

On Agent's Agreement and Partial-Equilibrium Pricing in Incomplete Markets

September 2, 2008

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Abstract. We consider two risk-averse financial agents who negotiate the price of an illiquid indivisible contingent claim in an incomplete semimartingale market environment. Under the assumption that the agents are exponential utility maximizers with non-traded random endowments, we provide necessary and sufficient conditions for negotiation to be successful, i.e., for the trade to occur. We also study the asymptotic case where the size of the claim is small compared to the random endowments and we give a full characterization in this case. Finally, we study a partial-equilibrium problem for a bundle of divisible claims and establish existence and uniqueness. A number of technical results on conditional indifference prices is provided.

Key words and phrases. exponential utility, incomplete markets, indifference prices, conditional indifference prices, partial equilibrium, random endowment, risk-aversion, semimartingales.

2000 Mathematics Subject Classification. Primary: 91B70; Secondary: 91B30, 60G35.

Both authors were supported in part by the National Science Foundation under award number DMS-0706947 during the preparation of this work. Any opinions, findings and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect those of the National Science Foundation.

The authors would like to thank Hans Föllmer, Lorenzo Garlappi, Stathis Tompaidis, Thaleia Zariphopoulou, and the participants of the 11th International Congress on Insurance: Mathematics and Economics, IME, Piraeus, Greece, July 2007 for fruitful discussions and good advice. The constructive and deep comments made by two anonymous referees are very much appreciated.

1. INTRODUCTION

1.1. A description of the problem. In an ideal complete market, each contingent claim can be perfectly replicated and, thus, a rational agent is indifferent between the (random) claim itself and its (deterministic) replication price. Abundant empirical evidence shows that the real financial markets are far from complete; only a small portion of contingent claims can be replicated in the market to a satisfactory degree. A non-specific abstract notion of rationality is no longer sufficient to single out a unique “fair price” of any contingent claim. This effect is demonstrable in over-the-counter transactions where (typically) two agents negotiate a price of a single, indivisible, not-perfectly-replicable contingent claim. The final outcome of such a negotiation eventually hinges upon two idiosyncratic factors - the agents’ attitude towards risk and their negotiation skills. The focus of the present paper is the former. We ask the following question: *Under what conditions on the claim whose price is being negotiated, the liquid-market environment and the agents’ risk attitudes will a mutually beneficial agreement be feasible?*

Our modelling choices are informed by simplicity, but we steer clear of oversimplification. In particular, we assume that the two agents are expected utility maximizers in the von Neumann-Morgenstern sense, with a common investment horizon T . For simplicity and analytic tractability we assume that both agents’ utility functions are exponential, possibly with different risk-aversion parameters. An important feature which is not present in a major part of the past work on the subject is the presence of random endowments - the agents are assumed to hold an illiquid portfolio and the risk assessment of any contingent claim will depend heavily upon its (co-)relation with this illiquid portfolio. In addition to the illiquid random endowments, both agents have access to a liquid incomplete financial market modelled by a general locally-bounded semimartingale. Also, we assume that all pay-offs are already discounted in time-0 terms; this way we can freely compare values corresponding to different points in time. Mathematical-finance literature abounds with information on the utility-maximization problem for a variety of utility concepts (see, for instance, Karatzas et al. (1990), Kramkov and Schachermayer (1999), Schachermayer (2001), Cvitanić et al. (2001), Owen (2002), Owen and Žitković (2006)).

Under the conditions described above, the two agents meet at time 0 when one of the agents (the seller) offers a contingent claim with time- T payoff B to the other one (the buyer) in exchange for a lump-sum payment p at time $t = 0$. Our central question, posed above, can now be made more precise and split into two separate components:

1. Is there a number $p \in \mathbb{R}$ such that the exchange of the contingent claim B for a lump sum p is (strictly) beneficial for both agents?
2. If more than one such p exists, can we determine the exact outcome of the negotiation?

The net gain $B - p$ will be beneficial for the buyer if he/she can find a trading strategy such that the resulting wealth at time T gives rise to a higher expected utility than the one he/she would be able to obtain without

$B - p$. A similar criterion applies to the seller. In case the answer to question 1. is positive, we say that the agents are *in agreement*.

While we give a fairly complete answer to question 1., we only touch upon the issues involved in question 2. In fact, it is not possible to give a definitive answer to this question without a precise model of the negotiation process (see, for instance, Bazerman and Neale (1992)). A partial answer is possible, however, when the indivisibility assumption is dropped (see Section 5).

1.2. Our results and how they relate to existing research. Our results are naturally split into 4 parts which correspond to Sections 3, 4, 5 and Appendix A in the present paper:

1. Abstract agreement: We start with the study of the class \mathcal{G}° of all (appropriately regular) contingent claims B for which the agents are in agreement. It is, perhaps, surprising that unless non-replicable random endowments are present, no contingent claims will lead to agreement, even for agents with different risk-aversion coefficients. When the random endowments are indeed present, we give a necessary and sufficient condition for the set \mathcal{G}° to be non-empty. This characterization is closely related to the notion of *optimal risk sharing* which was first studied in the context of insurance/reinsurance negotiation (see, for instance, Borch (1962), Bühlmann and Jewell (1979) and Dana and Scarsini (2007)) and in terms of the principal's problem (see, for instance, Holmstrom and Milgrom (1987)) and recently developed in more general settings (see Föllmer and Schied (2004), Barrieu and El Karoui (2004), Barrieu and El Karoui (2005), Jouini et al. (2006) and Filipović and Kupper (2008a)).

2. Agreement for a specific claim - residual risk and approximation: Next, we consider a question which is in a sense dual to the one tackled in the previous part: is there a criterion for an agreement about a *given* claim B ? We propose two approaches: one through the notion of residual risk and the other based on asymptotic approximation of conditional indifference prices for small quantities.

Residual risk (introduced in Musiela and Zariphopoulou (2004)) of a random liability is defined as the difference between the liability's payoff and the terminal value of the optimal risk-monitoring strategy at maturity. We establish the following criterion, made precise in the body of the paper: a claim is mutually agreeable if and only if it reduces the residual risk for both agents.

The other approach provides an explicit criterion in the asymptotic case when the size of the contingent claim B is small compared to the size of the agents' random endowments. It is possible to phrase the agreement problem in terms of a relationship between the buyer's and the seller's conditional indifference price for the claim, so it is not unusual that an asymptotic study of these quantities plays a major role. More precisely, we provide a rather general Taylor-type approximation of the conditional exponential indifference price for locally bounded semimartingales on continuous filtrations. These approximations are then used to give simple asymptotic criteria for agreeability, as well as the asymptotic size of the interval of mutually-agreeable prices. Since it is not possible to obtain closed-form representations of indifference prices in general market models, such asymptotic results can be very useful even beyond the agreement problem.

Asymptotic techniques are not new in utility maximization problems. Results on the second order approximation of the indifference price for the exponential and the power utility, in a market with Brownian motion dynamics, are given in Henderson (2002) (see also Henderson and Hobson (2002)). These results are generalized in Kramkov and Sirbu (2006) for semimartingale markets and general utility functions defined on the positive real line. However, the arguments presented there do not cover the case of the exponential utility (e.g., the situation when the optimal wealth is not necessarily bounded from below by zero). For the exponential utility in a semimartingale market, the first derivative of the indifference price for a vector of claims is given in Ilhan et al. (2005). By imposing the assumption of continuity on the filtration, we generalize their result (as well as the asymptotic approximation in Sircar and Zariphopoulou (2005)) by providing a second order approximation of the price for a vector of claims.

3. Partial equilibrium prices: In the third part of the paper we look into the following, related, question: *if the agents are allowed to choose not only the price of the claim, but also the quantity traded, can the market clearing (partial equilibrium) conditions be used to compute these two quantities?* We consider bundles of several contingent claims and prove existence and uniqueness of equilibrium price-quantities, as well as a formula for the partial equilibrium price. The existence results of various types of competitive equilibria are a staple of quantitative economics literature, and have recently made their way into mathematical finance (see, among others, Dana and Le Van (2000), Heath and Ku (2004), Žitković (2006), Burgert and Rüschenendorf (2008) and Filipović and Kupper (2008b)). Our incomplete partial-equilibrium setting is, however, new and not covered by any of the existing results. As we already mentioned above, it is only in the present setting that we can say something about question 2., i.e., about the realized price p of the offered contingent claim B .

4. Conditional indifference prices: Our structural results rely heavily on the notion of conditional indifference prices. Utility-indifference prices were first introduced in Hodges and Neuberger (1989), and then further investigated and developed by a large number of authors (see, for instance, Karatzas and Kou (1996), Davis (1997), Frittelli (2000), Rouge and El Karoui (2000), Zariphopoulou (2001), Hugonnier et al. (2005), Mania and Schweizer (2005), Klöppel and Schweizer (2007)). The special case (pertinent to the present paper) of exponential indifference prices was studied, e.g., in Frittelli (2000), Rouge and El Karoui (2000), Zariphopoulou (2001), Delbaen et al. (2002) and Mania and Schweizer (2005).

In the presence of an illiquid random endowment, we talk about a conditional indifference price (also known as relative indifference price in Musiela and Zariphopoulou (2005) and Stoikov (2006)). In the exponential world, some of its properties can be obtained by a simple change of measure which, effectively, removes the conditionality. Other properties, however, cannot be dealt with in that manner. The goal of the last part of this work is to establish certain properties of conditional indifference prices in a general semimartingale market setting. We show, for instance, a rather unexpected fact that conditional indifference prices (unlike their unconditional versions) do not have to be monotone in the risk-aversion parameter.

1.3. The structure of the paper. In Section 2, we describe the market model and introduce necessary notation. The notion of agreement is introduced and our main abstract results are proven in Section 3. In Section 4 we study the small-quantity asymptotics of conditional indifference prices, and use it to provide an agreement criterion. An example in the Brownian setting is also presented. The topics of Section 5 are existence and uniqueness of partial equilibrium price-quantities for vectors of contingent claims. In Appendix A we state some properties of conditional indifference prices, and in Appendix B we give an outline of some known results on residual risk.

2. SOME MODELLING AND NOTATIONAL PRELIMINARIES

2.1. The financial market. Our model of the financial market is based on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, $\mathbb{F} = (\mathcal{F}_t)_{t \in [0, T]}$, $T > 0$, which satisfies the usual conditions of right-continuity and completeness. There are $d + 1$ traded assets ($d \in \mathbb{N}$), whose discounted price processes are modelled by an \mathbb{R}^{d+1} -valued locally bounded semimartingale $(S_t^{(0)}; \mathbf{S}_t)_{t \in [0, T]} = (S_t^{(0)}; S_t^{(1)}, \dots, S_t^{(d)})_{t \in [0, T]}$. The first asset $S_t^{(0)}$ plays the role of a numéraire security or a discount factor. Operationally, but with some loss of generality, we simply set $S_t^{(0)} \equiv 1$, for all $t \in [0, T]$, a.s.

2.2. Agent behaviour. Placing ourselves in the von Neumann-Morgenstern framework, we assume that each market participant evaluates the risk of an uncertain position X at time T according to the expected utility $\mathbb{E}^{\mathbb{P}}[U(X + \mathcal{E})]$, where U is a utility function and \mathcal{E} is the *random endowment* (accumulated illiquid wealth) and \mathbb{P} is a subjective probability measure. For technical reasons, we restrict our attention to $\mathcal{E} \in \mathbb{L}^\infty(\mathcal{F})$ and the class of exponential utilities

$$U(x) = -\exp(-\gamma x), \quad x \in \mathbb{R}$$

where the constant $\gamma \in (0, \infty)$ is the (absolute) risk aversion coefficient.

2.3. Admissible strategies and the absence of arbitrage. A financial agent invests in the market by choosing a portfolio strategy $\boldsymbol{\vartheta}$ in an admissibility class Θ , to be specified below. The resulting *gains process* $(G_t^{\boldsymbol{\vartheta}})_{t \in [0, T]}$ is simply the stochastic integral $G_t^{\boldsymbol{\vartheta}} = (\boldsymbol{\vartheta} \cdot \mathbf{S})_t = \int_0^t \boldsymbol{\vartheta}_u d\mathbf{S}_u$. Due to the exponential nature of the utility functions considered here, we follow the setup introduced in Mania and Schweizer (2005) or Delbaen et al. (2002). Before we give a precise description of the aforementioned set Θ , we need to introduce several concepts related to the no-arbitrage requirement. We start with the set \mathcal{M}_a of *absolutely continuous local martingale measures*, where

$$\mathcal{M}_a = \{\mathbb{Q} \ll \mathbb{P} : \mathbf{S} \text{ is a local martingale under } \mathbb{Q}\}$$

The set \mathcal{M}_e of all elements \mathbb{Q} of \mathcal{M}_a which additionally satisfy $\mathbb{Q} \sim \mathbb{P}$ is called *the set of equivalent local martingale measures*. For a probability measure \mathbb{Q} on (Ω, \mathcal{F}) , we define

$$\mathcal{H}(\mathbb{Q}|\mathbb{P}) = \begin{cases} \mathbb{E}^{\mathbb{P}} \left[\frac{d\mathbb{Q}}{d\mathbb{P}} \ln \left(\frac{d\mathbb{Q}}{d\mathbb{P}} \right) \right] & \mathbb{Q} \ll \mathbb{P}, \\ +\infty, & \text{otherwise} \end{cases}$$

The (extended) positive number $\mathcal{H}(\mathbb{Q}|\mathbb{P})$ is called the *relative entropy* of the probability measure \mathbb{Q} with respect to probability measure \mathbb{P} . For details on the notion of relative entropy we refer the interested reader to Grandits and Rheinländer (2002) or Frittelli (2000). We set

$$\mathcal{M}_{e,f} = \{\mathbb{Q} \in \mathcal{M}_e : \mathcal{H}(\mathbb{Q}|\mathbb{P}) < \infty\}$$

and enforce the following assumption

Assumption 2.1. $\mathcal{M}_{e,f} \neq \emptyset$.

Assumption 2.1 trivially implies that $\mathcal{M}_e \neq \emptyset$ which, in turn, guarantees that no arbitrage opportunities exist in the market (a stronger statement of “no free lunch with vanishing risk” will hold, as well). The additional requirement in Assumption 2.1 is common in the literature and it ensures that the choice of the exponential function for the utility leads to a well-defined behavior for utility-maximizing agents (see, among others, Delbaen et al. (2002), Frittelli (2000), Becherer (2001) and Mania and Schweizer (2005)).

Having introduced the required families of probability measures, we turn back to definition of the class Θ of *admissible strategies*:

$$(2.1) \quad \Theta = \{\vartheta \in L(\mathbf{S}) : (\vartheta \cdot \mathbf{S}) \text{ is a } \mathbb{Q}\text{-martingale, } \forall \mathbb{Q} \in \mathcal{M}_{e,f}\}$$

where $L(\mathbf{S})$ is the set of all predictable $(d+1)$ -dimensional \mathbf{S} -integrable processes on $[0, T]$. More information about the set Θ of admissible strategies is given in Mania and Schweizer (2005) (see also remarks on the set Θ_2 in Delbaen et al. (2002)).

We remind the reader that $\mathbb{L}^0(\mathcal{F})$ denotes the set of all (\mathbb{P} -a.s. equivalent classes of) \mathcal{F} -measurable random variables. A random variable $B \in \mathbb{L}^0(\mathcal{F})$ is said to be *replicable* if there exists a constant c and an admissible strategy $\vartheta \in \Theta$ such that $B = c + (\vartheta \cdot \mathbf{S})_T$ a.s.; the set of replicable random variables will be denoted by \mathcal{R} . More generally, we introduce the following equivalence relation between random variables in $\mathbb{L}^0(\mathcal{F})$:

Definition 2.2. We call two random variables $B, C \in \mathbb{L}^0(\mathcal{F})$ *risk equivalent* or *equal up to replicability* and write $B \sim C$, if the difference $B - C$ is replicable.

It is clear that the relation \sim is an equivalence relation on $\mathbb{L}^0(\mathcal{F})$ (since Θ is a vector space). We note that the zero equivalence class coincides with the set \mathcal{R} of the replicable random variables. For future reference, we let $\mathcal{R}^\infty = \mathcal{R} \cap \mathbb{L}^\infty(\mathcal{F})$ denote the set of all (essentially) bounded replicable random variables.

2.4. Some special probability measures. The expectation operator under a probability measure \mathbb{Q} is denoted by $\mathbb{E}^{\mathbb{Q}}[\cdot]$, where the superscript \mathbb{Q} is omitted in the case of the (subjective) measure \mathbb{P} . Also, for a random vector $\mathbf{B} = (B_1, B_2, \dots, B_n)$, $\mathbb{E}^{\mathbb{Q}}[\mathbf{B}]$ stands for the vector $(\mathbb{E}^{\mathbb{Q}}[B_1], \mathbb{E}^{\mathbb{Q}}[B_2], \dots, \mathbb{E}^{\mathbb{Q}}[B_n]) \in \mathbb{R}^n$.

For a random variable $B \in \mathbb{L}^0(\mathcal{F})$ with $\mathbb{E}[\exp(B)] < \infty$, the probability measure whose Radon-Nikodym derivative with respect to \mathbb{P} is given by $\frac{\exp(B)}{\mathbb{E}[\exp(B)]}$, is denoted by \mathbb{P}_B . Furthermore, $\mathbb{Q}^{(0)}$ denotes the probability measure in \mathcal{M}_a with the minimal relative entropy with respect to \mathbb{P} i.e., the probability measure for which $\mathcal{H}(\mathbb{Q}^{(0)}|\mathbb{P}) \leq \mathcal{H}(\mathbb{Q}|\mathbb{P})$ for all $\mathbb{Q} \in \mathcal{M}_a$. It is a consequence of Assumption 2.1 that the probability measure $\mathbb{Q}^{(0)}$ exists, is unique and belongs to $\mathcal{M}_{e,f}$ (see Frittelli (2000), page 43, Theorem 2.2). Similarly, for every B such that $\mathbb{E}[\exp(B)] < \infty$, there exists a unique probability measure $\mathbb{Q}^{(B)} \in \mathcal{M}_a$ such that $\mathcal{H}(\mathbb{Q}^{(B)}|\mathbb{P}_B) \leq \mathcal{H}(\mathbb{Q}|\mathbb{P}_B)$ for all $\mathbb{Q} \in \mathcal{M}_a$ (see Delbaen et al. (2002), page 103).

3. A NOTION OF AGREEMENT BETWEEN FINANCIAL AGENTS

3.1. Utility maximization and indirect utility. Given their risk profiles, financial agents trade in the financial market with the goal of maximizing expected utility. More precisely, an agent with initial wealth $x \in \mathbb{R}$, risk-aversion coefficient γ and random endowment $\mathcal{E} \in \mathbb{L}^\infty$ will choose a portfolio process $\boldsymbol{\vartheta} \in \Theta$ so as to maximize the expected utility $\mathbb{E}[-\exp(-\gamma((x + \boldsymbol{\vartheta} \cdot \mathbf{S})_T + \mathcal{E}))]$. The value function $u_\gamma(x|\mathcal{E})$ of the corresponding optimization problem is given by

$$(3.1) \quad u_\gamma(x|\mathcal{E}) = \sup_{\boldsymbol{\vartheta} \in \Theta} \mathbb{E} \left[-\exp(-\gamma(x + (\boldsymbol{\vartheta} \cdot \mathbf{S})_T + \mathcal{E})) \right], \quad x \in \mathbb{R}.$$

Overloading the notation slightly, for any random variable $B \in \mathbb{L}^\infty(\mathcal{F})$ (interpreted as a contingent payoff with maturity T) we define the *indirect utility* of B by $u_\gamma(B|\mathcal{E}) = u_\gamma(0|\mathcal{E} + B)$, i.e.

$$(3.2) \quad u_\gamma(B|\mathcal{E}) = \sup_{\boldsymbol{\vartheta} \in \Theta} \mathbb{E} \left[-\exp(-\gamma((\boldsymbol{\vartheta} \cdot \mathbf{S})_T + \mathcal{E} + B)) \right].$$

Remark 3.1. Thanks to the choice of the exponential utility, the case where the agents have different subjective probability measures, say $\mathbb{P}_1 \neq \mathbb{P}_2$, is also covered. Indeed, if we assume that $\mathbb{P}_1 \approx \mathbb{P}_2$ and $\ln(\frac{d\mathbb{P}_1}{d\mathbb{P}_2}) \in \mathbb{L}^\infty$, we can reduce the analysis to the case of two agents with the same subjective measure, say \mathbb{P}_2 , by adding $\gamma_1 \ln(\frac{d\mathbb{P}_1}{d\mathbb{P}_2})$ to first agent's random endowment.

3.2. A preference relation and a notion of acceptability. The indirect utility $u_\gamma(\cdot|\mathcal{E})$ induces a preference relation $\preceq_{\gamma, \mathcal{E}}$, on $\mathbb{L}^\infty(\mathcal{F})$; for $B_1, B_2 \in \mathbb{L}^\infty(\mathcal{F})$, we set

$$B_1 \preceq_{\gamma, \mathcal{E}} B_2 \quad \text{if} \quad u_\gamma(B_1|\mathcal{E}) \leq u_\gamma(B_2|\mathcal{E}).$$

In words, the payoff B_2 is preferable to the payoff B_1 for the agent with random endowment \mathcal{E} and risk aversion coefficient γ , if the total payoff $\mathcal{E} + B_2$ yields more indirect utility than the payoff $\mathcal{E} + B_1$.

The set of all payoffs $B \in \mathbb{L}^\infty(\mathcal{F})$ such that $0 \preceq_{\gamma, \mathcal{E}} B$ is called the (γ, \mathcal{E}) -*acceptance set*, and we denote it by $\mathcal{A}_\gamma(\mathcal{E})$. Equivalently, we have

$$(3.3) \quad \mathcal{A}_\gamma(\mathcal{E}) = \left\{ B \in \mathbb{L}^\infty(\mathcal{F}) : \sup_{\vartheta \in \Theta} \mathbb{E} \left[-\exp(-\gamma((\vartheta \cdot S)_T + \mathcal{E})) \right] \leq \right. \\ \left. \leq \sup_{\vartheta \in \Theta} \mathbb{E} \left[-\exp(-\gamma((\vartheta \cdot S)_T + \mathcal{E} + B)) \right] \right\}.$$

Closely related to the relation $\preceq_{\gamma, \mathcal{E}}$ is its strict version $\prec_{\gamma, \mathcal{E}}$ defined by

$$B_1 \prec_{\gamma, \mathcal{E}} B_2 \text{ if } u_\gamma(B_1|\mathcal{E}) < u_\gamma(B_2|\mathcal{E}), \quad B_1, B_2 \in \mathbb{L}^\infty(\mathcal{F}).$$

The *strict* (γ, \mathcal{E}) -*acceptance set* $\mathcal{A}_\gamma^\circ(\mathcal{E})$, is defined by $\mathcal{A}_\gamma^\circ(\mathcal{E}) = \{B \in \mathbb{L}^\infty : u_\gamma(0|\mathcal{E}) < u_\gamma(B|\mathcal{E})\}$, and the representation analogous to (3.3) (with \leq replaced by $<$) holds.

It is a consequence of the choice of the exponential utility function that the addition of any constant initial wealth $x \in \mathbb{R}$ to the random endowment \mathcal{E} does not influence the acceptance sets $\mathcal{A}_\gamma(\mathcal{E})$ and $\mathcal{A}_\gamma^\circ(\mathcal{E})$. More generally, we have the following simple proposition, the proof of which is standard. We remind the reader that a set \mathcal{A} is called *monotone* if $B \geq C$, a.s. and $C \in \mathcal{A}$ imply $B \in \mathcal{A}$.

Proposition 3.2. *For every $\mathcal{E} \in \mathbb{L}^\infty$ and $\gamma \in (0, \infty)$, the sets $\mathcal{A}_\gamma(\mathcal{E})$ and $\mathcal{A}_\gamma^\circ(\mathcal{E})$ are convex, monotone and*

$$(3.4) \quad \mathcal{E}_1 \sim \mathcal{E}_2 \text{ implies } \mathcal{A}_\gamma(\mathcal{E}_1) = \mathcal{A}_\gamma(\mathcal{E}_2) \text{ and } \mathcal{A}_\gamma^\circ(\mathcal{E}_1) = \mathcal{A}_\gamma^\circ(\mathcal{E}_2)$$

3.3. Conditional indifference prices. The acceptance set $\mathcal{A}_\gamma(\mathcal{E})$ can be used to introduce the notion of a conditional indifference price. The *conditional writer's indifference price* $\nu^{(w)}(B; \gamma|\mathcal{E})$ of the contingent claim $B \in \mathbb{L}^\infty(\mathcal{F})$ is defined by

$$(3.5) \quad \nu^{(w)}(B; \gamma|\mathcal{E}) = \inf \{p \in \mathbb{R} : p - B \in \mathcal{A}_\gamma(\mathcal{E})\}.$$

i.e., $\nu^{(w)}(B; \gamma|\mathcal{E})$ is the minimum amount that the agent with preference relation $\preceq_{\gamma, \mathcal{E}}$ will be willing to sell the claim with payoff B for. Similarly, the *conditional buyer's indifference price* $\nu^{(b)}(B; \gamma|\mathcal{E})$ is defined by

$$(3.6) \quad \nu^{(b)}(B; \gamma|\mathcal{E}) = \sup \{p \in \mathbb{R} : B - p \in \mathcal{A}_\gamma(\mathcal{E})\}.$$

i.e., $\nu^{(b)}(B; \gamma|\mathcal{E})$ is the maximum amount that the agent with preference relation $\preceq_{\gamma, \mathcal{E}}$ will offer for a contingent claim with payoff B .

In the special case where $\mathcal{E} \sim 0$, the corresponding prices are called *unconditional indifference prices* (or, simply, indifference prices) and are denoted by $\nu^{(w)}(B; \gamma)$ and $\nu^{(b)}(B; \gamma)$. A compendium of relevant properties of both conditional and unconditional indifference prices is given in Appendix A.

The notion of the indifference price has been studied by many authors (see, among others, Hodges and Neuberger (1989), Rouge and El Karoui (2000), Henderson (2002) and Musiela and Zariphopoulou (2004)). The definition of the conditional indifference price under exponential utility was given in Becherer (2001) for general semimartingale model, in Musiela and Zariphopoulou (2005) for a binomial case model and in

Stoikov (2006) for a diffusion model (where the price is called relative indifference price). A discussion of the conditional indifference price under general utility functions is given in Owen and Žitković (2006).

3.4. Agreement. The present paper deals with the interaction between two financial agents, with risk aversion coefficients γ_1 and γ_2 and random endowments $\mathcal{E}_1, \mathcal{E}_2 \in \mathbb{L}^\infty(\mathcal{F})$.

Definition 3.3. A contingent claim $B \in \mathbb{L}^\infty(\mathcal{F})$ is said to be

- (1) *mutually agreeable* if there exists a number $p \in \mathbb{R}$ such that $p - B \in \mathcal{A}_{\gamma_1}(\mathcal{E}_1)$ and $B - p \in \mathcal{A}_{\gamma_2}(\mathcal{E}_2)$.
- (2) *strictly mutually agreeable* if there exists a number $p \in \mathbb{R}$ such that $p - B \in \mathcal{A}_{\gamma_1}^\circ(\mathcal{E}_1)$ and $B - p \in \mathcal{A}_{\gamma_2}^\circ(\mathcal{E}_2)$.

If a claim B is (strictly) mutually agreeable, the set of all $p \in \mathbb{R}$ such that the conditions in (1) (or (2) in the strict case) above hold is called the *set of (strictly) mutually agreeable prices* for B .

A discussion related to our notion of mutually agreeability is given in Jouini et al. (2006), subsection 3.6, for cash invariant monetary utility functions, but without the presence of a financial market.

Using the conditional writer's and buyer's indifference prices, $\nu^{(w)}(\cdot; \gamma_1 | \mathcal{E}_1)$ and $\nu^{(b)}(\cdot; \gamma_2 | \mathcal{E}_2)$ defined above, we can give a simple characterization of the set of mutually-agreeable prices.

Proposition 3.4. *A claim $B \in \mathbb{L}^\infty(\mathcal{F})$ is mutually agreeable if and only if*

$$(3.7) \quad \nu^{(w)}(B; \gamma_1 | \mathcal{E}_1) \leq \nu^{(b)}(B; \gamma_2 | \mathcal{E}_2).$$

In that case, the set of mutually-agreeable prices for B is given by

$$[\nu^{(w)}(B; \gamma_1 | \mathcal{E}_1), \nu^{(b)}(B; \gamma_2 | \mathcal{E}_2)].$$

Remark 3.5.

- (1) A version of Proposition 3.4 for strict mutually-agreeable prices with strict inequality in (3.7) and the interval $[\nu^{(w)}(B; \gamma_1 | \mathcal{E}_1), \nu^{(b)}(B; \gamma_2 | \mathcal{E}_2)]$ replaced by its interior $(\nu^{(w)}(B; \gamma_1 | \mathcal{E}_1), \nu^{(b)}(B; \gamma_2 | \mathcal{E}_2))$ holds.
- (2) For a contingent claim $B \in \mathbb{L}^\infty(\mathcal{F}) \setminus \mathcal{R}^\infty$, each (strictly) mutually agreeable price p of B satisfies $p \in (\inf_{\mathbb{Q} \in \mathcal{M}_a} \mathbb{E}^{\mathbb{Q}}[B], \sup_{\mathbb{Q} \in \mathcal{M}_a} \mathbb{E}^{\mathbb{Q}}[B])$ (see, e.g., Owen and Žitković (2006), Proposition 7.2) i.e., every mutually agreeable price is an arbitrage-free price. Trivially, every claim $B \in \mathcal{R}^\infty$ is mutually agreeable and the mutually agreeable price is unique and equal to the unique arbitrage-free price.

3.5. The set of all mutually agreeable claims. It will be important in the sequel to introduce separate notation for the set of all (strictly) mutually agreeable claims:

$$\begin{aligned} \mathcal{G} &= \{B \in \mathbb{L}^\infty : B \text{ is mutually agreeable}\}, \text{ and,} \\ \mathcal{G}^\circ &= \{B \in \mathbb{L}^\infty : B \text{ is strictly mutually agreeable}\}. \end{aligned}$$

We remind the reader that \mathcal{R}^∞ is the set of all replicable claims in $\mathbb{L}^\infty(\mathcal{F})$.

Proposition 3.6.

- (1) \mathcal{G} is convex and $\sigma(\mathbb{L}^\infty, \mathbb{L}^1)$ -closed.
- (2) $\mathcal{G} \cap (-\mathcal{G}) = \mathcal{R}^\infty$, $\mathcal{G}^\circ \cap (-\mathcal{G}^\circ) = \emptyset$,
- (3) $\mathcal{G} = \mathbb{L}^\infty$ if and only if $\mathcal{R}^\infty = \mathbb{L}^\infty$.

Proof.

- (1) The convexity of \mathcal{G} follows from the convexity of $\nu^{(w)}(\cdot; \gamma_1 | \mathcal{E}_1)$ and concavity of $\nu^{(b)}(\cdot; \gamma_2 | \mathcal{E}_2)$ (see Proposition A.2). As for the closedness, it will be enough to note that $\nu^{(w)}(\cdot; \gamma_1 | \mathcal{E}_1) : \mathbb{L}^\infty \rightarrow \mathbb{R}$ is lower semi-continuous and $\nu^{(b)}(\cdot; \gamma_2 | \mathcal{E}_2) : \mathbb{L}^\infty \rightarrow \mathbb{R}$ is upper semi-continuous with respect to the weak-* topology $\sigma(\mathbb{L}^\infty, \mathbb{L}^1)$ (see Corollary A.4).
- (2) Trivially, $\mathcal{R}^\infty \subseteq \mathcal{G} \cap (-\mathcal{G})$. For a claim $B \in \mathcal{G} \cap (-\mathcal{G})$, there exists $p, \hat{p} \in \mathbb{R}$ such that $p - B \in \mathcal{A}_{\gamma_1}(\mathcal{E}_1)$ and $B - p \in \mathcal{A}_{\gamma_2}(\mathcal{E}_2)$, as well as $\hat{p} + B \in \mathcal{A}_{\gamma_1}(\mathcal{E}_1)$ and $-B - \hat{p} \in \mathcal{A}_{\gamma_2}(\mathcal{E}_2)$. It follows, by convexity of $\mathcal{A}_{\gamma_1}(\mathcal{E}_1)$ that $\frac{1}{2}(p + \hat{p}) = \frac{1}{2}(p - B + \hat{p} + B) \in \mathcal{A}_{\gamma_1}(\mathcal{E}_1)$, i.e., $u_{\gamma_1}(0 | \mathcal{E}_1) \leq u_{\gamma_1}(\frac{1}{2}(p + \hat{p}) | \mathcal{E}_1)$. The strict monotonicity of the value-function $u_{\gamma_1}(\cdot | \mathcal{E}_1)$ for deterministic arguments implies that $\frac{1}{2}(p + \hat{p}) \geq 0$. Applying the same line of reasoning to $\mathcal{A}_{\gamma_2}(\mathcal{E}_2)$ and the value function $u_{\gamma_2}(\cdot | \mathcal{E}_2)$, we get that $-\frac{1}{2}(p + \hat{p}) \geq 0$, and, consequently, $p = -\hat{p}$. Using the definitions (3.5) and (3.6) of the conditional indifference prices we easily get that $\nu^{(b)}(B; \gamma_1 | \mathcal{E}_1) \geq p \geq \nu^{(w)}(B; \gamma_1 | \mathcal{E}_1)$, which, according to Corollary 3.9, implies that $B \in \mathcal{R}^\infty$.

To prove the second claim, it suffices to note that $\mathcal{G}^\circ \cap \mathcal{R}^\infty = \emptyset$. Indeed, $\nu^{(w)}(B; \gamma_1 | \mathcal{E}_1) = \nu^{(b)}(B; \gamma_2 | \mathcal{E}_2) = \mathbb{E}^{\mathbb{Q}}[B]$ for $B \in \mathcal{R}^\infty$ and all $\mathbb{Q} \in \mathcal{M}_a$.

- (3) If $\mathcal{G} = \mathbb{L}^\infty$ then $\mathbb{L}^\infty \subseteq \mathcal{G} \cap (-\mathcal{G})$ so $\mathbb{L}^\infty = \mathcal{R}^\infty$, by (2) above. Conversely, if $\mathbb{L}^\infty = \mathcal{R}^\infty$ then $\mathbb{L}^\infty = \mathcal{G} \cap (-\mathcal{G}) \subseteq \mathcal{G}$.

□

Remark 3.7. The weak-* topology $\sigma(\mathbb{L}^\infty, \mathbb{L}^1)$ in Proposition 3.6 can be replaced by an even weaker one, namely the coarsest topology τ on \mathbb{L}^∞ which makes the expectation mappings $\mathbb{E}^{\mathbb{Q}}[\cdot] : \mathbb{L}^\infty \rightarrow \mathbb{R}$ continuous for each $\mathbb{Q} \in \mathcal{M}_{e,f}$.

3.6. No agreement without random endowments. The following, at first glance surprising, result states that mere difference in risk-aversion is not enough for two exponential agents to agree on a price for *any* contingent claim. Qualitatively different random endowments are needed.

Proposition 3.8 (Non-agreement with replicable random endowments). *Suppose that $\mathcal{E}_1 \sim \mathcal{E}_2 \sim 0$. Then $\mathcal{G} = \mathcal{R}^\infty$ and $\mathcal{G}^\circ = \emptyset$.*

Proof. The limiting relationships in (A.10) and the monotonicity properties of the indifference prices (see Proposition A.11) imply that

$$(3.8) \quad \nu^{(w)}(B; \gamma_1 | \mathcal{E}_1) = \nu^{(w)}(B; \gamma_1) \geq \mathbb{E}^{\mathbb{Q}^{(0)}}[B] \geq \nu^{(b)}(B; \gamma_2) = \nu^{(b)}(B; \gamma_2 | \mathcal{E}_2), \forall B \in \mathbb{L}^\infty.$$

Therefore, the strict inequality $\nu^{(w)}(B; \gamma_1 | \mathcal{E}_1) < \nu^{(b)}(B; \gamma_2 | \mathcal{E}_2)$ - needed for the strong agreement - cannot hold. Consequently, $\mathcal{G}^\circ = \emptyset$.

If $B \in \mathcal{G}$, (3.8) implies that $\nu^{(w)}(B; \gamma_1 | \mathcal{E}_1) = \nu^{(w)}(B; \gamma_1) = \mathbb{E}^{\mathbb{Q}^{(0)}}[B] = \lim_{\gamma \rightarrow 0} \nu^{(w)}(B; \gamma)$. Therefore, the function $\gamma \mapsto \nu^{(w)}(B; \gamma)$ can not be strictly increasing on $(0, \infty)$, so, by Proposition A.11, we must have $B \in \mathcal{R}^\infty$. Hence, $\mathcal{G} \subseteq \mathcal{R}^\infty$, which implies that $\mathcal{G} = \mathcal{R}^\infty$. □

Our following result, Corollary 3.9, follows directly from Proposition 3.8 and the fact that the conditional indifference price becomes unconditional if the measure \mathbb{P} is changed to $\mathbb{P}_{-\gamma\mathcal{E}}$ (see Proposition A.1 and the discussion at the beginning of subsection A.2).

Corollary 3.9. *Suppose that $\mathcal{E}_1 \sim \mathcal{E}_2$. Then for every $\gamma > 0$ and $B \in \mathbb{L}^\infty$, we have*

$$\begin{cases} \nu^{(w)}(B; \gamma | \mathcal{E}_1) > \nu^{(b)}(B; \gamma | \mathcal{E}_2) & \text{for } B \notin \mathcal{R}^\infty, \\ \nu^{(w)}(B; \gamma | \mathcal{E}_1) = \nu^{(b)}(B; \gamma | \mathcal{E}_2) & \text{otherwise.} \end{cases}$$

Remark 3.10. A non-agreement problem in the case where the agents' utility functions are defined on the positive real line is addressed in Collin Dufresne and Hugonnier (2007). More precisely, the authors deal with the special case when the market incompleteness comes from the presence of "extraneous risk" and conclude that there is no agreement between the agents on non-replicable, not-necessarily-bounded claims. However, their arguments depend heavily on the special structure of the market and do not generalize directly to our setting.

3.7. Agreement with random endowments. Proposition 3.8 states that the absence of random endowments is a sufficient condition for the lack of (strict) agreement. Is it also necessary? Given the result of Proposition 3.4, the question of the existence of non-replicable mutually agreeable claims leads to the following optimization problem with value function $\Sigma : (0, \infty)^2 \times (\mathbb{L}^\infty)^2 \rightarrow [0, +\infty]$, where

$$(3.9) \quad \Sigma(\gamma_1, \gamma_2, \mathcal{E}_1, \mathcal{E}_2) = \sup_{B \in \mathbb{L}^\infty} (\nu^{(b)}(B; \gamma_2 | \mathcal{E}_2) - \nu^{(w)}(B; \gamma_1 | \mathcal{E}_1)).$$

It follows directly from the Definition 3.3 of the set \mathcal{G} that the following result holds:

Proposition 3.11. *For $\mathcal{E}_1, \mathcal{E}_2 \in \mathbb{L}^\infty$, $\gamma_1, \gamma_2 \in (0, \infty)$ and $\Sigma = \Sigma(\gamma_1, \gamma_2, \mathcal{E}_1, \mathcal{E}_2)$, the following two statements are equivalent*

- (a) $\mathcal{G}^\circ \neq \emptyset$, and
- (b) $\Sigma > 0$.

Remark 3.12. The optimization problem above permits an interpretation in terms of the so-called *optimal risk-sharing problem*. In the case where the agents do not have access to a financial market, this problem has been addressed by many authors (see, e.g., Borch (1962) for its relation to the insurance/reinsurance problem, Holmstrom and Milgrom (1987) and Barrieu and El Karoui (2004) for the exponential utility case,

Jouini et al. (2006) for monetary utility functionals and Barrieu and Scandolo (2008) for concave preference functionals). When a financial market is present, the problem of optimal risk sharing when both agents have exponential utility has been studied in Barrieu and El Karoui (2005), where the authors focus on the form of the optimal structure.

Before we proceed, we need to define several terms.

Definition 3.13.

- (1) The sum $\mathcal{E} = \mathcal{E}_1 + \mathcal{E}_2$ of the random endowments of the agents is called the *aggregate endowment*.
- (2) A pair (B_1, B_2) in $(\mathbb{L}^\infty)^2$ is called an *allocation*, while an allocation (B_1, B_2) such that $B_1 + B_2 = \mathcal{E}$ is called a *feasible allocation*; the set of all feasible allocations will be denoted by $F(\mathcal{E})$
- (3) For an allocation (B_1, B_2) , the sum $\nu^{(b)}(B_1; \gamma_1) + \nu^{(b)}(B_2; \gamma_2)$, denoted by $\sigma(B_1, B_2)$, is called the *score* of (B_1, B_2) . The difference $\sigma(B_1, B_2) - \sigma(\mathcal{E}_1, \mathcal{E}_2)$ is called the *excess score* (where, for simplicity, the parameters γ_1 and γ_2 are omitted from the notation).

By (A.8), the expression $\nu^{(b)}(B; \gamma_2 | \mathcal{E}_2) - \nu^{(w)}(B; \gamma_1 | \mathcal{E}_1)$ appearing in (3.9) above can be rewritten as

$$(3.10) \quad \begin{aligned} \nu^{(b)}(B; \gamma_2 | \mathcal{E}_2) - \nu^{(w)}(B; \gamma_1 | \mathcal{E}_1) &= \nu^{(b)}(B; \gamma_2 | \mathcal{E}_2) + \nu^{(b)}(-B; \gamma_1 | \mathcal{E}_1) = (\nu^{(b)}(B + \mathcal{E}_2; \gamma_2) - \nu^{(b)}(\mathcal{E}_2; \gamma_2)) \\ &\quad + (\nu^{(b)}(-B + \mathcal{E}_1; \gamma_1) - \nu^{(b)}(\mathcal{E}_1; \gamma_1)) = \sigma(\mathcal{E}_1 - B, \mathcal{E}_2 + B) - \sigma(\mathcal{E}_1, \mathcal{E}_2). \end{aligned}$$

So that

$$\Sigma(\gamma_1, \gamma_2, \mathcal{E}_1, \mathcal{E}_2) = \sup \{ \sigma(B_1, B_2) : (B_1, B_2) \in F(\mathcal{E}) \} - \sigma(\mathcal{E}_1, \mathcal{E}_2).$$

In words, Σ is the maximized the excess score. If we think of the aggregate endowment \mathcal{E} as the total wealth of our two-agent economy, the solution of (3.9) (if it exists) will provide a redistribution of wealth so as to maximize the (improvement in) the score. Even though there is no direct economic reason why the sum of individual indifference prices should be maximized, Proposition 3.14 - which is a mere restatement of the discussion above - explains why the score is a useful concept.

Proposition 3.14. *For each $B \in \mathbb{L}^\infty$, the following two statements are equivalent:*

- (1) $B \in \mathcal{G}^\circ$, and
- (2) $\sigma(\mathcal{E}_1 - B, \mathcal{E}_2 + B) > \sigma(\mathcal{E}_1, \mathcal{E}_2)$.

The following proposition, which characterizes the score-optimal allocation, is a variation of Theorem 2.3 in Barrieu and El Karoui (2005) (see also Borch (1962) and Holmstrom and Milgrom (1987)).

Proposition 3.15. *For any $\mathcal{E}_1, \mathcal{E}_2 \in \mathbb{L}^\infty$ and $\gamma_1, \gamma_2 > 0$ there exists $B^* \in \mathbb{L}^\infty$ such that*

$$\sigma(\mathcal{E}_1 - B^*, \mathcal{E}_2 + B^*) \geq \sigma(B_1, B_2), \text{ for all } (B_1, B_2) \in F(\mathcal{E}).$$

Moreover, B^* is unique up to replicability and

$$B^* \sim \frac{\gamma_1 \mathcal{E}_1 - \gamma_2 \mathcal{E}_2}{\gamma_1 + \gamma_2}.$$

Proof. By (3.10), it suffices to show that

$$(3.11) \quad \nu^{(w)}(-B^* - \mathcal{E}_2; \gamma_2) + \nu^{(w)}(B^* - \mathcal{E}_1; \gamma_1) \leq \nu^{(w)}(-B - \mathcal{E}_2; \gamma_2) + \nu^{(w)}(B - \mathcal{E}_1; \gamma_1)$$

for all $B \in \mathbb{L}^\infty$, which is, in turn, a consequence of Lemma A.7. Indeed, it states that

$$\nu^{(w)}(B - \mathcal{E}_1; \gamma_1) + \nu^{(w)}(-B - \mathcal{E}_2; \gamma_2) \geq \nu^{(w)}(-\mathcal{E}_1 - \mathcal{E}_2; \tilde{\gamma}),$$

with equality if and only if

$$\frac{\gamma_1}{\tilde{\gamma}}(B - \mathcal{E}_1) \sim \frac{\gamma_2}{\tilde{\gamma}}(-B - \mathcal{E}_2), \text{ i.e., } B \sim \frac{\gamma_1 \mathcal{E}_1 - \gamma_2 \mathcal{E}_2}{\gamma_1 + \gamma_2}.$$

□

Corollary 3.16. *The following statements are equivalent:*

- (1) $\mathcal{G}^\circ = \emptyset$,
- (2) $B^* = \frac{\gamma_1 \mathcal{E}_1 - \gamma_2 \mathcal{E}_2}{\gamma_1 + \gamma_2}$ is replicable, and
- (3) $\frac{\gamma_1}{\gamma_2} \mathcal{E}_1 \sim \mathcal{E}_2$.

Remark 3.17. We can relate the existence of mutually agreeable non-replicable claims with the well-known notion of *Pareto optimality*. More precisely, an allocation $(B_1, B_2) \in F(\mathcal{E})$ is called Pareto optimal if $\#(C_1, C_2) \in F(\mathcal{E})$ s.t. $B_i \preceq_{\gamma_i, \mathcal{E}_i} C_i$ for $i = 1, 2$ and $B_i \prec_{\gamma_i, \mathcal{E}_i} C_i$ for at least one $i = 1, 2$. It follows from Corollary 3.16, that the condition $\frac{\gamma_1}{\gamma_2} \mathcal{E}_1 \sim \mathcal{E}_2$, implies that the allocation $(\mathcal{E}_1, \mathcal{E}_2)$ is the unique (up to replicability) Pareto optimal one. If $\frac{\gamma_1}{\gamma_2} \mathcal{E}_1 \not\sim \mathcal{E}_2$, a transaction involving the optimal claim B^* will lead to a Pareto optimal allocation.

4. RESIDUAL RISK AND AN APPROXIMATION OF THE INDIFFERENCE PRICE

4.1. Agreement and the residual risk process. The notion of *residual risk* for the indifference price valuation was defined in Musiela and Zariphopoulou (2004) for the setting of our Example 4.5., in Stoikov and Zariphopoulou (2006) for the stochastic volatility model and in Musiela and Zariphopoulou (2005) for a binomial-type model. This notion can be used to give a characterization of contingent claims in \mathcal{G} (see Propositions 4.2 and 4.4 below). Before we state this characterization, we give a short introduction to residual risk in a static setting (see Appendix B for definitions and some properties in the dynamic setting).

4.1.1. Residual risk in a static setting. Let $\gamma > 0$ be a risk-aversion coefficient, and let $B \in \mathbb{L}^\infty$ be a contingent claim. It can be shown that the optimization problem with the value function $u_\gamma(x | \nu^{(w)}(B; \gamma) - B)$, introduced in (3.1) admits an essentially unique maximizer $\vartheta^{(B)} \in \Theta$ (see Theorem A.3 in Appendix A, Delbaen et al. (2002), page 105, Theorem 2.2 and Kabanov and Stricker (2002), page 127, Theorem 2.1). The corresponding wealth process

$$X_t^{(B)} = \nu^{(w)}(B; \gamma) + \int_0^t \vartheta_u^{(B)} d\mathbf{S}_u,$$

can be interpreted as the optimal risk-monitoring strategy for the writer of the claim B , compensated by $\nu^{(w)}(B; \gamma)$ at the initial time.

Let $\gamma > 0$ be a risk-aversion coefficient, and let $B \in \mathbb{L}^\infty$ be a contingent claim. It can be shown that the optimization problem with the value function $u_\gamma(x|\nu^{(w)}(B; \gamma) - B)$, introduced in (3.1), admits an essentially unique maximizer $\vartheta^{(B)} \in \Theta$ (see Theorem A.3 in Appendix A, Delbaen et al. (2002), page 105, Theorem 2.2 and Kabanov and Stricker (2002), page 127, Theorem 2.1). Following the same notation, $\vartheta^{(0)} \in \Theta$ stands for the optimal strategy without the claim. We then set $\bar{\vartheta}_t^{(B)} = \vartheta_t^{(B)} - \vartheta_t^{(0)}$, for all $t \in [0, T]$ (note that since Θ is a vector space, $\bar{\vartheta}^{(B)} \in \Theta$). The corresponding wealth process

$$X_t^{(B)} = \nu^{(w)}(B; \gamma) + \int_0^t \bar{\vartheta}_u^{(B)} d\mathbf{S}_u,$$

can be interpreted as the optimal risk-monitoring strategy for the writer of the claim B , compensated by $\nu^{(w)}(B; \gamma)$ at the initial time. The hedging error

$$(4.1) \quad R^{(w)}(B; \gamma) = B - X_T^{(B)}$$

is called the (*writer's*) *residual risk*. $R^{(w)}(B; \gamma)$ can be interpreted as the risk remaining after the optimal hedging has been performed. Note that $R^{(w)}(B; \gamma) = 0$, a.s., for all replicable claims $B \in \mathbb{L}^\infty$. In the conditional case, an analogous discussion and the decomposition formula (see (A.8))

$$\nu^{(w)}(B; \gamma|\mathcal{E}) = \nu^{(w)}(B - \mathcal{E}; \gamma) - \nu^{(w)}(-\mathcal{E}; \gamma),$$

allow us to define the *conditional residual risk* $R^{(w)}(B; \gamma|\mathcal{E})$ by

$$R^{(w)}(B; \gamma|\mathcal{E}) = R^{(w)}(B - \mathcal{E}; \gamma) - R^{(w)}(-\mathcal{E}; \gamma),$$

and obtain the following decomposition

$$(4.2) \quad B = \nu^{(w)}(B; \gamma|\mathcal{E}) + \int_0^T \bar{\vartheta}_t^{(B|\mathcal{E})} d\mathbf{S}_t + R^{(w)}(B; \gamma|\mathcal{E}),$$

where $\bar{\vartheta}_t^{(B|\mathcal{E})} = \bar{\vartheta}_t^{(B-\mathcal{E})} - \bar{\vartheta}_t^{(-\mathcal{E})} = \vartheta_t^{(B-\mathcal{E})} - \vartheta_t^{(-\mathcal{E})}$, $t \in [0, T]$. The process $(\bar{\vartheta}_t^{(B|\mathcal{E})})_{t \in [0, T]}$, as well as the decomposition (4.2), could have been derived equivalently using the optimization problems used to define the conditional indifference prices. All of the above concepts have natural analogues when seen from the buyer's side. Namely, we define the (buyer's) residual risk by $R^{(b)}(B; \gamma) = R^{(w)}(-B; \gamma)$ and by $R^{(b)}(B; \gamma|\mathcal{E}) = R^{(b)}(B + \mathcal{E}; \gamma) - R^{(b)}(\mathcal{E}; \gamma)$ in the conditional case.

Remark 4.1. Using decomposition (4.2) and the Proposition A.2, we observe that

$$\nu^{(b)}(R^{(w)}(B; \gamma|\mathcal{E}); \gamma|\mathcal{E}) = \nu^{(b)}(B; \gamma|\mathcal{E}) - \nu^{(w)}(B; \gamma|\mathcal{E}).$$

Hence, the agreement condition (3.7) can also be written as

$$B \in \mathcal{G} \Leftrightarrow \nu^{(b)}(R^{(w)}(B; \gamma|\mathcal{E}); \gamma|\mathcal{E}) \geq 0.$$

In particular, $\nu^{(b)}(R^{(w)}(B; \gamma|\mathcal{E}); \gamma|\mathcal{E}) < 0$ for all $\mathcal{E} \in \mathcal{R}^\infty$, and $B \in \mathbb{L}^\infty \setminus \mathcal{R}^\infty$. We also note that $\nu^{(w)}(R^{(w)}(B; \gamma|\mathcal{E}); \gamma|\mathcal{E}) = 0$, for all $\mathcal{E}, B \in \mathbb{L}^\infty$.

The following proposition gives a characterization of mutually agreeable contingent claims in terms of their residual risk.

Proposition 4.2. *For $B \in \mathbb{L}^\infty$, the following statements are equivalent:*

- (1) $B \in \mathcal{G}$,
- (2) the inequality

$$(4.3) \quad \mathbb{E}^{\mathbb{Q}} \left[R^{(w)}(B; \gamma_1 | \mathcal{E}_1) \right] + \mathbb{E}^{\mathbb{Q}} \left[R^{(b)}(B; \gamma_2 | \mathcal{E}_2) \right] \geq 0$$

holds for some $\mathbb{Q} \in \mathcal{M}_{e,f}$.

- (3) the inequality

$$(4.4) \quad \mathbb{E}^{\mathbb{Q}} \left[R^{(w)}(B - \mathcal{E}_1; \gamma_1) - R^{(w)}(-\mathcal{E}_1; \gamma_1) \right] + \mathbb{E}^{\mathbb{Q}} \left[R^{(w)}(-B - \mathcal{E}_2; \gamma_2) - R^{(w)}(-\mathcal{E}_2; \gamma_2) \right] \geq 0$$

holds for some $\mathbb{Q} \in \mathcal{M}_{e,f}$.

- (4) the inequality (4.3) holds for all $\mathbb{Q} \in \mathcal{M}_{e,f}$.
- (5) the inequality (4.4) holds for all $\mathbb{Q} \in \mathcal{M}_{e,f}$.

Proof. It suffices to make the following two observations

- (a) thanks to the definition (4.1) of residual risk, the differences $\nu^{(w)}(B; \gamma_1 | \mathcal{E}_1) - R^{(w)}(B; \gamma_1 | \mathcal{E}_1)$ and $\nu^{(b)}(B; \gamma_2 | \mathcal{E}_2) - R^{(b)}(B; \gamma_2 | \mathcal{E}_2)$ are both of the form $\int_0^T \boldsymbol{\vartheta}_t d\mathbf{S}_t$ with $\boldsymbol{\vartheta} \in \boldsymbol{\Theta}$, and
- (b) the following equality holds

$$\begin{aligned} R^{(w)}(B; \gamma_1 | \mathcal{E}_1) - R^{(b)}(B; \gamma_2 | \mathcal{E}_2) &= R^{(w)}(B - \mathcal{E}_1; \gamma_1) - R^{(w)}(-\mathcal{E}_1; \gamma_1) \\ &\quad + R^{(w)}(-B - \mathcal{E}_2; \gamma_2) - R^{(w)}(-\mathcal{E}_2; \gamma_2). \end{aligned} \quad \square$$

Remark 4.3. It should be pointed out that it is enough to check the above inequalities just for some probability measure in $\mathcal{M}_{e,f}$. Also, it follows from the definition of the residual risk that the inequality (4.1) implies that the transaction of claim B at any price p will decrease the sum of expected residual risks. If in addition $p \in (\nu^{(w)}(B; \gamma_1 | \mathcal{E}_1), \nu^{(b)}(B; \gamma_2 | \mathcal{E}_2))$, each agent's expected residual risk will be decreased.

Under the additional mild assumption of continuity for the filtration \mathbb{F} , we can replace the criterion given in Proposition 4.2 by the following one (see Appendix B for the additional notation), which sometimes is easier to check.

Proposition 4.4. *Suppose that \mathbb{F} is continuous. For $B \in \mathbb{L}^\infty$, the following two statements are equivalent:*

- (1) $B \in \mathcal{G}$, and
- (2)

$$\begin{aligned} \gamma_1 \mathbb{E}^{\mathbb{Q}^{(0)}} \left[\left\langle R^{(w)}(B - \mathcal{E}_1; \gamma_1) \right\rangle_T - \left\langle R^{(w)}(-\mathcal{E}_1; \gamma_1) \right\rangle_T \right] + \\ + \gamma_2 \mathbb{E}^{\mathbb{Q}^{(0)}} \left[\left\langle R^{(w)}(-B + \mathcal{E}_2; \gamma_2) \right\rangle_T - \left\langle R^{(w)}(\mathcal{E}_2; \gamma_2) \right\rangle_T \right] \geq 0 \end{aligned}$$

Proof. The equivalence follows from Proposition 4.2 and part (2) of Theorem B.1, which effectively state that $\langle R^{(w)}(B; \gamma) \rangle_t = \langle L^{(w)}(B; \gamma) \rangle_t$ for all $t \in [0, T]$, so that $R^{(w)}(B; \gamma) - \frac{\gamma}{2} \langle R^{(w)}(B; \gamma) \rangle_T$ is a $\mathbb{Q}^{(0)}$ -martingale, for any $\gamma > 0$ and $B \in \mathbb{L}^\infty$. \square

Example 4.5. This example is set in an incomplete financial market similar to the one considered in Musiela and Zariphopoulou (2004) (see, also, Henderson (2002)). The market consists of one risky asset $S = (S_t)_{t \in [0, T]}$ with dynamics

$$dS_t = S_t(\mu(t) dt + \sigma(t) dW_t^{(1)})$$

and an additional (non-traded) factor $Y = (Y_t)_{t \in [0, T]}$ which evolves is a unique strong solution of

$$dY_t = b(Y_t, t) dt + a(Y_t, t) \left(\rho dW_t^{(1)} + \rho' dW_t^{(2)} \right),$$

where $W^{(1)} = (W_t^{(1)})_{t \in [0, T]}$ and $W^{(2)} = (W_t^{(2)})_{t \in [0, T]}$ are two standard independent Brownian Motions defined on a probability space $(\Omega, \mathbb{F}, (\mathcal{F}_t), \mathbb{P})$. The constant $\rho \in (-1, 1)$ is the correlation coefficient and $\rho' := \sqrt{1 - \rho^2}$. We assume that the deterministic functions $\mu, \sigma : [0, T] \rightarrow \mathbb{R}$ are uniformly bounded ($\sigma > 0$).

By Theorem 3 in Musiela and Zariphopoulou (2004) (in Musiela and Zariphopoulou (2004) μ and σ are constants, but their arguments carry over to our setting), we have that

$$(4.5) \quad v^{(w)}(B; \gamma) = \frac{1}{\gamma(1 - \rho^2)} \ln \left\{ \mathbb{E}^{\mathbb{Q}^{(0)}} \left(e^{\gamma(1 - \rho^2)B} \right) \right\},$$

for any payoff $B \in \mathbb{L}^\infty$, such that $B = g(Y_T)$ for some bounded Borel function $g : \mathbb{R} \rightarrow \mathbb{R}$, where the Radon-Nikodym derivative of $\mathbb{Q}^{(0)}$ is given by

$$\frac{d\mathbb{Q}^{(0)}}{d\mathbb{P}} = \exp \left(\int_0^T \frac{1}{2} \lambda^2(t) dt + \int_0^T \lambda(t) dW_t^{(1)} \right),$$

and $\lambda(t) := \frac{\mu(t)}{\sigma(t)}$ is the *Sharpe ratio* of S .

Let us further suppose that $\mathcal{E}_1 = g_1(Y_T)$ and $\mathcal{E}_2 = g_2(Y_T)$ for some Borel bounded functions g_1 and g_2 . Proposition 3.4 and representation (4.5) imply that that $B = g(Y_T) \in \mathcal{G}$ if and only if

$$\left(\frac{\mathbb{E}^{\mathbb{Q}^{(0)}} \left(e^{\gamma_1 \tilde{B}} e^{-\gamma_1 \tilde{\mathcal{E}}_1} \right)}{\mathbb{E}^{\mathbb{Q}^{(0)}} \left(e^{-\gamma_1 \tilde{\mathcal{E}}_1} \right)} \right)^{\frac{\gamma_2}{\gamma_1}} \leq \frac{\mathbb{E}^{\mathbb{Q}^{(0)}} \left(e^{-\gamma_2 \tilde{\mathcal{E}}_2} \right)}{\mathbb{E}^{\mathbb{Q}^{(0)}} \left(e^{-\gamma_2 \tilde{\mathcal{E}}_2} e^{-\gamma_2 \tilde{B}} \right)}$$

where $\tilde{B} = (1 - \rho^2)B$ and $\tilde{\mathcal{E}}_i = (1 - \rho^2)\mathcal{E}_i$, $i = 1, 2$.

As we have seen, $\nu^{(w)}(B; \gamma_1) > \nu^{(b)}(B; \gamma_2)$, $\forall B \approx 0$. It is easy to verify that $\nu^{(w)}(B; \gamma_1 | \mathcal{E}_1) \leq \nu^{(w)}(B; \gamma_1)$ if and only if $\text{Cov}^{\mathbb{Q}^{(0)}}(\mathcal{E}_1, B) \geq 0$, where $\text{Cov}^{\mathbb{Q}^{(0)}}(., .)$ is the covariance under the measure $\mathbb{Q}^{(0)}$. This means that the presence of random endowment which is positively correlated with the claim payoff, reduces the writer's indifference price. Similarly, $\nu^{(b)}(B; \gamma_2 | \mathcal{E}_2) \geq \nu^{(b)}(B; \gamma_2)$ if and only if $\text{Cov}^{\mathbb{Q}^{(0)}}(\mathcal{E}_2, B) \leq 0$. Therefore, we infer that a necessary condition for a claim B to be mutually agreeable is that $\text{Cov}^{\mathbb{Q}^{(0)}}(\mathcal{E}_1, B) > 0$ or $\text{Cov}^{\mathbb{Q}^{(0)}}(\mathcal{E}_2, B) < 0$.

4.2. An asymptotic approximation of indifference prices. When the size of the claim whose price is negotiated is small compared to the sizes of agent's contingent claims (and this is typically the case in practice), one can use a Taylor-type expansion of the indifference price around 0, and obtain more precise quantitative answers to the agreement question. More precisely, we assume that claim under consideration has the form $\alpha \cdot \mathbf{B}$ for a some vector $\mathbf{B} = (B_1, \dots, B_n)$ in $(\mathbb{L}^\infty)^n$, where $\alpha \in \mathbb{R}^n$ should be interpreted as a small parameter. We start with a single agent's point of view and present an approximation result for the indifference price $\nu^{(w)}(\alpha \cdot \mathbf{B}; \gamma | \mathcal{E})$ of $\alpha \cdot \mathbf{B}$, where $\mathcal{E} \in \mathbb{L}^\infty$ and $\gamma > 0$. To alleviate the notation, we shorten $\nu^{(w)}(\alpha \cdot \mathbf{B}; \gamma | \mathcal{E})$ to $w(\alpha)$, for $\alpha \in \mathbb{R}^n$.

A straightforward extension of Theorem 5.1 on page 590 in Ilhan et al. (2005), where we use the fact that the conditional indifference prices are just the unconditional ones under a changed measure, yields the following result:

Proposition 4.6. *The function w is continuously differentiable on \mathbb{R}^n with*

$$\nabla w(\alpha) = \mathbb{E}^{\mathbb{Q}^{(\gamma \alpha \cdot \mathbf{B} - \gamma \mathcal{E})}}[\mathbf{B}] = (\mathbb{E}^{\mathbb{Q}^{(\gamma \alpha \cdot \mathbf{B} - \gamma \mathcal{E})}}[B_1], \dots, \mathbb{E}^{\mathbb{Q}^{(\gamma \alpha \cdot \mathbf{B} - \gamma \mathcal{E})}}[B_n]), \quad \alpha \in \mathbb{R}^n.$$

The concept of minimized variance, defined below, is important for the study of second derivatives of the function w .

Definition 4.7. Let $\mathbb{Q} \in \mathcal{M}_e$ be an arbitrary martingale measure.

(1) For $B \in \mathbb{L}^\infty$ we define the *projected variance* $\Delta^{\mathbb{Q}}(B)$ of B under \mathbb{Q} as :

$$(4.6) \quad \Delta^{\mathbb{Q}}(B) = \inf_{\vartheta \in \Theta_{\mathbb{Q}}^2} \mathbb{E}^{\mathbb{Q}}[(B - \mathbb{E}^{\mathbb{Q}}[B] - (\vartheta \cdot \mathbf{S})_T)^2],$$

where, $\Theta_{\mathbb{Q}}^2 = \{\vartheta \in L(\mathbf{S}) : (\vartheta \cdot \mathbf{S}) \text{ is square integrable martingale under } \mathbb{Q}\}$, so that $\bigcap_{\mathbb{Q} \in \mathcal{M}_{e,f}} \Theta_{\mathbb{Q}}^2 \subset \Theta$.

(2) For $B_1, B_2 \in \mathbb{L}^\infty$ we define the *projected covariance* $\Delta^{\mathbb{Q}}(B_1, B_2)$ of B_1 and B_2 by polarization:

$$\Delta^{\mathbb{Q}}(B_1, B_2) = \frac{1}{2}(\Delta^{\mathbb{Q}}(B_1 + B_2) - \Delta^{\mathbb{Q}}(B_1) - \Delta^{\mathbb{Q}}(B_2)).$$

(3) For a vector $\mathbf{B} = (B_1, \dots, B_n) \in (\mathbb{L}^\infty)^n$ and a probability measure $\mathbb{Q} \in \mathcal{M}_e$, we define the \mathbb{Q} -*projected variance-covariance matrix* $\Delta^{\mathbb{Q}}(\mathbf{B})$ by

$$\Delta_{ij}^{\mathbb{Q}}(\mathbf{B}) = \Delta^{\mathbb{Q}}(B_i, B_j), \quad i, j = 1, \dots, n.$$

Remark 4.8.

- (1) The projected variance $\Delta^{\mathbb{Q}}(B)$ is the square of the $\mathbb{L}^2(\mathbb{Q})$ -norm of the projection $P^{\mathbb{Q}}(B)$ of the random variable $B \in \mathbb{L}^{\infty} \subseteq \mathbb{L}^2(\mathbb{Q})$ onto the closed subspace $\mathbb{R} \oplus \{(\boldsymbol{\vartheta} \cdot \mathbf{S})_T : \boldsymbol{\vartheta} \in \Theta_{\mathbb{Q}}^2\}$ of $\mathbb{L}^2(\mathbb{Q})$ (the closeness of $\{(\boldsymbol{\vartheta} \cdot \mathbf{S})_T : \boldsymbol{\vartheta} \in \Theta_{\mathbb{Q}}^2\}$ in $\mathbb{L}^2(\mathbb{Q})$ is an immediate consequence of the $\mathbb{L}^2(d[\mathbf{S}])$ - $\mathbb{L}^2(\mathbb{Q})$ isometry of stochastic integration). It follows that the projected covariance $\Delta^{\mathbb{Q}}(B_1, B_2)$ can be represented as

$$\Delta^{\mathbb{Q}}(B_1, B_2) = \mathbb{E}^{\mathbb{Q}}[P^{\mathbb{Q}}(B_1)P^{\mathbb{Q}}(B_2)].$$

In particular, $\Delta^{\mathbb{Q}}(\cdot, \cdot)$ is a bilinear functional on $\mathbb{L}^{\infty} \times \mathbb{L}^{\infty}$ and the following equality holds

$$(4.7) \quad \Delta^{\mathbb{Q}}(\boldsymbol{\alpha} \cdot \mathbf{B}) = \boldsymbol{\alpha} \cdot \boldsymbol{\Delta}^{\mathbb{Q}}(\mathbf{B})\boldsymbol{\alpha} = \sum_{i,j=1}^n \alpha_i \Delta_{ij}^{\mathbb{Q}}(\mathbf{B}) \alpha_j,$$

for all $\mathbb{Q} \in \mathcal{M}_e$, $\mathbf{B} \in (\mathbb{L}^{\infty})^n$ and $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n$.

- (2) Details on the notion of the projected variance, which is closely related to mean-variance hedging, can be found in Föllmer and Sondermann (1986) or Schweizer (2001). Note that the existence of the minimizer in the Definition 4.7 for bounded claims can be established using Kunita-Watanabe decomposition of the uniformly integrable \mathbb{Q} -martingale $(B_t)_{t \in [0, T]}$ defined as $B_t = \mathbb{E}^{\mathbb{Q}}[B | \mathcal{F}_t]$. For details on the Kunita-Watanabe decomposition we refer the reader to Ansel and Stricker (1993).

We remind the reader that a filtration is called *continuous* if all adapted local martingales admit continuous versions.

Lemma 4.9. *Suppose that $n = 1$ and that the filtration \mathbb{F} is continuous. Then for $B, \mathcal{E} \in \mathbb{L}^{\infty}$, the function $w : \mathbb{R} \rightarrow \mathbb{R}$ is twice differentiable at any $\alpha \in \mathbb{R}$ and its second derivative is given by*

$$(4.8) \quad w''(\alpha) = \gamma \Delta^{\mathbb{Q}(\gamma \alpha B - \gamma \mathcal{E})}(B).$$

Proof. Without loss of generality we may suppose that $\mathcal{E} = 0$ - otherwise, we just work under the measure $\mathbb{P}_{-\gamma \mathcal{E}}$. Let us focus on the case $\alpha = 0$, where first derivative $w'(0)$ is equal to $\mathbb{E}^{\mathbb{Q}^{(0)}}[B]$ by Proposition 4.6, so a simple translation by a constant (under which Δ is clearly invariant) allows us to assume, in addition, that $\mathbb{E}^{\mathbb{Q}^{(0)}}[B] = 0$. It will be enough, therefore, to show that

$$\lim_{\alpha \rightarrow 0} \left| \frac{\nu^{(w)}(\alpha B; \gamma)}{\alpha^2} - \frac{\gamma}{2} \Delta^{\mathbb{Q}^{(0)}}(B) \right| = 0.$$

By the sign invariance of $\Delta(\cdot)$ and the scaling property (A.3) of the indifference prices, it suffices to consider only $\alpha > 0$, i.e., it is enough to establish that

$$(4.9) \quad \lim_{\alpha \searrow 0} \left| \frac{\nu^{(w)}(B; \alpha \gamma)}{\alpha} - \frac{\gamma}{2} \Delta^{\mathbb{Q}^{(0)}}(B) \right| = 0$$

Theorem B.1 and the definition of the residual risk in Appendix B state that

$$\frac{\nu^{(w)}(B; \alpha\gamma)}{\alpha} = \frac{\gamma}{2} \mathbb{E}^{\mathbb{Q}^{(0)}} [(L^{(w)}(B; \alpha\gamma))_T] = \frac{\gamma}{2} \mathbb{E}^{\mathbb{Q}^{(0)}} [L_T^{(w)}(B; \alpha\gamma)^2],$$

where $(L_t^{(w)}(B; \alpha\gamma))_{t \in [0, T]}$ is as in Theorem B.1. The BMO-convergence of the processes $(L_t^{(w)}(B; \alpha\gamma))_{t \in [0, T]}$ from the same theorem, implies, in particular, the $\mathbb{L}^2(\mathbb{Q}^{(0)})$ -convergence of their terminal values, i.e.,

$$L_T^{(w)}(B; \alpha\gamma) \rightarrow L_T^{(w)}(B; 0) \text{ in } \mathbb{L}^2(\mathbb{Q}^{(0)}).$$

Therefore, it remains to prove that

$$\mathbb{E}^{\mathbb{Q}^{(0)}} [L_T^{(w)}(B; 0)^2] = \Delta^{\mathbb{Q}^{(0)}}(B), \quad \forall B \in \mathbb{L}^\infty.$$

Thanks to the final part of Theorem B.1, $L^{(w)}(B; 0)$ is strongly orthogonal to any process of the form $(\boldsymbol{\vartheta} \cdot \mathbf{S})$, for $\boldsymbol{\vartheta} \in \Theta_{\mathbb{Q}^{(0)}}^2$. In particular, for $\boldsymbol{\vartheta} \in \Theta_{\mathbb{Q}^{(0)}}^2$ and $\hat{\boldsymbol{\vartheta}}^{(B)}$ as in Theorem B.1, we have

$$B - (\boldsymbol{\vartheta} \cdot \mathbf{S})_T = ((\hat{\boldsymbol{\vartheta}}^{(B)} - \boldsymbol{\vartheta}) \cdot \mathbf{S})_T + L_T^{(w)}(B; 0),$$

so that

$$\mathbb{E}^{\mathbb{Q}^{(0)}} [(B - (\boldsymbol{\vartheta} \cdot \mathbf{S})_T)^2] = \mathbb{E} \left[\left((\hat{\boldsymbol{\vartheta}}^{(B)} - \boldsymbol{\vartheta}) \cdot \mathbf{S} \right)_T^2 \right] + \mathbb{E} [L_T^{(w)}(B; 0)^2].$$

Consequently, the minimum in the definition of $\Delta^{\mathbb{Q}^{(0)}}[B]$ is attained at $\boldsymbol{\vartheta} = \hat{\boldsymbol{\vartheta}}^{(B)}$ so that $\Delta^{\mathbb{Q}}(B) = \mathbb{E}^{\mathbb{Q}^{(0)}} [L_T^{(w)}(B; 0)^2]$, which is the equality we set out to prove.

For $\alpha \neq 0$, we may again suppose that $w'(\alpha) = \mathbb{E}^{\mathbb{Q}^{(\gamma\alpha B)}}[B] = 0$. Hence, it is enough to show that $\lim_{\varepsilon \rightarrow 0} \frac{w(\alpha + \varepsilon) - w(\alpha)}{\varepsilon^2} = \frac{\gamma}{2} \Delta^{\mathbb{Q}^{(\gamma\alpha B)}}(B)$. For this, we note that $w(\alpha + \varepsilon) - w(\alpha) = v^{(w)}(\varepsilon B | -\alpha B; \gamma)$ i.e., we can rewrite the second derivative at some $\alpha \neq 0$ as the second derivative at $\alpha = 0$ with random endowment. This observation finishes the proof. \square

The case $n > 1$ is covered by the following lemma.

Lemma 4.10. For $\boldsymbol{\alpha}, \boldsymbol{\delta} \in \mathbb{R}^n$, $\mathbf{B} = (B_1, \dots, B_n) \in (\mathbb{L}^\infty)^n$ and $\mathcal{E} \in \mathbb{L}^\infty$, we have

$$(4.10) \quad \lim_{\varepsilon \rightarrow 0} \frac{w(\boldsymbol{\alpha} + \varepsilon \boldsymbol{\delta}) - w(\boldsymbol{\alpha}) - \varepsilon \nabla w(\boldsymbol{\alpha}) \cdot \boldsymbol{\delta}}{\varepsilon^2} = \frac{1}{2} \sum_{i,j=1}^n \delta_i \Delta_{ij}^{\mathbb{Q}}(\mathbf{B}) \delta_j.$$

Proof. We can assume that $\boldsymbol{\alpha} = (0, \dots, 0)$ by absorbing the term $-\boldsymbol{\alpha} \cdot \mathbf{B}$ into the random endowment \mathcal{E} . The left-hand side of (4.10) can now be understood as the second derivative at 0 of the function $\tilde{w} : \mathbb{R} \rightarrow \mathbb{R}$ given by

$$\tilde{w}(\varepsilon) = \nu^{(w)}(\varepsilon \boldsymbol{\delta} \cdot \mathbf{B}; \gamma | \mathcal{E}).$$

We can finish the proof by employing Lemma 4.9 and using the equality (4.7) with $\mathbb{Q} = \mathbb{Q}^{(-\gamma \mathcal{E})}$ and $\boldsymbol{\delta}$ substituted for $\boldsymbol{\alpha}$. \square

With the above results in our toolbox, we can give a second order directional Taylor-type approximation of the indifference price.

Proposition 4.11. *Choose $\mathbf{B} \in (\mathbb{L}^\infty)^n$, $\boldsymbol{\alpha} \in \mathbb{R}^n$, $\gamma > 0$ and $\mathcal{E} \in \mathbb{L}^\infty$, and assume that the filtration \mathbb{F} is continuous. With the notions of projected variance and covariance as in Definition 4.7, we have the following equality*

$$(4.11) \quad \nu^{(w)}(\varepsilon \boldsymbol{\alpha} \cdot \mathbf{B}; \gamma | \mathcal{E}) = \varepsilon \boldsymbol{\alpha} \cdot \mathbb{E}^{\mathbb{Q}^{(-\gamma \mathcal{E})}}[\mathbf{B}] + \frac{\varepsilon^2 \gamma}{2} \boldsymbol{\alpha} \cdot \Delta^{\mathbb{Q}^{(-\gamma \mathcal{E})}}(\mathbf{B}) \boldsymbol{\alpha} + o(\varepsilon^2), \text{ as } \varepsilon \rightarrow 0.$$

Remark 4.12. An approximation formula similar to (4.11) is given in Kramkov and Sirbu (2006), Theorem 10, where the authors consider utility functions defined on the positive real line.

Although the asymptotic expansion (4.11) in Proposition 4.11 above is important in its own right, its main application is the following criterion for mutual agreement for a small quantity of a given contingent claim.

Proposition 4.13. *Suppose that \mathbb{F} is continuous and that the random endowments $\mathcal{E}_1, \mathcal{E}_2 \in \mathbb{L}^\infty$ and risk-aversion coefficients $\gamma_1, \gamma_2 > 0$ are chosen. Let $B \in \mathbb{L}^\infty$ be a given contingent claim. The set \mathcal{G}° contains a segment of the form*

$$\{\alpha B : \alpha \in (0, \alpha_0)\} \text{ for some } \alpha_0 > 0 \text{ if and only if } \mathbb{E}^{\mathbb{Q}^{(-\gamma_1 \mathcal{E}_1)}}[B] < \mathbb{E}^{\mathbb{Q}^{(-\gamma_2 \mathcal{E}_2)}}[B],$$

and similarly, the set \mathcal{G}° contains a segment of the form

$$\{\alpha B : \alpha \in (-\alpha_0, 0)\} \text{ for some } \alpha_0 > 0 \text{ if and only if } \mathbb{E}^{\mathbb{Q}^{(-\gamma_1 \mathcal{E}_1)}}[B] > \mathbb{E}^{\mathbb{Q}^{(-\gamma_2 \mathcal{E}_2)}}[B].$$

Proof. We first note that the convexity of \mathcal{G}° implies that if $\exists \alpha_0 > 0$ such that $\alpha_0 B \in \mathcal{G}^\circ$, then $\alpha B \in \mathcal{G}^\circ, \forall \alpha \in [0, \alpha_0]$. By equation (4.11), $\nu^{(w)}(\alpha B; \gamma_1 | \mathcal{E}_1) = \alpha \mathbb{E}^{\mathbb{Q}^{(-\gamma_1 \mathcal{E}_1)}}[B] + o(\alpha)$ and $\nu^{(b)}(\alpha B; \gamma_2 | \mathcal{E}_2) = \alpha \mathbb{E}^{\mathbb{Q}^{(-\gamma_2 \mathcal{E}_2)}}[B] + o(\alpha)$. Hence, the inequality $\mathbb{E}^{\mathbb{Q}^{(-\gamma_1 \mathcal{E}_1)}}[B] < \mathbb{E}^{\mathbb{Q}^{(-\gamma_2 \mathcal{E}_2)}}[B]$ yields that there exists $\alpha_0 > 0$ small enough such that $\nu^{(w)}(\alpha_0 B; \gamma_1 | \mathcal{E}_1) < \nu^{(b)}(\alpha_0 B; \gamma_2 | \mathcal{E}_2)$, i.e., $\alpha_0 B \in \mathcal{G}^\circ$.

On the other hand, suppose that there exists $\alpha_0 > 0$ such that $\alpha_0 B \in \mathcal{G}^\circ$ and assume that $\mathbb{E}^{\mathbb{Q}^{(-\gamma_1 \mathcal{E}_1)}}[B] \geq \mathbb{E}^{\mathbb{Q}^{(-\gamma_2 \mathcal{E}_2)}}[B]$. It is easy to check that $\Delta^{\mathbb{Q}^{(-\gamma_i \mathcal{E}_i)}}(B) > 0$ for $i = 1, 2$ (since $B \notin \mathcal{R}^\infty$) and that by (4.11) and its buyer's version, we get

$$\alpha \mathbb{E}^{\mathbb{Q}^{(-\gamma_1 \mathcal{E}_1)}}[B] + \frac{\alpha^2 \gamma_1}{2} \Delta^{\mathbb{Q}^{(-\gamma_1 \mathcal{E}_1)}}(B) < \alpha \mathbb{E}^{\mathbb{Q}^{(-\gamma_2 \mathcal{E}_2)}}[B] - \frac{\alpha^2 \gamma_2}{2} \Delta^{\mathbb{Q}^{(-\gamma_2 \mathcal{E}_2)}}(B) + o(\alpha^2)$$

for every α close to zero such that $0 < \alpha \leq \alpha_0$ (note that thanks to the linearity of $\Theta_{\mathbb{Q}}^2$, we have $\Delta^{\mathbb{Q}}(B) = \Delta^{\mathbb{Q}}(-B), \forall B \in \mathbb{L}^\infty$).

This gives that for any such α

$$\frac{\alpha^2}{2} (\gamma_1 \Delta^{\mathbb{Q}^{(-\gamma_1 \mathcal{E}_1)}}(B) + \gamma_2 \Delta^{\mathbb{Q}^{(-\gamma_2 \mathcal{E}_2)}}(B)) + o(\alpha^2) < 0,$$

Dividing through by α^2 and letting $\alpha \rightarrow 0$, we get that $\gamma_1 \Delta^{\mathbb{Q}^{(-\gamma_1 \mathcal{E}_1)}}(B) + \gamma_2 \Delta^{\mathbb{Q}^{(-\gamma_2 \mathcal{E}_2)}}(B) \leq 0$, which is a contradiction. The proof of the second argument is similar and hence omitted. \square

Remark 4.14. Proposition 4.13 can be used to provide an approximation of the size of the set of the agreement prices for small number of units. More precisely, if $\mathbb{E}^{\mathbb{Q}^{(-\gamma_1 \mathcal{E}_1)}}[B] \neq \mathbb{E}^{\mathbb{Q}^{(-\gamma_2 \mathcal{E}_2)}}[B]$, it holds that

$$(4.12) \quad \begin{aligned} \nu^{(b)}(\alpha B; \gamma_2 | \mathcal{E}_2) - \nu^{(w)}(\alpha B; \gamma_1 | \mathcal{E}_1) &= \\ &= \alpha(\mathbb{E}^{\mathbb{Q}^{(-\gamma_2 \mathcal{E}_2)}}[B] - \mathbb{E}^{\mathbb{Q}^{(-\gamma_1 \mathcal{E}_1)}}[B]) - \frac{\alpha^2}{2}(\gamma_1 \Delta^{\mathbb{Q}^{(-\gamma_1 \mathcal{E}_1)}}(B) + \gamma_2 \Delta^{\mathbb{Q}^{(-\gamma_2 \mathcal{E}_2)}}(B)) + o(\alpha^2) \end{aligned}$$

for every $\alpha \in \mathbb{R}$ close to zero such that $\alpha B \in \mathcal{G}^\circ$. Another application of Proposition 4.13 is given at the first part of Remark 5.10.

A second order approximation of the optimal strategy has been recently given in Kramkov and Sirbu (2006) for general utilities defined on the positive real line. For the Example 4.5, the corresponding approximation result is easily obtained for the unconditional case by the formula (4.5) (see, e.g., Henderson (2002)). After changing the measure \mathbb{P} to $\mathbb{P}_{-\gamma \mathcal{E}}$, it is straightforward to show that

$$\nu^{(w)}(\alpha B; \gamma | \mathcal{E}) = \alpha \mathbb{E}^{\mathbb{Q}^{(-\gamma \mathcal{E})}}[B] + \frac{\alpha^2}{2} \gamma (1 - \rho^2) \text{Var}^{\mathbb{Q}^{(-\gamma \mathcal{E})}}(B) + o(\alpha^2), \forall B \in \mathbb{L}^\infty,$$

where $\text{Var}^{\mathbb{Q}}(B)$ denotes the variance of random variable B under the probability measure \mathbb{Q} .

Example 4.15. In the cases where there is no closed-form expression for the indifference price, the approximation (4.11) is rather useful. One of these cases is the stochastic volatility model studied in Sircar and Zariphopoulou (2005) and Ilhan et al. (2004) (see also Henderson (2007)). Proposition 4.13 is then a generalization of the approximation results of Ilhan et al. (2004), subsection 4.3.

5. PARTIAL EQUILIBRIUM PRICES

This section deals with the existence and the uniqueness of a partial equilibrium price of a contingent claim in the simplified two-agent economy. The discussion of mutual agreeability in previous sections assumed that the number of units is fixed and the claim is indivisible. If, however, the negotiation between agents involves the quantity traded as well as the price, and if this quantity is not constrained by quantization, a great deal more can be said about the outcome of the negotiation. The main advantage is that the methodology of equilibrium theory can be applied and a unique price-quantity pair singled out on the basis of the fundamental economic principle of market clearing. We stress that the agents take the form of the claims as given and do not engage in any form of optimal design, as is the case in many real-world situations in insurance, illiquid markets, bulk sales of shares, etc.

5.1. The partial equilibrium. A vector $\mathbf{B} = (B_1, \dots, B_n) \in (\mathbb{L}^\infty)^n$ of contingent claims is chosen and kept constant throughout this section. The extended set $\mathbb{R} \cup \{\pm\infty\}$ of real numbers is denoted by $\overline{\mathbb{R}}$.

It will be notationally convenient to define the “restrictions” $U_i : \bar{\mathbb{R}}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$, $i \in \{1, 2\}$ of the value functions $u_{\gamma_1}(\cdot|\mathcal{E}_1)$ and $u_{\gamma_2}(\cdot|\mathcal{E}_2)$ (see (3.1)) by

$$(5.1) \quad U_i(\boldsymbol{\alpha}; \mathbf{p}) = \begin{cases} u_{\gamma_i}(\boldsymbol{\alpha} \cdot (\mathbf{B} - \mathbf{p})|\mathcal{E}_i), & \boldsymbol{\alpha} \in \mathbb{R}^n, \\ \limsup_{\boldsymbol{\alpha}' \rightarrow \boldsymbol{\alpha}, \boldsymbol{\alpha}' \in \mathbb{R}^n} U_i(\boldsymbol{\alpha}'; \mathbf{p}), & \boldsymbol{\alpha} \in \bar{\mathbb{R}}^n \setminus \mathbb{R}^n, \end{cases} \quad i \in \{1, 2\}, \mathbf{p} \in \mathbb{R}^n.$$

i.e., $U_i(\cdot; \mathbf{p})$ is the extension of the continuous function $U_i(\cdot; \mathbf{p})|_{\mathbb{R}^n}$ to $\bar{\mathbb{R}}^n$ by upper semi-continuity and gives the indirect utility of agent i when she holds $\boldsymbol{\alpha}$ units of \mathbf{B} , purchased at price \mathbf{p} .

Definition 5.1. The *demand correspondence* $Z_i : \mathbb{R}^n \rightarrow 2^{\bar{\mathbb{R}}^n}$, for the agent $i \in \{1, 2\}$, is defined by

$$(5.2) \quad Z_i(\mathbf{p}) = \operatorname{argmax} \left\{ U_i(\boldsymbol{\alpha}, \mathbf{p}) : \boldsymbol{\alpha} \in \bar{\mathbb{R}}^n \right\}, \mathbf{p} \in \mathbb{R}^n.$$

Intuitively, the elements of $Z_i(\mathbf{p})$ give the numbers of units of \mathbf{B} that agent i is willing to purchase at price \mathbf{p} (the numbers of units that maximize her indirect utility). Using the above notation and definition, we are ready to introduce the central concept of this section:

Definition 5.2. A pair $(\mathbf{p}, \boldsymbol{\alpha}) \in \mathbb{R}^n \times \mathbb{R}^n$ is called a *partial-equilibrium price-quantity (PEPQ)* if

$$(5.3) \quad \boldsymbol{\alpha} \in Z_1(\mathbf{p}) \text{ and } -\boldsymbol{\alpha} \in Z_2(\mathbf{p}).$$

A vector $\mathbf{p} \in \mathbb{R}^n$ for which there exists $\boldsymbol{\alpha} \in \mathbb{R}^n$ such that $(\mathbf{p}, \boldsymbol{\alpha})$ is a PEPQ is called a *partial-equilibrium price (PEP)*.

In other words, the PEP is the price-vector of the contingent claim \mathbf{B} at which the quantity that one agent is willing to sell is equal to the quantity which the other agent is wants to buy.

When there exists $\boldsymbol{\alpha} \in \mathbb{R}^m$ such that the contingent claim $\boldsymbol{\alpha} \cdot \mathbf{B}$ is replicable, any PEP \mathbf{p} must have the property that $\boldsymbol{\alpha} \cdot \mathbf{p} = p_{NA}$, where p_{NA} is the replication price of $\boldsymbol{\alpha} \cdot \mathbf{B}$; this will hold no matter what the characteristics of the agents are. It is, therefore, only reasonable to assume that such claims do not enter the negotiation, i.e., we enforce the following assumption for the remainder of the section:

Assumption 5.3. There exists no $\boldsymbol{\alpha} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ such that $\boldsymbol{\alpha} \cdot \mathbf{B} \sim 0$.

5.2. Properties of the demand functions. Let $\mathcal{P}^{NA} \subseteq \mathbb{R}^n$ be the set of all arbitrage-free price-vectors of the contingent claims \mathbf{B} , i.e.,

$$\mathcal{P}^{NA} = \{ \mathbb{E}^{\mathbb{Q}}[\mathbf{B}] : \mathbb{Q} \in \mathcal{M}_e \},$$

where, as usual, $\mathbb{E}^{\mathbb{Q}}[\mathbf{B}] = (\mathbb{E}^{\mathbb{Q}}[B_1], \dots, \mathbb{E}^{\mathbb{Q}}[B_n]) \in \mathbb{R}^n$. To simplify the notation in the sequel, we introduce two n -dimensional families of measures in \mathcal{M}_e , parametrized by $\boldsymbol{\alpha} \in \mathbb{R}^n$:

$$\mathbb{Q}_i^{(\boldsymbol{\alpha})} = \mathbb{Q}^{(\gamma_i \boldsymbol{\alpha} \cdot \mathbf{B} - \gamma_i \mathcal{E}_i)}, \quad \boldsymbol{\alpha} \in \mathbb{R}^n, \quad i = 1, 2.$$

As we will see below (see Proposition 5.6), when we are looking for partial equilibrium prices, we can restrict ourselves to the sets

$$\mathcal{P}_i^U = \left\{ \mathbb{E}^{\mathbb{Q}_i^{(\alpha)}}[\mathbf{B}] : \alpha \in \mathbb{R}^n \right\}, \text{ for } i = 1, 2.$$

In general, $\mathcal{P}_i^U \subseteq \mathcal{P}^{NA}$, for $i = 1, 2$. The equality holds when $\mathcal{E}_i \sim 0$ (see Ilhan et al. (2005), Lemma 7.1). For future use, we define the function $\mathbf{u}_i : \mathbb{R}^n \rightarrow \overline{\mathbb{R}}$ by $\mathbf{u}_i(\mathbf{p}) = \sup_{\alpha \in \overline{\mathbb{R}}^n} \{U_i(\alpha; \mathbf{p})\}$, for $i \in \{1, 2\}$. Building on the notation of Section 4, we also introduce the following two shorthands:

$$(5.4) \quad \left. \begin{aligned} w_i(\alpha) &= \nu^{(w)}(\alpha \cdot \mathbf{B}; \gamma_i | \mathcal{E}_i) \\ b_i(\alpha) &= \nu^{(b)}(\alpha \cdot \mathbf{B}; \gamma_i | \mathcal{E}_i) \end{aligned} \right\} \alpha \in \mathbb{R}^n, i \in \{1, 2\}.$$

Lemma 5.4. *For $i = 1, 2$, w_i is strictly convex and b_i is strictly concave.*

Proof. A change of measure argument (where we replace \mathbb{P} by $\mathbb{P}_{-\gamma_i \mathcal{E}_i}$) can be employed to justify no loss of generality if we assume that $\mathcal{E}_i = 0$ in this proof. The fact that $w_i(\cdot)$ is convex follows from the convexity of the indifference price. In order to establish that the convexity is, in fact, strict, we assume, to the contrary, that there exist $\alpha_1, \alpha_2 \in \mathbb{R}^n$ with $\alpha_1 \neq \alpha_2$ and $\lambda \in (0, 1)$ such that

$$w_i(\lambda \alpha_1 + (1 - \lambda) \alpha_2) = \lambda w_i(\alpha_1) + (1 - \lambda) w_i(\alpha_2).$$

Equivalently, we have

$$\nu^{(w)}((\lambda \alpha_1 + (1 - \lambda) \alpha_2) \cdot \mathbf{B}; \gamma_i) = \nu^{(w)}(\lambda \alpha_1 \cdot \mathbf{B}; \frac{\gamma_i}{\lambda}) + \nu^{(w)}((1 - \lambda) \alpha_2 \cdot \mathbf{B}; \frac{\gamma_i}{1 - \lambda})$$

Since $(\frac{\gamma_i}{\lambda})^{-1} + (\frac{\gamma_i}{1 - \lambda})^{-1} = (\gamma_i)^{-1}$, we can use Lemma A.7 to conclude that

$$\alpha_1 \cdot \mathbf{B} \sim \alpha_2 \cdot \mathbf{B}, \text{ i.e. } \alpha \cdot \mathbf{B} \sim 0, \text{ where } \alpha = \alpha_1 - \alpha_2 \neq 0 \in \mathbb{R}^n,$$

a contradiction with Assumption 5.3. A similar argument can be employed to prove strict concavity of b_i , $i \in \{1, 2\}$. \square

Proposition 5.5. *For $i \in \{1, 2\}$, the functions $\mathbf{u}_i(\cdot)$ and $Z_i(\cdot)$ have the following properties*

- (1) *The maximum in (5.2) is always attained, i.e. $Z_i(\mathbf{p}) \neq \emptyset$, for all $\mathbf{p} \in \mathbb{R}^n$.*
- (2) *For $\mathbf{p} \in \mathbb{R}^n$, we have*

$$(5.5) \quad Z_i(\mathbf{p}) = \operatorname{argmax}_{\alpha \in \mathbb{R}^n} \{ \nu^{(b)}(\alpha \cdot \mathbf{B}; \gamma_i | \mathcal{E}_i) - \alpha \cdot \mathbf{p} \}.$$

- (3) *Either $Z_i(\mathbf{p}) = \{\alpha\}$ for some $\alpha \in \mathbb{R}^n$ or $Z_i \subseteq \overline{\mathbb{R}}^n \setminus \mathbb{R}^n$.*
- (4) *$Z_i(\mathbf{p}) = \{\alpha\}$ if and only if $\mathbb{E}^{\mathbb{Q}_i^{(\alpha)}}[\mathbf{B}] = \mathbf{p}$ (in particular, $\mathbf{p} \in \mathcal{P}_i^U$).*

Proof.

- (1) It follows from the fact that the function Z_i is upper semi-continuous on the compact space $\overline{\mathbb{R}}^n$.

(2) It suffices to observe that (3.6) implies that

$$(5.6) \quad \mathbf{u}_i(\mathbf{p}) = -\exp\{-\gamma_i \sup_{\boldsymbol{\alpha} \in \mathbb{R}^n} (\nu^{(b)}(\boldsymbol{\alpha} \cdot \mathbf{B}; \gamma_i | \mathcal{E}_i) - \boldsymbol{\alpha} \cdot \mathbf{p})\} \cdot (-u_{\gamma_i}(0 | \mathcal{E}_i)), \text{ for all } \mathbf{p} \in \mathbb{R}^n.$$

(3) The set $Z_i(\mathbf{p})$ is convex, so if it contains a point in \mathbb{R}^n and a point in $\overline{\mathbb{R}^n} \setminus \mathbb{R}^n$, it must contain infinitely many points in \mathbb{R}^n . This is in contradiction with the strict concavity of b_i on \mathbb{R}^n .

(4) Proposition 4.6 states that b_i is continuously differentiable on \mathbb{R}^n and that $\nabla b_i(\boldsymbol{\alpha}) = \mathbb{E}^{\mathbb{Q}_i^{(-\boldsymbol{\alpha})}}[\mathbf{B}]$. Therefore, $\nu^{(b)}(\boldsymbol{\alpha} \cdot \mathbf{B}; \gamma_i | \mathcal{E}_i) - \boldsymbol{\alpha} \cdot \mathbf{p}$ is a concave and differentiable function of $\boldsymbol{\alpha} \in \mathbb{R}^n$ and its derivative is given by $\mathbb{E}^{\mathbb{Q}_i^{(-\boldsymbol{\alpha})}}[\mathbf{B}] - \mathbf{p}$. Consequently, $\nu^{(b)}(\boldsymbol{\alpha} \cdot \mathbf{B}; \gamma_i | \mathcal{E}_i) - \boldsymbol{\alpha} \cdot \mathbf{p}$ attains its maximum on \mathbb{R}^n if and only if $\mathbb{E}^{\mathbb{Q}_i^{(-\boldsymbol{\alpha})}}[\mathbf{B}] = \mathbf{p}$ has a solution $\boldsymbol{\alpha} \in \mathbb{R}^n$. In that case, $Z_i(\mathbf{p}) = \{\boldsymbol{\alpha}\}$. □

Proposition 5.6. *A pair $(\hat{\mathbf{p}}, \hat{\boldsymbol{\alpha}})$ is a PEPQ if and only if $\hat{\mathbf{p}} \in \mathcal{P}_1^U \cap \mathcal{P}_2^U$, $\boldsymbol{\alpha} \in \mathbb{R}^n$ and*

$$(5.7) \quad \mathbb{E}^{\mathbb{Q}_1^{(\hat{\boldsymbol{\alpha}})}}[\mathbf{B}] = \mathbb{E}^{\mathbb{Q}_2^{(-\hat{\boldsymbol{\alpha}})}}[\mathbf{B}] = \hat{\mathbf{p}}.$$

Proof. If $(\hat{\mathbf{p}}, \hat{\boldsymbol{\alpha}})$ is a PEPQ, then $Z_i(\hat{\mathbf{p}}) \cap \mathbb{R}^n \neq \emptyset$ and, so, by Proposition 5.5, part (3), we must have $Z_i(\hat{\mathbf{p}}) = \{\boldsymbol{\alpha}_i\}$, for some $\boldsymbol{\alpha}_i \in \mathbb{R}^n$ and $\hat{\mathbf{p}} \in \mathcal{P}_i^U$, for $i = 1, 2$. By (5.3), we have $\boldsymbol{\alpha}_1 = -\boldsymbol{\alpha}_2$. The equalities in (5.7), with $\hat{\boldsymbol{\alpha}} = \boldsymbol{\alpha}_1$ follow directly from part (4) of Proposition 5.5.

Conversely, suppose that (5.7) holds. Then, by part (4) of Proposition 5.5, we have $Z_1(\mathbf{p}) = \{\hat{\boldsymbol{\alpha}}\}$ and $Z_2(\mathbf{p}) = \{-\hat{\boldsymbol{\alpha}}\}$, which, in turn, implies (5.3). □

We have also shown the following result which will be used shortly:

Corollary 5.7. *A pair $(\hat{\mathbf{p}}, \hat{\boldsymbol{\alpha}}) \in (\mathcal{P}_1^U \cap \mathcal{P}_2^U) \times \mathbb{R}^n$ is a PEPQ if and only if*

$$w_1(\hat{\boldsymbol{\alpha}}) - b_2(\hat{\boldsymbol{\alpha}}) \leq w_1(\boldsymbol{\alpha}) - b_2(\boldsymbol{\alpha}) \text{ for any } \boldsymbol{\alpha} \in \mathbb{R}^n, \text{ and } \hat{\mathbf{p}} = \nabla w_1(\hat{\boldsymbol{\alpha}}).$$

The main result of this Section is presented in the following Theorem:

Theorem 5.8. *Let $\mathcal{E}_1, \mathcal{E}_2 \in \mathbb{L}^\infty$, $\gamma_1, \gamma_2 > 0$ and $\mathbf{B} \in (\mathbb{L}^\infty)^n$ be arbitrary, and suppose that the Assumption 5.3 is satisfied. Then, there exists a unique partial equilibrium price-quantity $(\boldsymbol{\alpha}, \mathbf{p}) \in \mathbb{R}^n \times \mathbb{R}^n$. Moreover, $\mathbf{p} \in \mathcal{P}_1^U \cap \mathcal{P}_2^U$.*

Proof. If the PEPQ $(\boldsymbol{\alpha}, \mathbf{p})$ exists, then $\boldsymbol{\alpha}$ globally minimizes the strictly concave function $w_1 - b_2$, so it must be unique. To establish existence, it will be enough to solve the equation $\nabla f = 0$, where $f = w_1 - b_2$. Assume, to the contrary, that $\nabla f(\boldsymbol{\alpha}) \neq 0$, for all $\boldsymbol{\alpha} \in \mathbb{R}^n$. Continuity of f implies that for each $m \in \mathbb{N}$ there exists $\boldsymbol{\alpha}_m \in \overline{B}_m = \{\boldsymbol{\alpha} \in \mathbb{R}^n : \sum_{i=1}^n |\alpha_i| \leq m\}$ such that $f(\boldsymbol{\alpha}_m) \leq f(\boldsymbol{\alpha})$ for all $\boldsymbol{\alpha} \in \overline{B}_m$. Thanks to strict convexity of f and the fact that $\nabla f \neq 0$ on \overline{B}_m , we must have $\|\boldsymbol{\alpha}_m\|_1 = m$, where $\|\boldsymbol{\alpha}\|_1 = \sum_{i=1}^n |\alpha_i|$. In order to reach a contradiction, it will be enough to show that

$$(5.8) \quad \liminf_{m \rightarrow \infty} \frac{f(\boldsymbol{\alpha}_m)}{m} > 0.$$

Indeed, (5.8) would provide the following coercivity condition

$$\liminf_{m \rightarrow \infty} \inf \left\{ \frac{f(\boldsymbol{\alpha})}{\|\boldsymbol{\alpha}\|_1} : \boldsymbol{\alpha} \in \bar{B}_m \setminus \{\mathbf{0}\} \right\} > 0,$$

which, in turn, would guarantee existence of a global minimizer $\boldsymbol{\alpha}_0 \in \mathbb{R}^n$ for f (see Chapter 1 of Borwein and Lewis (2000)), at which $\nabla f(\boldsymbol{\alpha}_0) = 0$ holds.

The first step in the proof of (5.8) uses the representation (A.8) and the risk-measure properties of $\nu^{(w)}(\cdot; \gamma | \mathcal{E})$ to obtain the following:

$$\begin{aligned} \liminf_{m \rightarrow \infty} \frac{f(\boldsymbol{\alpha}_m)}{m} &= \liminf_{m \rightarrow \infty} \frac{1}{m} \left(\nu^{(w)}(\boldsymbol{\alpha}_m \cdot \mathbf{B} - \boldsymbol{\varepsilon}_1; \gamma_1) + \nu^{(w)}(-\boldsymbol{\alpha}_m \cdot \mathbf{B} - \boldsymbol{\varepsilon}_2; \gamma_2) \right) \\ (5.9) \quad &\geq \liminf_{m \rightarrow \infty} \frac{1}{m} \left(\nu^{(w)}(\boldsymbol{\alpha}_m \cdot \mathbf{B}; \gamma_1) - \|\boldsymbol{\varepsilon}_1\|_{\mathbb{L}^\infty} + \nu^{(w)}(-\boldsymbol{\alpha}_m \cdot \mathbf{B}; \gamma_2) - \|\boldsymbol{\varepsilon}_2\|_{\mathbb{L}^\infty} \right) \\ &= \liminf_{m \rightarrow \infty} \left(\nu^{(w)}\left(\frac{1}{m} \boldsymbol{\alpha}_m \cdot \mathbf{B}; m\gamma_1\right) + \nu^{(w)}\left(-\frac{1}{m} \boldsymbol{\alpha}_m \cdot \mathbf{B}; m\gamma_2\right) \right) \end{aligned}$$

Any subsequence of \mathbb{N} through which the last limit inferior in (5.9) above is realized admits a further subsequence $(m_k)_{k \in \mathbb{N}}$ such that the sequence $\frac{1}{m_k} \boldsymbol{\alpha}_{m_k}$ converges to some $\boldsymbol{\alpha}_0 \in \mathbb{R}^n$ with $\|\boldsymbol{\alpha}_0\|_1 = 1$; indeed, the full sequence $(\frac{1}{m} \boldsymbol{\alpha}_m)_{m \in \mathbb{N}}$ takes values in the compact set $\{\boldsymbol{\alpha} \in \mathbb{R}^n : \|\boldsymbol{\alpha}\|_1 = 1\}$. Proposition A.14 implies that

$$\begin{aligned} (5.10) \quad &\nu^{(w)}\left(\frac{1}{m_k} \boldsymbol{\alpha}_{m_k} \cdot \mathbf{B}; m_k \gamma_1\right) \rightarrow \sup_{\mathbb{Q} \in \mathcal{M}_e} \mathbb{E}^{\mathbb{Q}}[\boldsymbol{\alpha}_0 \cdot \mathbf{B}], \text{ and} \\ &\nu^{(w)}\left(-\frac{1}{m_k} \boldsymbol{\alpha}_{m_k} \cdot \mathbf{B}; m_k \gamma_2\right) \rightarrow - \inf_{\mathbb{Q} \in \mathcal{M}_e} \mathbb{E}^{\mathbb{Q}}[\boldsymbol{\alpha}_0 \cdot \mathbf{B}], \end{aligned}$$

as $k \rightarrow \infty$. Therefore,

$$\liminf_{m \rightarrow \infty} \frac{1}{m} f(\boldsymbol{\alpha}_m) = \sup_{\mathbb{Q} \in \mathcal{M}_e} \mathbb{E}^{\mathbb{Q}}[\boldsymbol{\alpha}_0 \cdot \mathbf{B}] - \inf_{\mathbb{Q} \in \mathcal{M}_e} \mathbb{E}^{\mathbb{Q}}[\boldsymbol{\alpha}_0 \cdot \mathbf{B}].$$

It remains to note that the equality $\sup_{\mathbb{Q} \in \mathcal{M}_e} \mathbb{E}^{\mathbb{Q}}[\boldsymbol{\alpha}_0 \cdot \mathbf{B}] = \inf_{\mathbb{Q} \in \mathcal{M}_e} \mathbb{E}^{\mathbb{Q}}[\boldsymbol{\alpha}_0 \cdot \mathbf{B}]$ cannot hold; if it did, Assumption 5.3 would be violated. \square

Remark 5.9. The concept of the partial equilibrium pricing as stated in Definition 5.2 has its roots in the classic market clearing equilibrium arguments (see e.g., Border (2003), Chapter 6 of Magill and Quinzii (1996) etc. for the mathematical overview). The above proof of the existence of the PEPQ (and its uniqueness guaranteed by Assumption 5.3) differs from the classical scheme in the following sense: the special properties of the exponential utility and, in particular, the form of the induced demand function, allow us to derive an explicit expression for the PEPQ; namely, it is given as the minimizer of the difference $w_1(\boldsymbol{\alpha}) - b_2(\boldsymbol{\alpha})$. Additionally, and unlike the classical approach, such a constructive expression opens a possibility of efficient numerical computation in many cases of interest.

Remark 5.10.

- (1) When $n = 1$, the proof above can be simplified considerably; one can show that

$$\lim_{\alpha \rightarrow \infty} w'_1(\alpha) > \lim_{\alpha \rightarrow \infty} b'_2(\alpha) \text{ and } \lim_{\alpha \rightarrow -\infty} w'_1(\alpha) < \lim_{\alpha \rightarrow -\infty} b'_2(\alpha),$$

and deduce the existence of the solution of the equation $w'_1(\alpha) = b'_2(\alpha)$ directly.

In addition, by Remark 4.14, we easily get that the quantity $\tilde{\alpha} = \frac{\mathbb{E}^{\mathbb{Q}^{(-\gamma_2 \mathcal{E}_2)}}[B] - \mathbb{E}^{\mathbb{Q}^{(-\gamma_1 \mathcal{E}_1)}}[B]}{\gamma_1 \Delta^{\mathbb{Q}^{(-\gamma_1 \mathcal{E}_1)}(B)} + \gamma_2 \Delta^{\mathbb{Q}^{(-\gamma_2 \mathcal{E}_2)}(B)}}$ minimizes the second order approximation of the difference $w_1(\alpha) - b_2(\alpha)$. In view of Corollary 5.7, we can heuristically consider $\tilde{\alpha}$ as an approximation of the partial equilibrium quantity (PEQ), provided that $\tilde{\alpha}$ is close to zero.

- (2) Corollary 3.16 and the discussion preceding it show that when $\frac{\gamma_1}{\gamma_2} \mathcal{E}_1 \sim \mathcal{E}_2$, the unique PEPQ must be of the form $(0, \mathbf{p})$, where $\mathbf{p} = \mathbb{E}^{\mathbb{Q}^{(-\gamma_1 \mathcal{E}_1)}}[\mathbf{B}] = \mathbb{E}^{\mathbb{Q}^{(-\gamma_2 \mathcal{E}_2)}}[\mathbf{B}]$ for every \mathbf{B} which satisfies the Assumption 5.3. In such cases \mathbf{p} should not be interpreted as a price of \mathbf{B} , since no transaction actually occurs. Furthermore, the strict agreement (in the sense of Definition 3.3) can then be reached for no contingent claim of the form $\alpha \cdot \mathbf{B}$, $\alpha \in \mathbb{R}^n$.

Even when $\frac{\gamma_1}{\gamma_2} \mathcal{E}_1 \approx \mathcal{E}_2$, there might exist claims for which the PEPQ is of the form $(0, \mathbf{p})$. In fact, PEPQ is of the form $(0, \mathbf{p})$ if and only if $\mathbb{E}^{\mathbb{Q}^{(-\gamma_1 \mathcal{E}_1)}}[\mathbf{B}] = \mathbb{E}^{\mathbb{Q}^{(-\gamma_2 \mathcal{E}_2)}}[\mathbf{B}]$ (see Proposition 4.13). As an example, consider a claim \mathbf{B} which is independent of the stochastic process \mathbf{S} , as well as the two random endowments. The partial equilibrium price is then simply a certainty equivalent $\mathbf{p} = \mathbb{E}[\mathbf{B}] = \mathbb{E}^{\mathbb{Q}^{(-\gamma_1 \mathcal{E}_1)}}[\mathbf{B}] = \mathbb{E}^{\mathbb{Q}^{(-\gamma_2 \mathcal{E}_2)}}[\mathbf{B}]$.

If a vector of claims \mathbf{B} satisfies the Assumption 5.3 and its PEPQ is of the form $(0, \mathbf{p})$, then $\nu^{(w)}(\alpha \cdot \mathbf{B}; \gamma_1 | \mathcal{E}_1) - \nu^{(b)}(\alpha \cdot \mathbf{B}; \gamma_2 | \mathcal{E}_2) > 0$, for every $\alpha \in \mathbb{R}^n \setminus \{0\}$, i.e. $\alpha \cdot \mathbf{B} \notin \mathcal{G}$ for all $\alpha \neq 0$. In other words, any trade in a nontrivial linear combination $\alpha \cdot \mathbf{B}$ must make at least one of the agents strictly worse off.

APPENDIX A. CONDITIONAL INDIFFERENCE PRICES

The subject of this Section is the conditional indifference price and some of its properties. The results stated below are not only very useful for our analysis mutually agreeability, they may also be seen as interesting in their own right since they describe some of the aspects of indifference evaluation under the presence of random endowment. Some new results about the unconditional indifference price (see Lemma A.7, Propositions A.10, A.11 and A.14), as well as several generalizations of existing results in the case of the conditional price (see Theorem A.3, Propositions A.5 and A.13) are exhibited.

A.1. First properties. We remind the reader that the *writer's and buyer's conditional (relative) indifference prices* $\nu^{(w)}(B; \gamma | \mathcal{E})$ and $\nu^{(b)}(B; \gamma | \mathcal{E})$, for $B \in \mathbb{L}^\infty$, are defined as

$$\nu^{(w)}(B; \gamma | \mathcal{E}) = \inf \{p \in \mathbb{R} : p - B \in \mathcal{A}_\gamma(\mathcal{E})\}, \quad \nu^{(b)}(B; \gamma | \mathcal{E}) = \sup \{p \in \mathbb{R} : B - p \in \mathcal{A}_\gamma(\mathcal{E})\},$$

where $\mathcal{A}_\gamma(\mathcal{E}) = \{B \in \mathbb{L}^\infty : u_\gamma(B | \mathcal{E}) \geq u_\gamma(0 | \mathcal{E})\}$, with the notation introduced on page 7 at the beginning of Section 3. Proposition A.1 collects some basic properties of the indifference prices and its proof is standard.

Proposition A.1.

- (1) $\nu^{(b)}(B; \gamma | \mathcal{E}) = -\nu^{(w)}(-B; \gamma | \mathcal{E})$, for $B \in \mathbb{L}^\infty$.
- (2) When $\mathcal{E} \in \mathcal{R}^\infty$ (in particular, when \mathcal{E} is constant) $\nu^{(w)}(\cdot; \gamma | \mathcal{E})$ and $\nu^{(b)}(\cdot; \gamma | \mathcal{E})$ coincide with their unconditional versions $\nu^{(w)}(\cdot; \gamma)$ and $\nu^{(b)}(\cdot; \gamma)$.
- (3) More generally, we have $\nu^{(w)}(\cdot; \gamma | \mathcal{E}) = \nu^{(w)}(\cdot; \gamma | \mathcal{E}')$ and $\nu^{(b)}(\cdot; \gamma | \mathcal{E}) = \nu^{(b)}(\cdot; \gamma | \mathcal{E}')$ as soon as $\mathcal{E} \sim \mathcal{E}'$.

When \mathcal{E} is constant or, more generally, when $\mathcal{E} \in \mathcal{R}^\infty$, $\nu^{(w)}(\cdot; \gamma | \mathcal{E})$ and $\nu^{(b)}(\cdot; \gamma | \mathcal{E})$ are usually denoted by $\nu^{(w)}(\cdot; \gamma)$ and $\nu^{(b)}(\cdot; \gamma)$, and are called the (writer's and buyer's) *unconditional indifference prices*.

A.2. Conditional indifference prices as convex risk measures. With the notation from subsection 2.4, the conditional indifference price $\nu^{(w)}(\cdot; \gamma | \mathcal{E})$ can be understood as an unconditional indifference price computed under the probability measure $\mathbb{P}_{-\gamma \mathcal{E}}$. In particular, using the terminology of Föllmer and Schied (2004), Section 4.8, the following statement holds:

Proposition A.2. *Maps $B \mapsto \nu^{(w)}(-B; \gamma | \mathcal{E})$ and $B \mapsto -\nu^{(b)}(B; \gamma | \mathcal{E})$ are replication-invariant convex risk measures on \mathbb{L}^∞ , where replication-invariance refers to the following property*

$$\nu^{(w)}(B + (\boldsymbol{\vartheta} \cdot \mathbf{S})_T; \gamma | \mathcal{E}) = \nu^{(w)}(B; \gamma | \mathcal{E}), \text{ for all } \boldsymbol{\vartheta} \in \boldsymbol{\Theta}.$$

Moreover, these measures admit a robust dual representation, as stated in the following theorem, which follows from Theorem 2.2 in Delbaen et al. (2002) and Theorem 2.1 in Kabanov and Stricker (2002):

Theorem A.3. *(Delbaen F., Grandits P., Rheinländer T., Samperi D., Schweizer M. and Stricker C., 2002, Kabanov Y. and Stricker C., 2002)*

For $B \in \mathbb{L}^\infty$, we have

$$(A.1) \quad \nu^{(w)}(B; \gamma | \mathcal{E}) = \sup_{\mathbb{Q} \in \mathcal{M}_a} \left\{ \mathbb{E}_{\mathbb{Q}}(B) - \frac{1}{\gamma} h_{-\gamma \mathcal{E}}(\mathbb{Q}) \right\},$$

where, for $C \in \mathbb{L}^\infty$, we define the map $h_C : \mathbb{L}^1 \mapsto [0, +\infty]$ as

$$h_C(\mathbb{Q}) = \begin{cases} \mathcal{H}(\mathbb{Q} | \mathbb{P}_C) - \mathcal{H}(\mathbb{Q}^{(C)} | \mathbb{P}_C) & \text{when } \mathbb{Q} \in \mathcal{M}_a, \\ +\infty & \text{otherwise.} \end{cases}$$

The supremum in (A.1) is uniquely attained by the measure $\mathbb{Q}^{(-\gamma \mathcal{E} + \gamma B)}$, which belongs in $\mathcal{M}_{e,f}$ and its Radon-Nikodym derivative with respect to $\mathbb{P}_{-\gamma \mathcal{E} + \gamma B}$ can be written as

$$(A.2) \quad \frac{d\mathbb{Q}^{(-\gamma \mathcal{E} + \gamma B)}}{d\mathbb{P}_{-\gamma \mathcal{E} + \gamma B}} = k e^{(-\gamma \boldsymbol{\vartheta}^{(-\gamma \mathcal{E} + \gamma B)} \cdot \mathbf{S})_T},$$

where $\boldsymbol{\vartheta}^{(-\gamma \mathcal{E} + \gamma B)} \in \boldsymbol{\Theta}$ is the maximizer of the control problem associated with the value function $u_\gamma(-B | \mathcal{E})$.

Corollary A.4. *The maps $B \mapsto \nu^{(w)}(B; \gamma | \mathcal{E})$ and $B \mapsto \nu^{(b)}(B; \gamma | \mathcal{E})$ are, respectively, lower and upper semi-continuous with respect to the weak-* topology $\sigma(\mathbb{L}^\infty, \mathbb{L}^1)$.*

Proof. It suffices to note that (A.1) represents $\nu^{(w)}(\cdot; \gamma|\mathcal{E})$ as a supremum of $\sigma(\mathbb{L}^\infty, \mathbb{L}^1)$ -continuous and linear functionals on \mathbb{L}^∞ . \square

The function $h_{-\gamma\mathcal{E}}(\cdot)$ in Theorem A.3 is sometimes called the *penalty function* for the indifference price $\nu^{(w)}(\cdot; \gamma|\mathcal{E})$, and is clearly convex (strictly convex on its effective domain \mathcal{M}_e, f). It is well known (see, e.g., Föllmer and Schied (2004), Lemma 3.29) that the conjugate representation,

$$\mathbb{E}[X \log X] = \sup_{Y \in \mathbb{L}^\infty} (\mathbb{E}[Y X] - \log \mathbb{E}[e^Y]),$$

where we use the convention that $x \log(x) = +\infty$, for $x < 0$, is valid for all $X \in \mathbb{L}^1$. Using this representation and the natural identification of finite measures equivalent to \mathbb{P} with their Radon-Nikodym derivatives in \mathbb{L}^1 , we can readily establish the following properties of the penalty function h :

Proposition A.5. *For $C \in \mathbb{L}^\infty$, $h_C : \mathbb{L}^1 \mapsto [0, +\infty]$ is convex (strictly on its effective domain) and $\sigma(\mathbb{L}^1, \mathbb{L}^\infty)$ -lower semicontinuous.*

An immediate corollary of Proposition A.5 and the Hahn-Banach Theorem in the separation form (see Föllmer and Schied (2004) for details on convex analysis and Jouini et al. (2006), Theorem 2.1) is the following result:

Proposition A.6. *The map $h_{-\gamma\mathcal{E}}$ is the minimal penalty function for $\nu^{(w)}(\cdot; \gamma|\mathcal{E})$, i.e.*

$$h_{-\gamma\mathcal{E}}(\mathbb{Q}) \leq \tilde{h}(\mathbb{Q}), \text{ for all } \mathbb{Q} \in \mathcal{M}_a,$$

whenever the function \tilde{h} satisfies

$$\nu^{(w)}(B; \gamma|\mathcal{E}) = \sup_{\mathbb{Q} \in \mathcal{M}_a} \left(\mathbb{E}_{\mathbb{Q}}(B) - \frac{1}{\gamma} \tilde{h}(\mathbb{Q}) \right), \text{ for all } B \in \mathbb{L}^\infty.$$

Moreover, we have the following, dual, conjugate representation

$$\frac{1}{\gamma} h_{-\gamma\mathcal{E}}(\mathbb{Q}) = \sup_{B \in \mathbb{L}^\infty} \left(\mathbb{E}^{\mathbb{Q}}[B] - \nu^{(w)}(B; \gamma|\mathcal{E}) \right), \quad \forall \mathbb{Q} \in \mathbb{L}^1(\Omega, \mathcal{F}_T, \mathbb{P}_{-\gamma\mathcal{E}}).$$

A.3. Some auxiliary results. Using the linearity of the set Θ of the admissible trading strategies and the properties of the exponential utility, one can deduce (see Becherer (2001), Chapter 1) that the following scaling property holds true:

$$(A.3) \quad \alpha \nu^{(w)}(B; \alpha\gamma) = \nu^{(w)}(\alpha B; \gamma), \text{ for } B \in \mathbb{L}^\infty, \gamma, \alpha > 0.$$

The following Lemma (which is used several times in the present paper) states that the risk measures induced by the indifference price has a certain subadditive property, with true additivity holding only in exceptional cases.

Lemma A.7. For $B_1, B_2 \in \mathbb{L}^\infty$ and $\gamma_1, \gamma_2 > 0$, let $\tilde{\gamma} > 0$ be given by $\frac{1}{\tilde{\gamma}} = \frac{1}{\gamma_1} + \frac{1}{\gamma_2}$. Then,

- (a) $\nu^{(w)}(B_1; \gamma_1) + \nu^{(w)}(B_2; \gamma_2) \geq \nu^{(w)}(B_1 + B_2; \tilde{\gamma})$, and
- (b) the following two conditions are equivalent
 - (1) $\nu^{(w)}(B_1; \gamma_1) + \nu^{(w)}(B_2; \gamma_2) = \nu^{(w)}(B_1 + B_2; \tilde{\gamma})$,
 - (2) $\frac{\gamma_1}{\tilde{\gamma}} B_1 \sim \frac{\gamma_2}{\tilde{\gamma}} B_2$.

Proof.

- (a) Using the dual representation (A.1), the inequality in (a) above is equivalent to the following inequality

$$(A.4) \quad \sup_{\mathbb{Q} \in \mathcal{M}_a} \left(\mathbb{E}^{\mathbb{Q}}[B_1] - \frac{1}{\gamma_1} h(\mathbb{Q}) \right) + \sup_{\mathbb{Q} \in \mathcal{M}_a} \left(\mathbb{E}^{\mathbb{Q}}[B_2] - \frac{1}{\gamma_2} h(\mathbb{Q}) \right) \geq \sup_{\mathbb{Q} \in \mathcal{M}_a} \left(\mathbb{E}^{\mathbb{Q}}[B_1 + B_2] - \frac{1}{\tilde{\gamma}} h(\mathbb{Q}) \right),$$

which always holds for elementary reasons.

- (b) (1) \Rightarrow (2). If the equality in (1) above holds, then it also holds in (A.4). By strict convexity of the function $h(\cdot)$ in this effective domain, i.e. on $\mathcal{M}_{e,f}$, and the scaling property (A.3), this is equivalent to equality of dual minimizers

$$\mathbb{Q}^{(\frac{\gamma_1}{\tilde{\gamma}} B_1)} = \mathbb{Q}^{(\frac{\gamma_2}{\tilde{\gamma}} B_2)} = \mathbb{Q}^{(B_1 + B_2)}.$$

By the representation (A.2) of the Radon-Nikodym derivatives of the above measures, we get

$$\begin{aligned} k_1 e^{(\boldsymbol{\vartheta}^{(\frac{\gamma_1}{\tilde{\gamma}} B_1)} \cdot \mathbf{S})_T} e^{\frac{\gamma_1}{\tilde{\gamma}} B_1} &= \frac{d\mathbb{Q}^{(\frac{\gamma_1}{\tilde{\gamma}} B_1)}}{d\mathbb{P}^{\frac{\gamma_1}{\tilde{\gamma}} B_1}} \frac{d\mathbb{P}^{\frac{\gamma_1}{\tilde{\gamma}} B_1}}{d\mathbb{P}} = \frac{d\mathbb{Q}^{(\frac{\gamma_1}{\tilde{\gamma}} B_1)}}{d\mathbb{P}} = \\ &= \frac{d\mathbb{Q}^{(\frac{\gamma_2}{\tilde{\gamma}} B_2)}}{d\mathbb{P}} = \frac{d\mathbb{Q}^{(\frac{\gamma_2}{\tilde{\gamma}} B_2)}}{d\mathbb{P}^{\frac{\gamma_2}{\tilde{\gamma}} B_2}} \frac{d\mathbb{P}^{\frac{\gamma_2}{\tilde{\gamma}} B_2}}{d\mathbb{P}} = k_2 e^{(\boldsymbol{\vartheta}^{(\frac{\gamma_2}{\tilde{\gamma}} B_2)} \cdot \mathbf{S})_T} e^{\frac{\gamma_2}{\tilde{\gamma}} B_2}, \end{aligned}$$

and so $\frac{\gamma_1}{\tilde{\gamma}} B_1 - \frac{\gamma_2}{\tilde{\gamma}} B_2 = (\boldsymbol{\vartheta} \cdot \mathbf{S})_T + k$, where $k = \log(k_2) - \log(k_1)$ and $\boldsymbol{\vartheta} = \boldsymbol{\vartheta}^{(\frac{\gamma_2}{\tilde{\gamma}} B_2)} - \boldsymbol{\vartheta}^{(\frac{\gamma_1}{\tilde{\gamma}} B_1)}$.

(2) \Rightarrow (1). Conversely, suppose that $\frac{\gamma_1}{\tilde{\gamma}} B_1 - \frac{\gamma_2}{\tilde{\gamma}} B_2 = (\boldsymbol{\vartheta} \cdot \mathbf{S})_T + k$, for some $k \in \mathbb{R}$ and $\boldsymbol{\vartheta} \in \Theta$. Using the scaling property (A.3), the equality in (1) is equivalent to

$$(A.5) \quad \frac{1}{\gamma_1} \nu^{(w)} \left(\frac{\gamma_1}{\tilde{\gamma}} B_1; \tilde{\gamma} \right) + \frac{1}{\gamma_2} \nu^{(w)} \left(\frac{\gamma_2}{\tilde{\gamma}} B_2; \tilde{\gamma} \right) = \frac{1}{\tilde{\gamma}} \nu^{(w)}(B_1 + B_2; \tilde{\gamma})$$

By the risk equivalence between $\frac{\gamma_1}{\tilde{\gamma}} B_1$ and $\frac{\gamma_2}{\tilde{\gamma}} B_2$ and the replication invariance of $\nu^{(w)}(\cdot; \tilde{\gamma})$, we have

$$(A.6) \quad \begin{aligned} \frac{1}{\gamma_1} \nu^{(w)} \left(\frac{\gamma_1}{\tilde{\gamma}} B_1; \tilde{\gamma} \right) + \frac{1}{\gamma_2} \nu^{(w)} \left(\frac{\gamma_2}{\tilde{\gamma}} B_2; \tilde{\gamma} \right) &= \frac{1}{\gamma_1} \nu^{(w)} \left(\frac{\gamma_1}{\tilde{\gamma}} B_1; \tilde{\gamma} \right) + \frac{1}{\gamma_2} \nu^{(w)} \left(\frac{\gamma_1}{\tilde{\gamma}} B_1 + k + (\boldsymbol{\vartheta} \cdot \mathbf{S})_T; \tilde{\gamma} \right) \\ &= \frac{1}{\tilde{\gamma}} \nu^{(w)} \left(\frac{\gamma_1}{\tilde{\gamma}} B_1; \tilde{\gamma} \right) + \frac{k}{\gamma_2}. \end{aligned}$$

On the other hand,

$$(A.7) \quad \frac{1}{\tilde{\gamma}} \nu^{(w)}(B_1 + B_2; \tilde{\gamma}) = \frac{1}{\tilde{\gamma}} \nu^{(w)} \left(B_1 + \frac{\gamma_1}{\gamma_2} B_1 + \frac{\tilde{\gamma}}{\gamma_2} (k + (\boldsymbol{\vartheta} \cdot \mathbf{S})_T); \tilde{\gamma} \right) = \frac{1}{\tilde{\gamma}} \nu^{(w)} \left(\frac{\gamma_1}{\tilde{\gamma}} B_1; \tilde{\gamma} \right) + \frac{k}{\gamma_2}.$$

The equality in (A.5) now follows directly from (A.6) and (A.7). \square

The conjugacy between (affine transformations of) $\nu^{(w)}(\cdot; \gamma|\mathcal{E})$ and $h(\cdot)$, as displayed in Theorem A.3 and Proposition A.6, yields directly the following auxiliary result:

Lemma A.8. *For $\mathcal{E}, \tilde{\mathcal{E}} \in \mathbb{L}^\infty$, $\gamma > 0$, the following two statements are equivalent*

- (1) $\nu^{(w)}(B; \gamma|\mathcal{E}) \geq \nu^{(w)}(B; \gamma|\tilde{\mathcal{E}})$, for all $B \in \mathbb{L}^\infty$,
- (2) $h_{-\gamma\mathcal{E}}(\mathbb{Q}) \leq h_{-\gamma\tilde{\mathcal{E}}}(\mathbb{Q})$, for all $\mathbb{Q} \in \mathcal{M}_a$.

We use Lemma A.8 in the proof of the following proposition:

Proposition A.9. *For $\mathcal{E} \in \mathbb{L}^\infty$ and $\gamma > 0$, the following statements are equivalent:*

- (1) $\nu^{(w)}(B; \gamma) \geq \nu^{(w)}(B; \gamma|\mathcal{E})$, for all $B \in \mathbb{L}^\infty$,
- (2) $\nu^{(w)}(B; \gamma) = \nu^{(w)}(B; \gamma|\mathcal{E})$, for all $B \in \mathbb{L}^\infty$,
- (3) $\mathcal{E} \in \mathcal{R}^\infty$, and
- (4) $\mathbb{Q}^{(0)} = \mathbb{Q}^{(-\gamma\mathcal{E})}$.

Proof. (4) \Rightarrow (3) Just like in the proof of implication (1) \Rightarrow (2) in Lemma A.7, we can use the equation (A.2) in Theorem A.3 to show that (4) implies (3).

(3) \Rightarrow (2) Follows immediately from statement (3) in Proposition A.1.

(2) \Rightarrow (1) Clearly, (1) is weaker than (2).

(1) \Rightarrow (4) By Lemma A.8, the equality in (2) implies that $h_{-\gamma\mathcal{E}}(\mathbb{Q}) \geq h(\mathbb{Q})$, for all $\mathbb{Q} \in \mathcal{M}_a$, i.e.

$$\mathcal{H}(\mathbb{Q}|\mathbb{P}_{-\gamma\mathcal{E}}) - \mathcal{H}(\mathbb{Q}^{(-\gamma\mathcal{E})}|\mathbb{P}_{-\gamma\mathcal{E}}) \geq \mathcal{H}(\mathbb{Q}|\mathbb{P}) - \mathcal{H}(\mathbb{Q}^{(0)}|\mathbb{P}), \quad \forall \mathbb{Q} \in \mathcal{M}_a.$$

In particular, for $\mathbb{Q} = \mathbb{Q}^{(-\gamma\mathcal{E})}$, we get

$$\mathcal{H}(\mathbb{Q}^{(-\gamma\mathcal{E})}|\mathbb{P}) \leq \mathcal{H}(\mathbb{Q}^{(0)}|\mathbb{P}).$$

Therefore, $\mathbb{Q}^{(-\gamma\mathcal{E})} = \mathbb{Q}^{(0)}$, by the strict convexity of the relative entropy $\mathcal{H}(\cdot|\mathbb{P})$ on its effective domain. \square

Considered as convex risk measure, the indifference price is not homogeneous. In fact, the homogeneity holds only for replicable claims as the following proposition states.

Proposition A.10. *For $B, \mathcal{E} \in \mathbb{L}^\infty$ and $\gamma > 0$, the following statements are equivalent:*

- (1) $\nu^{(w)}(\alpha B; \gamma|\mathcal{E}) = \alpha \nu^{(w)}(B; \gamma|\mathcal{E})$, for some $\alpha \in \mathbb{R} \setminus \{0, 1\}$,
- (2) $B \in \mathcal{R}^\infty$.

Proof. We assume, for simplicity, that $\mathcal{E} = 0$ (otherwise, we simply change the underlying probability to $\mathbb{P}_{-\gamma\mathcal{E}}$).

(2) \Rightarrow (1) If $B \in \mathcal{R}^\infty$, then $\alpha B \in \mathcal{R}^\infty$, so (1) follows from the replication-invariance of $\nu^{(w)}(\cdot; \gamma)$.

(1) \Rightarrow (2) Suppose, first, that (1) holds with $\alpha > 0$. Then

$$\sup_{\mathbb{Q} \in \mathcal{M}_a} \left(\mathbb{E}^{\mathbb{Q}}[B] - \frac{1}{\gamma} h(\mathbb{Q}) \right) = \sup_{\mathbb{Q} \in \mathcal{M}_a} \left(\mathbb{E}^{\mathbb{Q}}[B] - \frac{1}{\alpha\gamma} h(\mathbb{Q}) \right).$$

The two maximized functions are strictly concave, ordered and agree only for \mathbb{Q} such that $h(\mathbb{Q}) = 0$. Therefore, the equality of their (attained) suprema forces the relation $h(\mathbb{Q}^{(\gamma B)}) = h(\mathbb{Q}^{(\alpha\gamma B)}) = 0$, which, in turn, implies that $\mathbb{Q}^{(\gamma B)} = \mathbb{Q}^{(\alpha\gamma B)} = \mathbb{Q}^{(0)}$. We can conclude that $B \in \mathcal{R}^\infty$ by using the implication (4) \Rightarrow (3) in Proposition A.9.

It remains to treat the case $\alpha < 0$. By considering the random variable $|\alpha|B$ instead of B , it is clear that we can safely assume that $\alpha = -1$, i.e., $\nu^{(w)}(-B; \gamma) = -\nu^{(w)}(B; \gamma)$. Equivalently, we have

$$\inf_{\mathbb{Q} \in \mathcal{M}_a} \left(\mathbb{E}^{\mathbb{Q}}[B] + \frac{1}{\gamma} h(\mathbb{Q}) \right) = \sup_{\mathbb{Q} \in \mathcal{M}_a} \left(\mathbb{E}^{\mathbb{Q}}[B] - \frac{1}{\gamma} h(\mathbb{Q}) \right),$$

which, by positivity of $h(\cdot)$, implies that $h(\mathbb{Q}^{(\gamma B)}) = 0$. We continue as above to conclude that $B \in \mathcal{R}^\infty$. \square

A.4. Relationship between conditional and unconditional indifference prices. It has been observed (see Remark 1.3.2 in Becherer (2001)) that the conditional indifference price can be written as difference of two unconditional ones:

$$(A.8) \quad \nu^{(w)}(B; \gamma | \mathcal{E}) = \nu^{(w)}(B - \mathcal{E}; \gamma) - \nu^{(w)}(-\mathcal{E}; \gamma) = \nu^{(w)}(B - \mathcal{E}; \gamma) + \nu^{(b)}(\mathcal{E}; \gamma).$$

A similar relationship, namely $\nu^{(b)}(B; \gamma) = \nu^{(b)}(B + \mathcal{E}; \gamma) - \nu^{(b)}(\mathcal{E}; \gamma) = \nu^{(b)}(B + \mathcal{E}; \gamma) + \nu^{(w)}(-\mathcal{E}; \gamma)$, holds for the buyer's conditional indifference prices.

A.5. $\nu^{(w)}(B; \gamma | \mathcal{E})$ as a function of γ . It is a known property of the *unconditional* indifference price that the mappings $\gamma \mapsto \nu^{(w)}(B; \gamma)$ and $\gamma \mapsto -\nu^{(b)}(B; \gamma)$ are non-decreasing. In fact, we have the following, more precise, statement

Proposition A.11. *For $\gamma > 0$ and $B \in \mathbb{L}^\infty$, the mapping $\gamma \mapsto \nu^{(w)}(B; \gamma)$ ($\gamma \mapsto \nu^{(b)}(B; \gamma)$) is*

- (1) *constant and equal to the value $\mathbb{E}^{\mathbb{Q}}[B]$, constant over $\mathbb{Q} \in \mathcal{M}_a$, when $B \in \mathcal{R}^\infty$, and*
- (2) *strictly increasing (decreasing), otherwise.*

Proof. We only deal with the writer's price $\nu^{(w)}(B; \gamma)$. The case of the buyer's price is parallel.

- (1) By the replication invariance of the $\nu^{(w)}(\cdot; \gamma)$, the value of $\nu^{(w)}(B; \gamma)$ equals to the value $\mathbb{E}^{\mathbb{Q}}[B]$, $\mathbb{Q} \in \mathcal{M}_a$, when $B \in \mathcal{R}^\infty$.
- (2) Suppose now that $\nu^{(w)}(B; \gamma_1) \leq \nu^{(w)}(B; \gamma_2)$, for some $0 < \gamma_1 < \gamma_2$. By the dual representation (A.1), we have $\nu^{(w)}(B; \gamma_1) = \nu^{(w)}(B; \gamma_2)$, and using the scaling property (A.3), we get

$$\alpha \nu^{(w)}(B; \gamma_2) = \nu^{(w)}(\alpha B; \gamma_2),$$

where $\alpha = \gamma_2/\gamma_1 > 1$. By Proposition A.10, $B \in \mathcal{R}^\infty$. \square

A similar proposition in the conditional case fails. Indeed, here is a simple example. Pick $\mathcal{E} \notin \mathcal{R}^\infty$, and set $B = \mathcal{E}$. Then $\nu^{(w)}(\mathcal{E}; \gamma | \mathcal{E}) = \nu^{(b)}(\mathcal{E}; \gamma)$ - a *strictly decreasing* function of γ . An even more instructive example in which the dependence of γ ceases to be monotone *at all* is given below.

Example A.12. We adopt the setting of Example 4.5, and assume that the coefficients b and a are chosen in such a way that the distribution of the random variable Y_T is diffuse (under \mathbb{P} , and, therefore, under every equivalent martingale measure). Let $\mathbb{Q}^{(0)}$ be the minimal-entropy martingale measure and let $g_i : \mathbb{R} \rightarrow \mathbb{R}$, $i = 1, 2$ be two bounded Borel-measurable functions. We set $\mathcal{E} = -g_1(Y_T)$ and $B = g_2(Y_T) - g_1(Y_T)$, and compute the conditional indifference price $\nu^{(w)}(B; \gamma | \mathcal{E})$ as a difference $\nu^{(w)}(B; \gamma | \mathcal{E}) = \nu^{(w)}(B - \mathcal{E}; \gamma) - \nu^{(w)}(-\mathcal{E}; \gamma)$. By the expression (A.8) and the formula (4.5), we have

$$(A.9) \quad \begin{aligned} \nu^{(w)}(B; \gamma | \mathcal{E}) &= \nu^{(w)}(g_2(Y_T); \gamma) - \nu^{(w)}(g_1(Y_T); \gamma) \\ &= \frac{1}{\gamma(1 - \rho^2)} \left(\ln \mathbb{E}^{\mathbb{Q}^{(0)}}[\exp(\gamma(1 - \rho^2)g_1(Y_T))] - \ln \mathbb{E}^{\mathbb{Q}^{(0)}}[\exp(\gamma(1 - \rho^2)g_2(Y_T))] \right). \end{aligned}$$

The intervals of monotonicity of the mapping $\gamma \mapsto \nu^{(w)}(B; \gamma | \mathcal{E})$ therefore coincide with the intervals of monotonicity of the function $f : (0, \infty) \rightarrow \mathbb{R}$ given by

$$f(\gamma) = \frac{1}{\gamma} \left(\ln \mathbb{E}^{\mathbb{Q}^{(0)}}[X_1^\gamma] - \ln \mathbb{E}^{\mathbb{Q}^{(0)}}[X_2^\gamma] \right),$$

where the bounded and positive random variables X_i , are given by $X_i = \exp((1 - \rho^2)g_i(Y_T))$, $i = 1, 2$. It is clear that, thanks to the assumption of diffusivity of the random variable Y_T , any pair of probability distributions with compact support in $(0, \infty)$ can be chosen for X_1 and X_2 by the appropriate choice of the functions g_1 and g_2 .

Thanks to the boundedness of X_1 and X_2 , we can easily obtain the following asymptotic expansion for the function f around $\gamma = 0$:

$$f(\gamma) = \mathbb{E}^{\mathbb{Q}^{(0)}}[X_1] - \mathbb{E}^{\mathbb{Q}^{(0)}}[X_2] + \frac{1}{2}\gamma(\text{Var}_{\mathbb{Q}^{(0)}}[X_1] - \text{Var}_{\mathbb{Q}^{(0)}}[X_2]) + o(\gamma).$$

In a similar manner, we have

$$\lim_{\gamma \rightarrow \infty} f(\gamma) = \ln \|X_1\|_{\mathbb{L}^\infty} - \ln \|X_2\|_{\mathbb{L}^\infty}.$$

Therefore, if X_1 and X_2 satisfy

- (1) $\mathbb{E}^{\mathbb{Q}^{(0)}}[X_1] < \mathbb{E}^{\mathbb{Q}^{(0)}}[X_2]$, and
- (2) $\text{Var}_{\mathbb{Q}^{(0)}}[X_1] < \text{Var}_{\mathbb{Q}^{(0)}}[X_2]$,

the function f is strictly decreasing and negative in a neighborhood of $\gamma = 0$. If, in addition

- (3) $\|X_1\|_{\mathbb{L}^\infty} > \|X_2\|_{\mathbb{L}^\infty}$,

holds, this trend cannot continue for all γ since $f(+\infty) = \ln(\|X_1\|_{\mathbb{L}^\infty}/\|X_2\|_{\mathbb{L}^\infty}) > 0 > \mathbb{E}[X_1] - \mathbb{E}[X_2] = f(0+)$. The straightforward construction of examples of the random variables X_1 and X_2 having the above properties is left to the reader.

A.6. Asymptotics of the conditional indifference prices. The asymptotics of the unconditional indifference prices in the risk-aversion parameter γ are well-known (see, for instance, Corollary 5.1 in Delbaen et al. (2002) or Proposition 1.3.4 in Becherer (2001)):

$$(A.10) \quad \begin{aligned} \lim_{\gamma \rightarrow 0} \nu^{(w)}(B; \gamma) &= \mathbb{E}^{\mathbb{Q}^{(0)}}[B], & \lim_{\gamma \rightarrow +\infty} \nu^{(w)}(B; \gamma) &= \sup_{\mathbb{Q} \in \mathcal{M}_{e,f}} \mathbb{E}^{\mathbb{Q}}[B], \\ \lim_{\gamma \rightarrow 0} \nu^{(b)}(B; \gamma) &= \mathbb{E}^{\mathbb{Q}^{(0)}}[B], & \lim_{\gamma \rightarrow +\infty} \nu^{(b)}(B; \gamma) &= \inf_{\mathbb{Q} \in \mathcal{M}_{e,f}} \mathbb{E}^{\mathbb{Q}}[B]. \end{aligned}$$

Using the decomposition (A.8), these are easily extended to the conditional case:

Proposition A.13. *For $B, \mathcal{E} \in \mathbb{L}^\infty$, we have*

$$(A.11) \quad \begin{aligned} \lim_{\gamma \rightarrow 0} \nu^{(w)}(B; \gamma | \mathcal{E}) &= \mathbb{E}^{\mathbb{Q}^{(0)}}[B], & \lim_{\gamma \rightarrow +\infty} \nu^{(w)}(B; \gamma | \mathcal{E}) &= \sup_{\mathbb{Q} \in \mathcal{M}_{e,f}} \mathbb{E}^{\mathbb{Q}}[B - \mathcal{E}] + \inf_{\mathbb{Q} \in \mathcal{M}_{e,f}} \mathbb{E}^{\mathbb{Q}}[\mathcal{E}], \text{ and} \\ \lim_{\gamma \rightarrow 0} \nu^{(b)}(B; \gamma | \mathcal{E}) &= \mathbb{E}^{\mathbb{Q}^{(0)}}[B], & \lim_{\gamma \rightarrow +\infty} \nu^{(b)}(B; \gamma | \mathcal{E}) &= \inf_{\mathbb{Q} \in \mathcal{M}_{e,f}} \mathbb{E}^{\mathbb{Q}}[B - \mathcal{E}] + \sup_{\mathbb{Q} \in \mathcal{M}_{e,f}} \mathbb{E}^{\mathbb{Q}}[\mathcal{E}]. \end{aligned}$$

One can, further, establish the continuous differentiability of the map $\gamma \mapsto \nu^{(w)}(B; \gamma | \mathcal{E})$, for $\gamma \in (0, \infty)$, by noting the fact that

$$(A.12) \quad \nu^{(w)}(B; \gamma | \mathcal{E}) = \frac{1}{\gamma} \left(\nu^{(w)}(\gamma(B - \mathcal{E}); 1) - \nu^{(w)}(-\gamma \mathcal{E}; 1) \right)$$

and using it together with the result of Theorem 5.3 in Ilhan et al. (2005) which states that the function $\gamma \mapsto \nu^{(w)}(\gamma C; 1)$ is continuously differentiable on $(0, \infty)$ for $C \in \mathbb{L}^\infty$.

For $n \in \mathbb{N}$, let $(\mathbb{L}^\infty)^n$ denote the set of all n -tuples $\mathbf{B} = (B_1, \dots, B_n)$ of elements of \mathbb{L}^∞ , with $\|\mathbf{B}\|_{(\mathbb{L}^\infty)^n} = \max_{k \leq n} \|B_k\|_{\mathbb{L}^\infty}$. For $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_n) \in \mathbb{R}^n$, we write $\boldsymbol{\alpha} \cdot \mathbf{B} = \sum_{k=1}^n \alpha_k B_k \in \mathbb{L}^\infty$ and set $|\boldsymbol{\alpha}| = \max_{k \leq n} |\alpha_k|$.

Proposition A.14. *For $\mathcal{E} \in \mathbb{L}^\infty$ and $\mathbf{B} \in (\mathbb{L}^\infty)^n$, the function $w : \mathbb{R}^n \times (0, \infty] \rightarrow \mathbb{R}$ given by*

$$(A.13) \quad w(\boldsymbol{\alpha}, \gamma) = \begin{cases} \nu^{(w)}(\boldsymbol{\alpha} \cdot \mathbf{B}; \gamma | \mathcal{E}), & \gamma < \infty \\ \sup_{\mathbb{Q} \in \mathcal{M}_{e,f}} \mathbb{E}^{\mathbb{Q}}[\boldsymbol{\alpha} \cdot \mathbf{B} - \mathcal{E}] + \inf_{\mathbb{Q} \in \mathcal{M}_{e,f}} \mathbb{E}^{\mathbb{Q}}[\mathcal{E}], & \gamma = +\infty, \end{cases}$$

is jointly continuous, and Lipschitz continuous on every subset D of its domain of the form $D = [\gamma_0, \infty) \times \mathbb{R}^n$, $\gamma_0 > 0$.

Proof. The functional $B \mapsto \nu^{(w)}(B; \gamma | \mathcal{E})$ is positive and coincides with identity on constants, so for $\gamma \in (0, \infty)$,

$$(A.14) \quad \left| \nu^{(w)}(\boldsymbol{\alpha}_1 \cdot \mathbf{B}; \gamma | \mathcal{E}) - \nu^{(w)}(\boldsymbol{\alpha}_2 \cdot \mathbf{B}; \gamma | \mathcal{E}) \right| \leq \|(\boldsymbol{\alpha}_1 - \boldsymbol{\alpha}_2) \cdot \mathbf{B}\|_{\mathbb{L}^\infty} \leq |\boldsymbol{\alpha}_1 - \boldsymbol{\alpha}_2| \|\mathbf{B}\|_{(\mathbb{L}^\infty)^n}.$$

For $\gamma = +\infty$, the validity of (A.14) follows by passing to the limit $\gamma \rightarrow \infty$. On the other hand, by (A.12), for $B \in \mathbb{L}^\infty$, $\gamma_0 > 0$ and $\gamma_1, \gamma_2 \in [\gamma_0, \infty)$, we have

$$(A.15) \quad \begin{aligned} \left| \nu^{(w)}(B; \gamma_1 | \mathcal{E}) - \nu^{(w)}(B; \gamma_2 | \mathcal{E}) \right| &\leq \frac{1}{\gamma_0} \left(\left| \nu^{(w)}(\gamma_1(B - \mathcal{E}); 1) - \nu^{(w)}(\gamma_2(B - \mathcal{E}); 1) \right| \right. \\ &\quad \left. + \left| \nu^{(w)}(-\gamma_1 \mathcal{E}; 1) - \nu^{(w)}(-\gamma_2 \mathcal{E}; 1) \right| \right) \\ &\leq \frac{1}{\gamma_0} (\|B - \mathcal{E}\|_{\mathbb{L}^\infty} + \|\mathcal{E}\|_{\mathbb{L}^\infty}) |\gamma_1 - \gamma_2|. \end{aligned}$$

Therefore, for each $\gamma_0 > 0$, there exists a constant $C = C(\gamma_0) > 0$ such that

$$|w(\boldsymbol{\alpha}_1, \gamma_1) - w(\boldsymbol{\alpha}_2, \gamma_2)| \leq C (|\gamma_1 - \gamma_2| + \|\boldsymbol{\alpha}_1 - \boldsymbol{\alpha}_2\|), \text{ for } \gamma_1, \gamma_2 \in [\gamma_0, \infty), \boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2 \in \mathbb{R}^n.$$

The existence of the limit $w(\boldsymbol{\alpha}, \infty) = \lim_{\gamma \rightarrow \infty} w(\boldsymbol{\alpha}, \gamma)$ is the final ingredient in the proof. \square

APPENDIX B. THE RESIDUAL RISK PROCESS

In this Section, we deal with the the notion of the residual risk in the dynamics setting, i.e., the residual-risk process. We first recall the definition of dynamic version of the (conditional) indifference price.

B.1. A dynamic version of the indifference price. In addition to the study of the indifference prices $\nu^{(w)}(B; \gamma | \mathcal{E})$ and $\nu^{(b)}(B; \gamma | \mathcal{E})$ defined at time $t = 0$, one can restrict attention to any subinterval $[t, T]$ of $[0, T]$, and consider the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_u\}_{u \in [t, T]}, \mathbb{P})$ and the stock-price process $\{\mathbf{S}_u\}_{u \in [t, T]}$. The (conditional) indifference price of the contingent claim B , defined on this restricted model, is denoted by $\nu_t^{(w)}(B; \gamma | \mathcal{E})$. More precisely (see Mania and Schweizer (2005), Proposition 12, page 2127 for details), $\nu_t^{(w)}(B; \gamma | \mathcal{E})$ can be defined to be the a.s.-unique solution of the following equation

$$(B.1) \quad \begin{aligned} \text{esssup}_{\boldsymbol{\vartheta} \in \Theta} \mathbb{E} \left[- \exp \left(- \gamma \left(\mathcal{E} + \int_t^T \boldsymbol{\vartheta}_u d\mathbf{S}_u + \nu_t^{(w)}(B; \gamma | \mathcal{E}) - B \right) \right) \middle| \mathcal{F}_t \right] \\ = \text{esssup}_{\boldsymbol{\vartheta} \in \Theta} \mathbb{E} \left[- \exp \left(- \gamma \left(\mathcal{E} + \int_t^T \boldsymbol{\vartheta}_u d\mathbf{S}_u \right) \right) \middle| \mathcal{F}_t \right]. \end{aligned}$$

One can show using standard dynamic-programming methods (see e.g. Mania and Schweizer (2005)) that, when seen as a stochastic process, $(\nu_t^{(w)}(B; \gamma | \mathcal{E}))_{t \in [0, T]}$ admits a càdlàg modification. The process $(\nu_t^{(w)}(B; \gamma | \mathcal{E}))_{t \in [0, T]}$, modified so as to become càdlàg, is called the *writer's indifference price process* for the claim B . A natural analogue corresponding to the buyer's price can be introduced in a similar fashion.

B.2. The residual risk process. Having defined the dynamic version $(\nu_t^{(b)}(B; \gamma | \mathcal{E}))_{t \in [0, T]}$ of the indifference price process, one can render the notion of the residual risk introduced in Section 3, dynamic, too. More precisely, the writer's residual risk process $(R_t^{(w)}(B; \gamma | \mathcal{E}))_{t \in [0, T]}$ is defined by

$$R_t^{(w)}(B; \gamma | \mathcal{E}) = \nu_t^{(w)}(B; \gamma | \mathcal{E}) - \nu^{(w)}(B; \gamma | \mathcal{E}) - \int_0^t \boldsymbol{\vartheta}_u^{(B| \mathcal{E})} d\mathbf{S}_u.$$

(note that $R_T^{(w)}(B; \gamma|\mathcal{E}) = R^{(w)}(B; \gamma|\mathcal{E})$). We can define the buyer's residual risk process by $R_t^{(b)}(B; \gamma|\mathcal{E}) = R_t^{(w)}(-B; \gamma|\mathcal{E})$. It is straightforward that

$$(B.2) \quad R_t^{(w)}(B; \gamma|\mathcal{E}) = R_t^{(w)}(B - \mathcal{E}; \gamma|\mathcal{E}) - R_t^{(w)}(-\mathcal{E}; \gamma|\mathcal{E}), \quad t \in [0, T].$$

and that the process $(R_t^{(w)}(B; \gamma|\mathcal{E}))_{t \in [0, T]}$ admits a càdlàg modification. It has been shown in Mania and Schweizer (2005) (see Theorem 13) that when \mathbb{F} is continuous, the residual risk process admits a representation in terms of a martingale orthogonal to \mathbf{S} . We state the straightforward extension of this result to the conditional case below.

Theorem B.1 (Mania M. and Schweizer M. (2005)). *Suppose that the filtration \mathbb{F} is continuous, and let the process $(R_t^{(w)}(B; \gamma|\mathcal{E}))_{t \in [0, T]}$ be as above. Then there exists a process $(L_t^{(w)}(B; \gamma|\mathcal{E}))_{t \in [0, T]}$ such that*

- (1) $(L_t^{(w)}(B; \gamma|\mathcal{E}))_{t \in [0, T]}$ is a $\mathbb{Q}^{(-\gamma\mathcal{E})}$ -martingale in the space $BMO(\mathbb{Q}^{(-\gamma\mathcal{E})})$, and
- (2) $R_t^{(w)}(B; \gamma|\mathcal{E}) = L_t^{(w)}(B; \gamma|\mathcal{E}) - \frac{\gamma}{2} \langle L^{(w)}(B; \gamma|\mathcal{E}) \rangle_t$.

When $\mathcal{E} \sim 0$, the family $\{L^{(w)}(B; \gamma)\}_{\gamma > 0}$ admits a limit $L^{(w)}(B; 0)$, as $\gamma \searrow 0$, in $BMO(\mathbb{Q}^{(0)})$. The process $L^{(w)}(B; 0)$ can be identified as a term in the Kunita-Watanabe decomposition

$$(B.3) \quad B_t = E_{\mathbb{Q}^{(0)}}[B] + \int_0^t \hat{\boldsymbol{\vartheta}}_u^{(B)} d\mathbf{S}_u + L_t^{(w)}(B; 0), \quad t \in [0, T],$$

of the $\mathbb{Q}^{(0)}$ -martingale $B_t = \mathbb{E}^{\mathbb{Q}^{(0)}}[B|\mathcal{F}_t]$, where $\hat{\boldsymbol{\vartheta}}^{(B)}$ is an \mathbf{S} -integrable predictable process for which $(\hat{\boldsymbol{\vartheta}}^{(B)}, \mathbf{S})$ a $\mathbb{Q}^{(0)}$ -square integrable martingale. In particular, $L^{(w)}(B; 0)$ is strongly orthogonal to any $\mathbb{Q}^{(0)}$ -local martingale of the form $(\boldsymbol{\vartheta} \cdot \mathbf{S})$, $\boldsymbol{\vartheta} \in L(\mathbf{S})$.

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