathcal{X}_{U,B}(x,t)} \mathbb{E}[U\left(X+B\right) \mid \mathcal{F}_t] \tag{32}$$

where $\mathcal{X}_{U,B}(x,t)$ is defined similarly as in (3), taking into account the market conditions from time t onwards and x stands for the agent's (random) wealth at time t. Using the notation $u(x,t) = u(x,t;0)$, we can define the indirect utility differential at time t of the writer and the buyer of the given claim B, by

$$\varepsilon_{s,t} := u\left(x,t\right) - u\left(x+P,t;-B\right) \quad \text{and} \quad \varepsilon_{b,t} := u\left(x,t\right) - u_t\left(x-P,t;B\right), \tag{33}$$

so that the reservation prices for the seller and the buyer at time t given that utility differential ε is undertaken is given by the solutions of the equations

$$\varepsilon = u\left(x,t\right) - u\left(x+P^{(s)}_t(\varepsilon),t;-B\right) \text{ and } \varepsilon = u\left(x,t\right) - u\left(x-P^{(b)}_t(\varepsilon),t;B\right).$$

Hence, we can consider the family of minimization problems

$$\min_{(\varepsilon_s,\varepsilon_b)\in A_{s,t}\times A_{b,t}} \{\lambda\varepsilon_s + (1-\lambda)\varepsilon_b\} \tag{34}$$

$$\text{subject to } P_t^{(s)}(\varepsilon_s) \leq P_t^{(b)}(\varepsilon_b)$$

where, $A_{s,t} := (u_s(x_s, t) - U(+\infty), +\infty)$ and $A_{b,t} := (u_b(x_b, t) - U(+\infty), +\infty)$.

By similar assumptions and arguments as in problem (18), one can conclude the existence of (unique) solutions P_t^* for the family of problems (34).

Instead of providing the proof for the above arguments, which follows closely that of Theorem 3.1, we prefer to give a representative example of how problem (34) can be used to determine the price of claims that can be traded at any time before maturity in Section 4.

4. The Case of the Exponential Utility

The exponential utility, defined as $U(x) = -e^{-\gamma x}$, where $\gamma > 0$ is the risk aversion coefficient, is a widely used utility function in the literature, because it offers closed form solutions to a variety of utility maximization problems. In this section, we show that in the case where both agents have exponential utilities, problem (18), can be solved explicitly, thus leading to a closed form expression for the utility sharing price P^*. This allows us to perform a detailed analysis of the properties of P^*.

In the special case of the exponential utility, the problem of choosing the set of admissible strategies $\mathbf{\Theta}$ simplifies considerably to

$$\mathbf{\Theta}_{exp} := \left\{\vartheta : (\vartheta \cdot \mathbf{S})_{t\in[0,T]} \text{ is a true-martingale under any } \mathbb{Q} \in \mathcal{M}_{e,f}\right\}$$

where $\mathcal{M}_{e,f} := \{\mathbb{Q} \in \mathcal{M}_e : \mathcal{H}(\mathbb{Q}|\mathbb{P}) < +\infty\}$ (assumed to be non-empty) and $\mathcal{H}(\mathbb{Q}|\mathbb{P})$ is the relative entropy with respect to the probability measure $\mathbb{P}$, which is defined as follows

$$\mathcal{H}(\mathbb{Q}|\mathbb{P}) = \begin{cases} \mathbb{E}\left[\frac{d\mathbb{Q}}{d\mathbb{P}} \ln\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)\right], & \mathbb{Q} \ll \mathbb{P}, \\ +\infty, & \text{otherwise.} \end{cases}$$

(see e.g., [14], [28], [36]). The occurrence of the relative entropy in the definition of the set of the admissible strategies is not coincidental, since as showed in [14] and [22], the entropy minimization problem is the dual of the exponential utility maximization problem.

For every claim $B \in \mathbb{L}^\infty$, the utility maximization (4) is well-defined, the supremum is attained, the indirect utility, $u(x;B)$ is of the form

$$u(x;B) = e^{-\gamma x} u(0;B), \tag{35}$$

is finite, for every initial wealth $x \in \mathbb{R}$ and both the optimal strategy and the indifference prices are independent of x.

4.1. **On Price P^* and its Properties.** Let γ_s and $\gamma_b > 0$ be the seller's and buyer's risk aversion coefficients respectively. Using (35), (14) and (15) simplify to:

$$P^{(s)}(\varepsilon) = \frac{1}{\gamma_s} \ln\left(\frac{u_s(x_s; -B)}{u_s(x_s) - \varepsilon}\right) = v^{(s)}(B) - \frac{1}{\gamma_s} \ln\left(1 - \frac{\varepsilon}{u_s(x_s)}\right), \text{ for } \varepsilon \in A_s = (u_s(x_s), +\infty) \quad (36)$$

and

$$P^{(b)}(\varepsilon) = \frac{1}{\gamma_b} \ln\left(\frac{u_b(x_b) - \varepsilon}{u_b(x_b;B)}\right) = v^{(b)}(B) + \frac{1}{\gamma_b} \ln\left(1 - \frac{\varepsilon}{u_b(x_b)}\right), \text{ for } \varepsilon \in A_b = (u_b(x_b), +\infty), \quad (37)$$

where $v^{(s)}(B)$ and $v^{(b)}(B)$ denote the seller's and the buyer's indifference prices and $u_s(x_s)$, $u_b(x_b)$ their respective indirect utilities when no position on the claim B is undertaken.

Remark 4.1. Using equations (36) and (37), and straight-forward algebra, we observe that

$$(P^{(s)})'(\varepsilon) = \frac{1}{\gamma_s} \frac{1}{u_s(x_s) - \varepsilon} \quad \text{and} \quad (P^{(b)})'(\varepsilon) = -\frac{1}{\gamma_b} \frac{1}{u_b(x_b) - \varepsilon}, \quad (38)$$

while,

$$\frac{\partial P^{(s)}}{\partial x_s} = \frac{\varepsilon}{u_s(x_s) - \varepsilon} \quad \text{and} \quad \frac{\partial P^{(b)}}{\partial x_b} = \frac{\varepsilon}{\varepsilon - u_b(x_b)}$$

i.e., the sensitivity of the reservation prices with respect to the risk undertaken and the initial wealth are independent of the nature of the claim B. The above calculations indicates that given a loss of utility ($\varepsilon > 0$), a seller with greater initial wealth asks for lower prices, while given a gain of utility ($\varepsilon < 0$), she asks for higher prices. Similar results can be drawn for the buyer's side.

For the case of exponential utility, we can solve the problem (18) explicitly.

Proposition 4.1. *The utility sharing price is given by*

$$P^* = P^{(s)}(\varepsilon_s^*) = P^{(b)}(\varepsilon_b^*) = \frac{\gamma_s v^{(s)}(B) + \gamma_b v^{(b)}(B)}{\gamma_s + \gamma_b} + \frac{1}{\gamma_s + \gamma_b} \ln\left(\frac{u_s(x_s)\gamma_s\lambda}{u_b(x_b)\gamma_b(1-\lambda)}\right) \quad (39)$$

Proof. Consider again the Lagrangian

$$L(\varepsilon_s, \varepsilon_b, m) = \lambda\varepsilon_s + (1-\lambda)\varepsilon_b + m\left(P^{(s)}(\varepsilon_s) - P^{(b)}(\varepsilon_b)\right)$$

where, $(\varepsilon_s, \varepsilon_b) \in A_s \times A_b$ and $m \in \mathbb{R}_+$. Using (38), the first order conditions yield

$$\varepsilon_s^*(m) = u_s(x_s) + \frac{m}{\lambda\gamma_s} \quad \text{and} \quad \varepsilon_b^*(m) = u_b(x_b) + \frac{m}{(1-\lambda)\gamma_b}.$$

The value of the Lagrange multiplier m^* such that the constraint is satisfied is given by the solution of the algebraic equation

$$P^{(s)}(\varepsilon_s^*(m^*)) = P^{(b)}(\varepsilon_b^*(m^*))$$

which using (36) and (37) readily gives

$$m^* = (-u_s(x_s)\lambda\gamma_s)^{\frac{\gamma_b}{\gamma_s+\gamma_b}} (-u_b(x_b)(1-\lambda)\gamma_b)^{\frac{\gamma_s}{\gamma_s+\gamma_b}} \exp\left(\frac{\gamma_s\gamma_b}{\gamma_s+\gamma_b}(v^{(s)}(B) - v^{(b)}(B))\right).$$

Therefore, the utility sharing price is

$$P^* = P^{(s)}(\varepsilon_s^*) = P^{(b)}(\varepsilon_b^*) = \frac{\gamma_s v^{(s)}(B) + \gamma_b v^{(b)}(B)}{\gamma_s + \gamma_b} + \frac{1}{\gamma_s + \gamma_b} \ln\left(\frac{u_s\left(x_s\right)\gamma_s \lambda}{u_b\left(x_b\right)\gamma_b(1-\lambda)}\right) \tag{40}$$

Since, the indirect utilities have the necessary smoothness, we can conclude that the critical point is a minimum if the determinant of the matrix

$$H = \begin{pmatrix} \frac{\partial^2 L}{\partial \varepsilon_s^2}\left(\varepsilon_s^*, \varepsilon_b^*\right) & \frac{\partial^2 L}{\partial \varepsilon_s \partial \varepsilon_b}\left(\varepsilon_s^*, \varepsilon_b^*\right) & \frac{\partial \phi}{\partial \varepsilon_s}\left(\varepsilon_s^*, \varepsilon_b^*\right) \\ \frac{\partial^2 L}{\partial \varepsilon_s \partial \varepsilon_b}\left(\varepsilon_s^*, \varepsilon_b^*\right) & \frac{\partial^2 L}{\partial \varepsilon_b^2}\left(\varepsilon_s^*, \varepsilon_b^*\right) & \frac{\partial \phi}{\partial \varepsilon_b}\left(\varepsilon_s^*, \varepsilon_b^*\right) \\ \frac{\partial \phi}{\partial \varepsilon_s}\left(\varepsilon_s^*, \varepsilon_b^*\right) & \frac{\partial \phi}{\partial \varepsilon_b}\left(\varepsilon_s^*, \varepsilon_b^*\right) & 0 \end{pmatrix}$$

is negative, where $\phi\left(\varepsilon_s, \varepsilon_b\right) := P^{(s)}\left(\varepsilon_s\right) - P^{(b)}\left(\varepsilon_b\right)$. Indeed,

$$Det(H) = -P^{(s)\prime\prime}\left(\varepsilon_s^*\right)\frac{(1-\lambda)^2}{m^*} + \frac{\lambda}{m^*}P^{(b)\prime\prime}\left(\varepsilon_b^*\right) < 0$$

since $P^{(s)}$ is strictly convex and $P^{(b)}$ is strictly concave. □

Remark 4.2. In the special case where the agents have the same characteristics ($x_s = x_b = x$ and $\gamma_s = \gamma_b = \gamma$), the utility sharing price simplifies to

$$P^* = \frac{v^{(s)}(B) + v^{(b)}(B)}{2} + \frac{1}{2\gamma}\ln\left(\frac{\lambda}{1-\lambda}\right). \tag{41}$$

Therefore, the proposed price is the midpoint between the two agents' indifference prices plus a correction term reflecting the risk aversion and the utility sharing rule λ. This correction becomes zero if and only if $\lambda = 1/2$, i.e., if the corresponding utility is equally shared among the agents. The difference between their indirect utilities differentials that leads to the price P^* is given by

$$\varepsilon_s^* - \varepsilon_b^* = \sqrt{u\left(x;B\right)u\left(x;-B\right)}\left(\frac{1-2\lambda}{\sqrt{\lambda\left(1-\lambda\right)}}\right)$$

This difference is positive for $\lambda < 1/2$, negative for $\lambda > 1/2$ and zero if $\lambda = 1/2$.

Remark 4.3. Formula (40), can easily reveal the sensitivity of the utility sharing price on the risk aversion coefficients of the agents. Namely, P^* is an increasing function of γ_s and a decreasing function of γ_b, reflecting the fact that as the seller becomes less risk averse, she is willing to ask lower prices while the opposite happens for the bid prices of the buyer.

In the special case where the agents have the same risk aversion coefficient, i.e., $\gamma_s = \gamma_b = \gamma$, P^* is an increasing function of γ when $\lambda < 1/2$ and decreasing when $\lambda > 1/2$. For $\lambda = 1/2$ and for $\gamma_s = \gamma_b = \gamma$ we have that $P^* = \frac{v^{(s)}(B)+v^{(b)}(B)}{2} + \frac{x_b - x_s}{2}$, which is generally not a monotonic function of γ. An interesting limit is the limit as $\gamma \to 0$; $\lim_{\gamma\to 0} P^* = \mathbb{E}_{\mathbb{Q}^{(0)}}[B] + \frac{x_b - x_s}{2}$, where $\mathbb{Q}^{(0)}$ is the measure that minimizes the relative entropy $\mathcal{H}(\mathbb{Q}|\mathbb{P})$ (see [14] on the behavior of $v^{(s)}(B)$ and $v^{(b)}(B)$ as γ goes to zero).

Remark 4.4. Explicit bounds on λ can be provided such that the utility sharing price is a non-arbitrage price; Using equation (40), one can see through straight-forward calculations that $b(\inf_{\mathbb{Q}\in\mathcal{M}_e}\{\mathbb{E}_{\mathbb{Q}}[B]\}) < \lambda < b(\sup_{\mathbb{Q}\in\mathcal{M}_e}\{\mathbb{E}_{\mathbb{Q}}[B]\})$, where

$$b(a) = \frac{K(a)}{K(a)+1},$$

while $K(a) = e^{a(\gamma_s+\gamma_b)}M_B(x_s,x_b,\gamma_s,\gamma_b)$ and $M_B(x_s,x_b,\gamma_s,\gamma_b) = e^{(-\gamma_s(v^{(s)}(B)-x_s)-\gamma_b(v^{(b)}(B)+x_b))}\frac{\gamma_b u_b(0)}{\gamma_s u_s(0)}$. These explicit bounds allow us to draw some interesting conclusions. For example, by keeping the rest of the parameters constant, we get that as γ_s (γ_b) goes to zero, the function $\frac{K(a)}{K(a)+1}$ goes to one (zero), for every $a \in \mathbb{R}$, i.e., as an agent reduces her aversion level, her loss of utility weight approaches one.

4.2. **Example: A European Claim on a non-traded Asset.** In this subsection, we build on the model proposed in [38], concerning the utility pricing of a European claim written on a non-traded asset. This model considers two traded assets, a riskless one (the numéraire) and a risky one, whose discounted price S_t follows the dynamics

$$dS_t \;=\; \mu S_t dt + \sigma S_t dW_t^{(1)}$$

where $W^{(1)}_{t\in[0,T]}$ is a standard Wiener process, $\mu \in \mathbb{R}$ and $\sigma \in \mathbb{R}^*_+$. It further assumes the existence of a non-traded asset, whose discounted price dynamics follow the stochastic differential equation

$$dY_t \;=\; b(Y_t,t)dt + a(Y_t,t)\left(\rho dW_t^{(1)} + \rho' dW_t^{(2)}\right)$$

where $(W_t^{(2)})_{t\in[0,T]}$ is another standard Wiener process (independent of $W_t^{(1)}$). Constant $\rho \in (-1,1)$ is the correlation coefficient between the factors driven the dynamics of the prices of the traded and the non-traded asset.

Consider now a European contingent claim B, whose payoff depends on the value Y_T of the non-traded asset at time T, through $g(Y_T)$, where $g : \mathbb{R} \to \mathbb{R}$ is a bounded Borel function, so that $B \in \mathbb{L}^\infty$. In [38], expressions for indirect utilities and the bid and ask indifference prices in the case of exponential utilities are stated. In general, these prices do not lead to an agreement concerning a commonly acceptable price in the the option can be traded. We now propose an alternative approach to this problem using the definition of the utility sharing price as extended in Remark 3.4. The following theorem, which quotes the relevant results of [38] concerning the indirect utilities achieved, needed in this work, is included to enhance the readability of the present paper.

Theorem 4.1. *(M. Musiela and T. Zariphopoulou, 2004)*
Let $\mathbb{Q}^{(0)}$ be the minimal relative entropy martingale probability measure and

$$\widetilde{\mathcal{X}}_{exp}(x,t) := \left\{ x + \int_t^T \vartheta_u dS_u : \text{ for } (\vartheta_u)_{u\in[t,T]} \text{ such that } \mathbb{E}\left[\int_t^T \vartheta_u^2 d_u\right] < \infty \right\}.$$

Then,

(i) *The indirect utility of an agent with is no position on the contingent claim is*

$$u(x,t) = \sup_{X \in \widetilde{\mathcal{X}}_{exp}(x,t)} \mathbb{E}\left[-e^{-\gamma X} \,\middle|\, \mathcal{F}_t\right] = -e^{-\gamma x}\, e^{-\frac{\mu^2 (T-t)}{2\sigma^2}}$$

(ii) *The indirect utility for the seller, $u(x,y,t;-g(Y_T))$ is given by*

$$u(x,y,t;-g(Y_T)) = -e^{-\gamma_s x}\left(\mathbb{E}_{\mathbb{Q}^{(0)}}\left[e^{\gamma_s (1-\rho^2) g(Y_T)} e^{-\frac{(1-\rho^2)\mu^2 (T-t)}{2\sigma^2}} \,\middle|\, Y_t = y, \mathcal{F}_t\right]\right)^{\frac{1}{(1-\rho^2)}}$$

(iii) *The indirect utility for the buyer, $u(x,y,t;g(Y_T))$ is given by*

$$u(x,y,t;g(Y_T)) = -e^{-\gamma_b\, x}\left(\mathbb{E}_{\mathbb{Q}^{(0)}}\left[e^{-\gamma_b (1-\rho^2) g(Y_T)} e^{-\frac{(1-\rho^2)\mu^2 (T-t)}{2\sigma^2}} \,\middle|\, Y_t = y, \mathcal{F}_t\right]\right)^{\frac{1}{(1-\rho^2)}}.$$

For details on the measure $\mathbb{Q}^{(0)}$, we refer the interested reader to [38], Theorem 2.

We are now able to characterize the price functions $P^{(s)}$ and $P^{(b)}$ and the utility sharing price at time t .

Proposition 4.2.

(i) *The seller's price $P_t^{(s)}(\varepsilon)$ is given by*

$$P_t^{(s)}(\varepsilon) = v_t^{(s)}(B) - \frac{1}{\gamma_s} \ln\left(1 + \varepsilon \delta_t^{(s)}\right)$$

where

$$v_t^{(s)}(B) = \frac{1}{(1-\rho^2)\gamma_s} \ln\left(\mathbb{E}_{\mathbb{Q}^{(0)}}\left[e^{\gamma_s (1-\rho^2) g(Y_T)} \,\middle|\, Y_t = y, \mathcal{F}_t\right]\right)$$

is the seller's indifference price at time t and $\delta_t^{(s)} := e^{\gamma_s\, x_s + \frac{\mu^2 (T-t)}{2\sigma^2}}$.

(ii) *The buyers price $P_t^{(b)}(\varepsilon)$ is given by*

$$P_t^{(b)}(\varepsilon) = v_t^{(b)}(B) + \frac{1}{\gamma_b} \ln\left(1 + \varepsilon \delta_t^{(b)}\right)$$

where

$$v_t^{(b)}(B) = -\frac{1}{(1-\rho^2)\gamma_b} \ln\left(\mathbb{E}_{\mathbb{Q}^{(0)}}\left[e^{-\gamma_b (1-\rho^2) g(Y_T)} \,\middle|\, Y_t = y, \mathcal{F}_t\right]\right)$$

is the buyer's indifference price at time t and $\delta_t^{(b)} := e^{\gamma_b\, x_b + \frac{\mu^2 (T-t)}{2\sigma^2}}$.

(iii) *The utility sharing price P_t^* for any time $t \in [0,T]$, is given by*

$$P_t^* = \frac{\gamma_s v_t^{(s)}(B) + \gamma_b v_t^{(b)}(B)}{\gamma_s + \gamma_b} + \frac{1}{\gamma_s + \gamma_b} \ln\left(\frac{\gamma_s \lambda \delta_t^{(b)}}{\gamma_b (1-\lambda) \delta_t^{(s)}}\right) \tag{42}$$

Proof. Using (14) and (15) and Theorem 4.1, the proof of (i) and (ii) is straight-forward. Taking into account Theorem 4.1 and formula (40), we obtain the solution of the dynamic version of problem (34) as stated in (iii). □

Remark 4.5. The function $P_t^{(s)}(\varepsilon)$ can be considered as a function of time t and the market condition at this time, $y = Y_t$, and as such it can be showed to be the solution of a deterministic quasilinear PDE. For fixed ε, define

$$P_t^{(s)}(\varepsilon) =: P^{(s)}(y,t) =: \frac{1}{\gamma_s(1-\rho^2)} \ln \Phi^{(s)}(y,t).$$

Using the Feynman-Kac representation, one may show by extension of the arguments in [38], that $\Phi^{(s)}(y,t)$ is the solution of the linear backward Cauchy problem

$$\frac{\partial \Phi^{(s)}}{\partial t} + \frac{1}{2} a(y,t)^2 \frac{\partial^2 \Phi^{(s)}}{\partial y^2} + \left(b(y,t) - \frac{\rho\mu}{\sigma} a(y,t) \right) \frac{\partial \Phi^{(s)}}{\partial y} + \frac{R^{(s)\prime}(t)}{R^{(s)}(t)} W^{(s)} = 0$$

where

$$W^{(s)}(y,T) := \frac{e^{\gamma_s(1-\rho^2)g(y)}}{(1+\varepsilon e^{\gamma_s x_s})^{(1-\rho^2)}},$$

and

$$R^{(s)}(t) := (1+\varepsilon\delta_t^{(s)})^{(1-\rho^2)}.$$

By straight-forward algebraic manipulation, we get that the price is the solution of a quasilinear deterministic PDE of the form

$$\frac{\partial P^{(s)}}{\partial t} + \frac{1}{2} a(y,t)^2 \frac{\partial^2 P^{(s)}}{\partial y^2} + \left(b(y,t) - \frac{\rho\mu}{\sigma} a(y,t) \right) \frac{\partial P^{(s)}}{\partial y} + \frac{1}{2}\gamma_s(1-\rho^2) a(y,t)^2 \left(\frac{\partial P^{(s)}}{\partial y} \right)^2 + \Lambda^{(s)\prime}(t) = 0$$

with final condition

$$P^{(s)}(y,T) = g(y) - \frac{1}{\gamma_s} \ln(1+\varepsilon e^{\gamma_s x_s})$$

where

$$\Lambda^{(s)}(t) := \frac{1}{\gamma_s} \ln(1+\varepsilon\delta_t^{(s)}).$$

The indifference prices are recovered when setting $R^{(s)}(t) = 0$ and $\Lambda^{(s)}(t) = 0$ for all $t \in [0,T]$ in the above PDEs. Similar arguments give a quasilinear PDE for the evolution of the buyer's price after undertaking some risk. However, even though the utility sharing price is a linear combination of the indifference prices plus the addition of a known time dependent only factor, the quasilinear nature of the PDEs that the indifference prices satisfy, do not allow us to write down a single PDE that the utility sharing price as a function of $Y_t = y$ and t satisfies.

5. Extensions, Directions for Future Research and Conclusion

In this section, we discuss some possible extensions of the pricing scheme proposed in this paper and conclude.

5.1. **Utility Sharing Price Using Risk Measures.** Consider that the agents model their risk preferences using convex risk measures ρ_s and ρ_b respectively, rather than expected utility functions. Since they have access to a liquid market, their investment goal is to minimize their risk, as quantified by the employed risk measures, through trading into this market. More precisely, for a contingent claim B, the agents' *marketed risk measures* (as defined in [21], [23], [51]) are given by

$$\hat{\rho}_s(x_s; B) := \inf_{X \in \mathcal{X}(x_s)} \rho_s(X + B) \quad \text{and} \quad \hat{\rho}_b(x_b; B) := \inf_{X \in \mathcal{X}(x_b)} \rho_b(X + B).$$

Along the lines of equations (10) and (11), we can define the analogue to the reservation prices $P^{(s)}$ and $P^{(b)}$, when the risk is represented by $\hat{\rho}_s$ and $\hat{\rho}_b$, as the solutions of the following equations

$$\varepsilon_s = -\hat{\rho}_s(x_s; 0) + \hat{\rho}_s(x_s + P^{(s)}(\varepsilon_s); -B) \quad \text{and} \quad \varepsilon_b = -\hat{\rho}_b(x_b; 0) + \hat{\rho}_b(x_b - P^{(b)}(\varepsilon_b); B).$$

Using these definitions and similar arguments as in Section 3, we define a pricing scheme as dictated by the solution of the minimization problem (18). The convexity of the risk measures guarantees that problem (18) is well-posed. Notice however, that if the marketed risk measures are cash invariant, the above general arguments may break down since then problem (18) is equivalent to the minimization of

$$\lambda\varepsilon_s + (1-\lambda)\varepsilon_b = c(B) + P^{(s)}(\varepsilon_s)(2\lambda - 1) \tag{43}$$

under the constraint $P^{(s)}(\varepsilon_s) = P^{(b)}(\varepsilon_b)$, where $c(B) := \lambda(\hat{\rho}_s(0;0) - \hat{\rho}_s(0;-B)) + (1-\lambda)(\hat{\rho}_b(0;0) - \hat{\rho}_b(0;B))$ which is ill-posed.

However, even though the risk measures ρ_s and ρ_b are by definition cash invariant, the corresponding property for the marketed risk measures requires that $\mathcal{X}(x+y) = \mathcal{X}(x) + y$, for every possible initial wealths x and y. This clearly does not hold in most cases of some practical interest, (e.g. when borrowing constraints are imposed). Therefore in general, minimization of $\lambda\varepsilon_s + (1-\lambda)\varepsilon_b$ is a well-defined problem leading to meaningful risk sharing prices.

5.2. **Optimal Trading Time.** The family of utility sharing prices for the asset at time t as given by equation (42) is a stochastic process, on account of the stochastic nature of Y_t. At the cost of considerable algebraic manipulations, one may apply Itô's rule and determine the evolution of the process P_t^* as a stochastic differential equation. For every time $t \in [0, T]$ the price P_t^*, as given by equation (42), is the price corresponding to the minimal total risk that the two agents have to undertake, at any time t, so that the transaction may take place. However, this total risk is a stochastic process itself since it depends on the market conditions at time t. We may then consider the stochastic process $\varepsilon_t^* := \lambda\varepsilon_{s,t}^* + (1-\lambda)\varepsilon_{b,t}^* = \mathcal{E}(t, Y_t)$ which is the minimal total risk allowing the transaction at time t. The function $\mathcal{E}$ is a deterministic function the form of which is known explicitly (see Proposition 4.1). In principle, a straight-forward application of Itô's lemma gives us the stochastic differential equation this process satisfies. One may now consider the following problem: Suppose that we can find a stopping time $\tau \in [0, T]$ such that the expectation of the total risk ε_τ^* is minimized, over all such stopping times. When choosing to trade the contingent claim at this time, clearly both agents undertake the minimum possible total expected risk. We may then define this time τ as the *optimal trading time* for the contract, and the price P_τ^* is then the optimal trading price. The resolution of this problem can be based on optimal stopping techniques, using variational inequalities, the solution of which will help us to determine the optimal stopping rule.

5.3. **Different Forms of the Total Risk.** The expression of the total risk undertaken by the agents, can be generalized to $\lambda\psi_s(\varepsilon_s) + (1-\lambda)\psi_b(\varepsilon_b)$, where $\lambda \in (0,1)$ and $\psi_s(.)$, $\psi_b(.)$ are strictly convex increasing functions. We then may consider the problem

$$\min_{(\varepsilon_s,\varepsilon_b)\in A_s\times A_b} \{\lambda\psi_s(\varepsilon_s) + (1-\lambda)\psi_b(\varepsilon_b)\} \tag{44}$$

$$\text{subject to } P^{(s)}(\varepsilon_s) \leq P^{(b)}(\varepsilon_b)$$

In the special case where $\psi_s(\varepsilon) = \psi_b(\varepsilon) = \varepsilon$, for every ε, problem (44) reduces to problem (18).

Through the solution of problem (44), we can define a risk sharing price. The solvability of (44) follows along the same lines of the proof of Theorem 3.1.

This generalization is important in its own right, since it may offer the ground for relaxing the cardinality assumption on the utility functions.

5.4. **Conclusion.** In this work, we considered two agents, a seller and a buyer, who are interested in trading a given, non-divisible, non-replicable contingent claim. In order to agree on a commonly accepted price, out of the infinity of possible non-arbitrage prices, they have to proceed to some kind of bargaining. We proposed a pricing mechanism that establishes natural and correspondence between the transaction price and the relative bargaining power of the two agents According to this mechanism, the transaction price is determined by the minimization of the total risk undertaken according to a fixed sharing rule, under the constraint that the transaction is feasible. We proved that such a problem is well-posed and we illustrated in detail this pricing mechanism in the special case of the exponential utility.

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Appendix A. Utility Functions on the Positive Real Line

In this Appendix, we provide the necessary technical details that guarantee the validity of our results in the case where one or both agents have utility function defined on the positive real line.

A.1. **Assumptions on the Utility Functions.** The second type of utilities includes the functions that are defined only for positive wealth, i.e.,

$$U : \mathbb{R}_+ \to \mathbb{R}$$

where we set $U(x) = -\infty$ for every $x < 0$. U is taken to be strictly increasing, strictly concave and continuously differentiable on $\mathbb{R}$ and to satisfy the Inada conditions:

$$\lim_{x\to 0} U'(x) := U'(0) = +\infty \text{ and } \lim_{x\to+\infty} U'(x) := U'(+\infty) = 0. \tag{45}$$

Moreover, we assume reasonable asymptotic elasticity of U, meaning that

$$\limsup_{x\to+\infty} \frac{xU'(x)}{U(x)} < 1 \tag{46}$$

(see [11], [25] and [33]). Again, $U(+\infty)$ stands for $\lim_{x\to+\infty} U(x)$.

A.2. **Admissible Strategies and Initial Wealths.** For utilities that are defined only for positive wealths, the properties of indirect utility have been given in the seminal works of D. Kramkov and W. Schachermayer, [33], for the case of no random endowment (i.e., no random liability B) and J. Cvitanić, W. Schachermayer and H. Wang, [11], under the presence of random endowment. In this case, for every initial wealth $x \in \mathbb{R}_+$, the set of admissible wealths $\mathcal{X}(x)$ is

$$\mathcal{X}(x) = \{X \in \mathbb{L}^0(\Omega, \mathcal{F}_T, \mathbb{P}) : X \leq (\vartheta \cdot S)_T + x \text{ for some } \vartheta_s \in \mathbf{\Theta}\} \tag{47}$$

where $\mathbf{\Theta}$ is the set of admissible strategies given in Section 2. We also need to define the set

$$\mathcal{D} := \{\mathbb{Q} \in (\mathbb{L}^\infty)^* : \|\mathbb{Q}\| = 1 \text{ and } \langle \mathbb{Q}, X\rangle \leq 0,\ \forall X \in \mathcal{C}\} \tag{48}$$

where $\mathcal{C} = \mathbf{\Theta} \cap \mathbb{L}^\infty$ and $\langle .,. \rangle$ is the duality pairing of $\mathbb{L}^\infty$ and the space of finitely additive measures, $(\mathbb{L}^\infty)^*$. Finally, given any claim $B \in \mathbb{L}^\infty$, we set $x_0(B) := \sup_{\mathbb{Q}\in\mathcal{D}} \langle \mathbb{Q}, B\rangle$ (notice that in [15], Theorem 5.6, it is stated that $\mathcal{M}_e \subseteq \mathcal{D}$).

The following theorem, quoted from [11] (Theorem 3.1 and Lemma 4.3) and [33] (Theorem 2.2) gives the properties which are needed in the present work.

Theorem A.1. *(K. Kramkov and W. Schachermayer, 1999, J. Cvitanić, W. Schachermayer and H. Wang, 2001)*
Assume that:

(i) $B \in \mathbb{L}^\infty$.
(ii) *Assumption 2.1 holds (see Section 2).*
(iii) *There exists* $y \in \mathbb{R}_+$ *such that* $u(y;B) < U(\infty)$ *(where* $u(.,B)$ *is defined (5)).*

Then, there exists a unique optimal solution in the problem (5) for every $x > x_0(-B)$. Furthermore, the value function $u(.;B)$ is finitely valued, strictly increasing and strictly concave and continuously differentiable on $(x_0(-B), +\infty)$, $u(x;B) = -\infty$ for every $x < x_0(-B)$ with $\lim_{x \to x_0(-B)} u(x;B) = -\infty$. Finally,

$$\lim_{x \to +\infty} u'(x;B) = 0 \text{ and } \lim_{x \to x_0(-B)} u'(x;B) = \infty. \tag{49}$$

Remark A.1. Since, $u(x;B)$ is strict concave, $C^1(x_0(-B), +\infty)$ function of x, we obtain that $\frac{du(x;B)}{dx}$ is strictly decreasing for every $x > x_0(-B)$ and its range is $(0, +\infty)$. We also observe that $\lim_{x \to +\infty} u(x;B) = U(+\infty)$.

A.3. **Utilities of Both Types and the Utility Sharing Price.** Taking into account Theorem A.1 and the intervals (16) and (17), in the case of utility functions of this type, we impose the following assumption on the agents' initial wealths, in order to exclude infinite valued indirect utilities and to guarantee that the intersections of the images of the pricing rules $P^{(s)}$ and $P^{(b)}$ and the non-arbitrage prices for the fixed claim B are not empty.

Assumption A.1. *If the seller has utility of the second type, we assume that her initial wealth $x_s \in \mathbb{R}_+$ satisfies the following inequality*

$$x_s > \max\{x_0(B), x_0(B) - \sup_{\mathbb{Q} \in \mathcal{M}_e} \{\mathbb{E}_{\mathbb{Q}}[B]\}\} \tag{50}$$

Similarly, if the buyer has utility of the second type, we assume that her initial wealth $x_b \in \mathbb{R}_+$ satisfies the following inequality

$$x_b > \max\{x_0(-B), x_0(-B) + \inf_{\mathbb{Q} \in \mathcal{M}_e} \{\mathbb{E}_{\mathbb{Q}}[B]\}\} \tag{51}$$

Under the above assumptions, the pricing scheme induced by the problem (18) is valid even in the case where one or both of the agents have utility function of the second type. More precisely, we define the functions $P^{(s)}$ and $P^{(b)}$ in exactly the same way as in (14) and (15) with the difference that $P^{(s)}(A_{x_s}) = (x_0(B) - x_s, +\infty)$ and $P^{(b)}(A_{x_b}) = (-\infty, x_b - x_0(-B))$ and also that $\lim_{\varepsilon \to +\infty} P^{(s)}(\varepsilon) = x_0(B) - x_s$ and $\lim_{\varepsilon \to +\infty} P^{(b)}(\varepsilon) = x_b - x_0(-B)$.

Without repeating the proof of Theorem 3.1, we state its modification in this slightly generalized case.

Theorem A.2. *Let Assumptions 2.1 and A.1 hold. For any choice of utilities U_s, U_b (of either type) and for any weight $\lambda \in (0,1)$, the minimization problem (18) has a unique solution $(\varepsilon_s^*, \varepsilon_b^*) \in A_s \times A_b$. This provides a unique price $P^* = P^{(s)}(\varepsilon_s^*) = P^{(b)}(\varepsilon_b^*)$.*

Remark A.2. It should be mentioned that the only difference between the proof of Theorem 3.1 and Theorem A.2 is that in the case agents have utility function of the second type the limits that guarantee the existence of the Lagrange multiplier m^* have to be modified to

$$\lim_{m\to 0} P^{(s)}\left(\varepsilon_s^*(m)\right) = +\infty \text{ and } \lim_{m\to 0} P^{(b)}\left(\varepsilon_b^*(m)\right) = -\infty$$

$$\lim_{m\to +\infty} P^{(s)}\left(\varepsilon_s^*(m)\right) = x_0(B) - x_s \text{ and } \lim_{m\to +\infty} P^{(b)}\left(\varepsilon_b^*(m)\right) = x_b - x_0(-B)$$

(compare to limits given in (27)).

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