PARTIAL EQUILIBRIA WITH CONVEX CAPITAL REQUIREMENTS: EXISTENCE, UNIQUENESS AND STABILITY

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August 2, 2009

Abstract. In an incomplete semimartingale model of a financial market, we consider several risk-averse financial agents who negotiate the price of a bundle of contingent claims. Assuming that the agents' risk preferences are modelled by convex capital requirements, we define and analyze their demand functions and propose a notion of a partial equilibrium price. In addition to sufficient conditions for the existence and uniqueness, we also show that the equilibrium prices are stable with respect to misspecifications of agents' risk preferences.

Key words and phrases. acceptance sets, convex capital requirements, incomplete markets, mutually agreeable claims, partial equilibrium allocation, partial equilibrium price, stability of equilibria

2000 Mathematics Subject Classification. Primary: 91B70; Secondary: 91B30, 60G35.

1. Introduction

In complete market models, the price of a contingent claim is simply given by its replication cost. In the more realistic, incomplete models, the arbitrage-free paradigm typically fails to produce a unique price and yields only a price-interval. The presence of *unhedgeable* claims - due to the aforementioned market incompleteness - necessitates the introduction of another fundamental principle whenever one wants to produce a unique value for a given contingent claim. The long history of empirical inquiry into human behavior under risk dictates that this additional component is related to some numerical measure of risk-aversion, idiosyncratic to the agent valuing the claim. The majority of the existing literature uses agents' risk preferences to induce a *subjective* "pricing" mechanism which provides bid and ask prices for a claim payoff (consider for instance the indifference- or marginal-utility-based price concepts; see, e.g., the references in [15]). In reality, however, the observed price of any claim is always a result of interaction among a number of agents. In fact, the very notion of a "price" makes sense only as the observed quantity at which a transaction between two (or more) agents already took place; consequently, what is called *pricing* in the

Both authors were supported in part by the National Science Foundation under award number DMS-0706947 during the preparation of this work. Any opinions, findings and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect those of the National Science Foundation.

The authors would like to thank Ioannis Karatzas, Constantinos Kardaras, Mihai Sirbu, Stathis Tompaidis, Thaleia Zariphopoulou, and the participants of the Fifth World Congress of the Bachelier Finance Society, London, UK, 2008, for fruitful discussions and good advice.

bulk of the contemporary literature should rather be referred to as *valuation*. We abstain from such a renaming in order to keep in line with the already established terminology.

In the present paper, we consider several risk-averse financial agents who negotiate the price of a fixed bundle of claims, and we propose a *partial-equilibrium* pricing scheme in the spirit of the classical general-equilibrium theory. We place ourselves in a (liquid) financial market model driven by a *locally-bounded semimartingale*, fix a time horizon $T > 0$ and assume that each agent's risk preferences and investment goals are abstracted in the notion of an *acceptance set*. Roughly speaking, this set includes all the positions with maturity up to time T that the agent deems acceptable at time 0. Following the literature of convex risk measures, we assume that each acceptance set satisfies certain standard properties, such as monotonicity and convexity. An additional property that relates the agents' acceptance sets to the liquid market and the set of admissible strategies is also imposed (see Axiom *Ax4* on page 5). Thus axiomatized acceptance sets are naturally identified with *capital requirements* (or *risk measures*); intuitively, the (convex) capital requirement of a payoff is the minimum amount of money which, when added to the payoff, creates a position in the acceptance set.

The notion of risk measure was introduced to mathematical finance in the late nineties (see [3]) and has captured a large part of the research activity in this field since (see, among others, [18] and [25], as well as Chapter 4 in [27] and references therein). Convex risk measures in the context of a liquid financial market were first studied in [16] and [25] (see also [26] and section 4.8 of [27]). In [25], a convex risk measure is defined in the presence of a financial market, where the agent is allowed to trade in discrete time and under specific convex constrains. In [16], a pricing scheme for non-replicable claims based on risk measures is proposed in a finite-probability-space model. The abstract definition of a convex capital requirement and its dual representation for a large family of models was given in [28] and [5]. For the dynamic version of convex-risk-measure-based pricing in an incomplete market setting, we refer the reader to [4] and [36] and, for the sufficient conditions for existence of an optimal trading strategy that makes a contingent claim acceptable, to [46] and [41].

Having described the decision-theoretic set up, we focus on the interaction of $I \geq 2$ agents, who have access to (possibly) different financial markets, and we define *mutually agreeable bundle*. Given a bundle of contingent claims $\boldsymbol{B} = (B_1, B_2, ..., B_n)$, we start by introducing the set of its *allocations*, i.e., the set of matrices that represent the feasible ways of sharing $\boldsymbol{B}$ among agents. Then, we say that a pair $(\boldsymbol{B}, \boldsymbol{a})$, of a bundle $\boldsymbol{B}$ and its allocation $\boldsymbol{a}$, is *mutually agreeable* if there exists a price vector $\mathbf{p}$, at which re-allocation of $\boldsymbol{B}$ according to $\boldsymbol{a}$ is acceptable to every agent at price $\mathbf{p}$. This is a generalized version of the notion of mutually agreeable claims given in [2]. In section 3, we study its properties and relate it to the well-known notion of *Pareto optimality.*

For models that include uncertainty, the concept of a Pareto optimal allocation was first analyzed in the insurance/reinsurance context in [8], [29], and [12] and further developed in [10], [11] and [45]. More recently, the issues related to Pareto optimality and design of an optimal contract were studied in the more general settings of convex (coherent) risk measures (see, e.g., [4], [6], [13], [24], [31] and [34]). In the presence of a financial market, this problem was addressed in [5] and [36]. Recently, in [23] (see also [31]), the concept of Pareto-optimality has been used to determine an equilibrium pricing rule, where the term "pricing rule" refers to a finitely additive measure (an element of the dual of $\mathbb{L}^\infty$). More precisely, the authors provide sufficient and necessary conditions for the existence of a Pareto optimal allocation of agents' endowments, from which an equilibrium pricing rule is induced (in fact, the equilibrium pricing rule is the super-gradient of the representative agent's risk measure).

In this work, instead of establishing an equilibrium pricing rule from a Pareto optimal allocation, we take a more direct approach and apply the classical market-clearing arguments to derive a *partial equilibrium* *price* for a bundle $\boldsymbol{B}$ of claims (in addition to a liquid incomplete financial market). Provided that the agents are not already in a Pareto-optimal configuration, an agent's demand of the vector $\boldsymbol{B}$ at a price $\mathbf{p}$ is defined as the number of units of $\boldsymbol{B}$ that the agent is willing to buy at price $\mathbf{p}$. An equilibrium price for $\boldsymbol{B}$ is, then, the price at which the sum of agents' demands is equal to zero for each component of $\boldsymbol{B}$ and the resulting re-allocation of the bundle $\boldsymbol{B}$ is called the *partial equilibrium allocation.* In section 4, we give sufficient conditions for existence and uniqueness of the partial-equilibrium price and allocation and we relate it to the notion of *agents' agreement.* This result generalizes Theorem 5.8 in [2], where the case of two agents with exponential utility functions is considered.

Having settled the problem of existence and uniqueness of the partial-equilibrium price, we turn to the following question: *How is the equilibrium price-allocation affected by (small) perturbations of the agents' decision criteria?* This problem is of considerable importance, since estimation of the shape of each agent's acceptance set is an extremely difficult task. It is therefore reasonable - in the spirit of Hadamard's requirements (see [30]) - to ask that any result, which uses acceptance sets as exogenously given, should satisfy adequate stability criteria. Despite its importance, the problem of stability of equilibrium prices with respect various problem primitives has not been previously studied in the context of continuous-time finance. Well-posedness of various "single-agent" optimization problems, on the other hand, has been extensively studied and has always been an important part of the optimization theory (standard references on stability, and, more generally, well-posedness of variational problems are [38], [43] and [21]). However, stability of the agent's investment decisions in the presence of a financial market has only recently been investigated, and only for cases of utility function maximizers (see [14], [33], [35], [37]). Stability of problems related to the more general notion of a convex risk measure has still not been studied. In our setting, as demonstrated in Theorem 4.10, the problem of existence of the partial-equilibrium price can be viewed as a minimization problem of the sum of the agents' capital requirements. Considered as such, its stability can be guaranteed by certain conditions on allowed perturbations of the agents' acceptance sets. The central notion in this analysis is the one of *Kuratowski convergence*; it is applied to sets of "acceptable" numbers of units of the given bundle of claims and provides a framework for sufficient conditions for stability. As special cases, we consider the set-ups of [32] and [35], where agents' risk preferences are modelled by utility functions.

The structure of the paper is as follows: In section 2, we describe the market model, introduce necessary notation and state some properties of the agents' acceptance sets and capital requirements. In section 3, we define and discuss the notion of mutually agreeable claim-allocations and analyze its relation to the Pareto optimality. Partial-equilibrium price-allocation is introduced in section 4, where an existence and uniqueness result is provided and discussed. Finally, in section 5 we exhibit conditions on specification of the agents' acceptance sets that yield stability of the equilibrium price.

2. The Market Set-up

2.1. **The Liquid Part of the Financial Market.** Our model of the liquid part of the financial market is based on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$, $T > 0$, which satisfies the usual conditions of right-continuity and completeness. There are $d+1$ traded assets ($d \in \mathbb{N}$), whose discounted price processes are modelled by an $\mathbb{R}^{d+1}-$valued locally bounded semimartingale $(S_t^{(0)}; \mathbf{S}_t)_{t\in[0,T]} = (S_t^{(0)}; S_t^{(1)}, \dots, S_t^{(d)})_{t\in[0,T]}$. The first asset $S_t^{(0)}$ plays the role of a numéraire security or a discount factor. Operationally, we simply set $S_t^{(0)} \equiv 1$, for all $t \in [0,T]$, $\mathbb{P}-$a.s. We also impose the assumption of no free lunch with vanishing risk (see [20]). Namely, we define

$$\mathcal{M}_a = \{\mathbb{Q} \ll \mathbb{P} : \mathbf{S} \text{ is a local martingale under } \mathbb{Q}\}$$

and

$$\mathcal{M}_e = \{\mathbb{Q} \text{ is equivalent to } \mathbb{P} : \mathbf{S} \text{ is a local martingale under } \mathbb{Q}\}$$

and make the following standing assumption.

Assumption 2.1. $\mathcal{M}_e \neq \emptyset$.

We allow the possibility that the liquid part of the financial market is incomplete, i.e., that $\mathcal{M}_e$ is not a singleton.

Remark 2.2. Speaking broadly, the local boundedness condition imposed on the the price process is not necessary for a large portion of our results to hold (mutatis mutandis). The main reason we do impose it has to do with the technical, terminological and notational overhead that we would have to bear otherwise. In particular, we would need to introduce more complicated, and significantly less economically defensible admissibility constraints (see the following subsection). Such a generalization - while requiring much more effort on the part of the reader - would not, in our opinion, bring forth anything essentially new.

2.2. **Admissible Strategies.** For σ-algebra $\mathcal{G} \subseteq \mathcal{F}$, $\mathbb{L}^0(\mathcal{G})$ denotes the set of all $\mathbb{P}-$a.s. equivalence classes of $\mathcal{G}$-measurable random variables, and $\mathbb{L}^\infty(\mathcal{G})$ the set of all (classes of) essentially bounded elements of $\mathbb{L}^0(\mathcal{G})$. When the underlying σ-algebra $\mathcal{G}$ is omitted, it should be assumed that $\mathcal{G} = \mathcal{F}$. Shortcuts $\mathcal{B} + \mathcal{C} = \{X + Y \,:\, X \in \mathcal{B},\ Y \in \mathcal{C}\}$, $-\mathcal{B} = \{-X \,:\, X \in \mathcal{B}\}$, $\mathcal{B}_+ = \{X \in \mathcal{B} \,:\, X \geq 0, \text{ a.s.}\}$, $\mathcal{B}_- = \{X \in \mathcal{B} \,:\, X \leq 0, \text{ a.s.}\}$ for $\mathcal{B}, \mathcal{C} \subseteq \mathbb{L}^0 = \mathbb{L}^0(\mathcal{F})$, will be used throughout.

A financial agent (with initial wealth x) invests in the market by choosing a portfolio strategy $\boldsymbol{\vartheta} \in L(\mathbf{S})$, where $L(\mathbf{S})$ denotes the set of predictable stochastic processes integrable with respect to $\mathbf{S}$. The resulting *wealth process*, $(X_t^{x,\boldsymbol{\vartheta}})_{t\in[0,T]}$, is simply the stochastic integral:

$$X_t^{x,\boldsymbol{\vartheta}} = x + (\boldsymbol{\vartheta} \cdot \mathbf{S})_t = x + \int_0^t \boldsymbol{\vartheta}_u \, d\mathbf{S}_u. \tag{2.1}$$

We say that a strategy $\boldsymbol{\vartheta}$ is *admissible* if the induced wealth process is uniformly bounded from below by a constant and we denote the set of admissible strategies by $\boldsymbol{\Theta}$, i.e.,

$$\boldsymbol{\Theta} = \{\boldsymbol{\vartheta} \in L(\mathbf{S}) : \exists\, c \in \mathbb{R} \text{ such that } c \leq (\boldsymbol{\vartheta} \cdot \mathbf{S})_t,\ \forall t \in [0,T], \text{ a.s.}\} \tag{2.2}$$

The collection of all terminal values of wealth processes corresponding to the initial wealth x and admissible portfolio strategies is denoted by $\mathcal{X}(x)$, i.e.,

$$\mathcal{X}(x) = \left\{ X_T^{x,\boldsymbol{\vartheta}} \,:\, \boldsymbol{\vartheta} \in \boldsymbol{\Theta} \right\}.$$

Furthermore, we define the sets $\mathcal{X} = \bigcup_{x\in\mathbb{R}} \mathcal{X}(x)$, $\mathcal{X}^\infty = \mathcal{X} \cap \mathbb{L}^\infty$ and $\mathcal{R} = \{X \in \mathcal{X} \,:\, -X \in \mathcal{X}\}$.

Remark 2.3.

(1) A simple argument based solely on the absence of arbitrage implies that $X \in \mathcal{R}$ if and only if there exists $x \in \mathbb{R}$ and $\boldsymbol{\vartheta} \in \boldsymbol{\Theta}$ such that $X = x + (\boldsymbol{\vartheta} \cdot \mathbf{S})_T$ and $(\boldsymbol{\vartheta} \cdot \mathbf{S})_t$ is uniformly bounded. In particular, $\mathcal{R} \subseteq \mathbb{L}^\infty$.

(2) The lower bound on the losses of the admissible strategies is imposed to avoid pathologies that the so-called *doubling strategies* create. Moreover, Assumption 2.1 excludes the existence of arbitrage opportunities in the liquid market (see [20], Corollary 1.2). Note also that $X \in \mathcal{X}^\infty$ does not imply that $-X \in \mathcal{X}$, since there exist admissible strategies such that $(\boldsymbol{\vartheta} \cdot \mathbf{S})_T \in \mathbb{L}^\infty$ but $(-\boldsymbol{\vartheta} \cdot \mathbf{S})_t$ is not uniformly bounded from below.

2.3. **The Acceptance Sets.** Given the financial market $(S^{(0)};\mathbf{S})$ and the set of admissible strategies $\mathbf{\Theta}$, we suppose that each agent's risk preferences, investment goals, possible stochastic income, etc., are incorporated in a set $\tilde{\mathcal{A}} \subseteq \mathbb{L}^0(\mathcal{F})$ called the *acceptance set*. We interpret $\tilde{\mathcal{A}}$ as the set that contains the discounted net wealths of investment positions with maturity T that the agent deems acceptable at time $t = 0$.

In concordance with the standard postulates of the risk-measure theory, we assume that $\tilde{\mathcal{A}}$ satisfies the following axioms:

Ax1. $\tilde{\mathcal{A}} + \mathbb{L}^0_+ \subseteq \tilde{\mathcal{A}}$.
Ax2. $\tilde{\mathcal{A}}$ is convex.
Ax3. $\tilde{\mathcal{A}} \cap \mathbb{L}^0_-(\mathcal{F}) = \{0\}$.
Ax4. $\tilde{\mathcal{A}} - \mathcal{X}(0) \subseteq \tilde{\mathcal{A}}$. .

For future use we set $\mathcal{A} = \tilde{\mathcal{A}} \cap \mathbb{L}^\infty$.

Remark 2.4. Axiom *Ax1* simply states that every investment with payoff a.s. above the payoff of an acceptable claim is also acceptable. Axiom *Ax2* reflects the fact that diversified portfolios of acceptable investments should also be acceptable, while Axiom *Ax3* means that the "status quo" (i.e., no investment at all) is an acceptable position and that the non-trivial investments which never make money are not acceptable. Finally, axiom *Ax4* is the one that provides a link between the liquid market and the agent's acceptable positions. One should think of $\tilde{\mathcal{A}}$ as an "already-optimized" representation of agent's preferences, in the sense that the fact that the liquid market stands at the agent's disposal has already been taken into account. One of the direct consequences of *Ax4*, and a more mathematical reformulation of the last sentence, is the following property:

$$\text{If there exists } X \in \mathcal{X}(0) \text{ such that } B + X \in \tilde{\mathcal{A}}, \text{ then } B \in \tilde{\mathcal{A}}. \tag{2.3}$$

More directly, if a position can be improved to acceptability by costless trading, it should already be considered acceptable. The reader should note that the situation is not entirely symmetric: it can happen that $B - X$ is acceptable for some $X \in \mathcal{X}(0)$, but B is not. The reason is that X may not be bounded from above so that there is no admissible strategy (with $-X \notin \mathcal{X}(0)$ being the prime candidate) which will bring B into acceptability.

An important, but by no means only, example of an acceptable set which satisfies *Ax1*-*Ax4* can be constructed using utility functions:

Example 2.5. A classical example of an acceptance set that satisfies the axioms *Ax1*-*Ax4* is the one induced by a utility function, i.e., a mapping $U : (a,\infty) \to \mathbb{R}$, $a \in [-\infty, 0]$, which is strictly concave, strictly increasing, continuously differentiable and satisfies the Inada conditions

$$\lim_{x\to a^+} U'(x) = +\infty \text{ and } \lim_{x\to+\infty} U'(x) = 0.$$

We also include a random endowment (illiquid investments, stochastic income) whose value at time T is given by $\mathcal{E} \in \mathbb{L}^\infty(\mathcal{F}_T)$. The agent's investment goal is to maximize the expected utility by trading in the market assets and for every contingent claim $B \in \mathbb{L}^0_+ - \mathbb{L}^\infty_+$, the resulting indirect utility is defined by

$$u(x|B) = \sup_{X\in\mathcal{X}(0)} \mathbb{E}[U(x + \mathcal{E} + X + B)], \tag{2.4}$$

where $x > 0$ is the agent's initial wealth. For sufficient assumptions that lead to the existence of the optimal trading strategy, we refer the interested reader to [17], [32] for the case $a > -\infty$ and [7] and [40] for $a = -\infty$. The set of acceptable claims is then given by

$$\tilde{\mathcal{A}}_U(x) = \left\{B \in \mathbb{L}^0_+ - \mathbb{L}^\infty_+ \,:\, u(x|B) \geq u(x|0)\right\}. \tag{2.5}$$

It is straightforward to check that $\tilde{\mathcal{A}}_U(x)$ indeed satisfies the axioms *Ax1*-*Ax4*, for $x > 0$.

One can replace the expected utility by a general convex risk measure ρ in the above discussion. For $B \in \mathbb{L}^\infty$, we can introduce the function

$$\hat{\rho}(B) = \inf_{X \in \mathcal{X}(0)} \rho(B + X)$$

and use Lemma 2.4 in [46] to guarantee that $\hat{\rho}$ is itself a convex risk measure whose acceptance set $\hat{\mathcal{A}} = \{B \in \mathbb{L}^\infty : \hat{\rho}(B) \leq 0\}$ satisfies Ax.4.

2.4. **The Convex Capital Requirement.** Given an acceptance set $\tilde{\mathcal{A}}$, we call the map $\rho_{\mathcal{A}} : \mathbb{L}^\infty \to \bar{\mathbb{R}}$, defined by

$$\rho_{\mathcal{A}}(B) = \inf\{m \in \mathbb{R} : m + B \in \tilde{\mathcal{A}}\}, \text{ for every } B \in \mathbb{L}^\infty, \tag{2.6}$$

the agent's *convex capital requirement* or *convex risk measure* induced by the acceptance set $\tilde{\mathcal{A}}$. It follows that $\rho_{\mathcal{A}}(\cdot)$ is convex, non-increasing and cash invariant, i.e., $\rho_{\mathcal{A}}(B + m) = \rho_{\mathcal{A}}(B) - m$, for every $B \in \mathbb{L}^\infty$ and $m \in \mathbb{R}$.

Axioms *Ax1* and *Ax2* imply that $\rho_{\mathcal{A}}(0) = 0$ and the inequality $-\|B\|_\infty \leq B \leq \|B\|_\infty$ together with axiom *Ax1* force $\rho_{\mathcal{A}}(B) \in [-\|B\|_\infty, \|B\|_\infty] \subseteq \mathbb{R}$ for every $B \in \mathbb{L}^\infty$. The inclusion $\mathcal{A} \subseteq \{B \in \mathbb{L}^\infty : \rho_{\mathcal{A}}(B) \leq 0\}$ holds trivially. If, in addition, the set $\mathcal{A}$ satisfies the following mild closedness property

$$\{\lambda \in [0, 1] : \lambda m + (1 - \lambda)B \in \mathcal{A}\} \text{ is closed in } [0, 1], \text{ for every } m \in \mathbb{R}_+ \text{ and } B \in \mathbb{L}^\infty, \tag{2.7}$$

the inverse inclusion also holds (see Proposition 4.7 in [27]). Property (2.7) holds, in particular, if $\tilde{\mathcal{A}} \cap V$ is closed (with respect to any linear topology) for any finite-dimensional subspace $V \subseteq \mathbb{L}^\infty$. In what follows, with a slight abuse of terminology, when we mention the term acceptance set we will refer to the set $\mathcal{A} = \tilde{\mathcal{A}} \cap \mathbb{L}^\infty$, for $\tilde{\mathcal{A}}$ that satisfies *Ax1*-*Ax4*.

Remark 2.6. Similar definitions of the convex capital requirement have been given in [27] (page 207) and [28]. In the former, a given acceptance set $\mathcal{A}$ is related to the market through a larger acceptance set $\hat{\mathcal{A}}$, defined by

$$\hat{\mathcal{A}} = \{B \in \mathbb{L}^\infty : \exists\, \boldsymbol{\vartheta} \in \boldsymbol{\Theta}, A \in \mathcal{A} \text{ such that } (\boldsymbol{\vartheta} \cdot \mathbf{S})_T + B \geq A, \mathbb{P} - \text{a.s.}\}. \tag{2.8}$$

In our case, (2.3) implies that $\hat{\mathcal{A}} = \mathcal{A}$, which is yet another reformulation of the "already-optimized" property of Remark 2.4. In [28], the authors define the generalized capital requirement by

$$\hat{\rho}_{\mathcal{A}}(B) = \inf\{m \in \mathbb{R} : \exists\, X \in \mathcal{X}(m) \text{ such that } X + B \in \mathcal{A}\}.$$

If the acceptance set $\tilde{\mathcal{A}}$ satisfies the axioms *Ax1*-*Ax4*, it is straightforward to show that $\rho_{\mathcal{A}}(B) = \hat{\rho}_{\mathcal{A}}(B)$. The existence of an admissible strategy in the definitions of $\hat{\rho}_{\mathcal{A}}(\cdot)$ and $\hat{\mathcal{A}}$ has been established in [46], Theorem 2.6.

2.5. **A Robust Representation.** It is shown in [25] that under the assumption that $\mathcal{A}$ is weak-$*$ closed (closed in the weak topology $\sigma(\mathbb{L}^\infty, \mathbb{L}^1)$), the convex risk measure $\rho_{\mathcal{A}}(\cdot)$ admits a robust representation in the sense of [3] and [18]. The additionally imposed axiom *Ax4* provides some further information about the penalty function. The following proposition is similar, but not identical, to the results in [27] and [36].

Proposition 2.7. *If $\mathcal{A}$ is a weak-$*$ closed acceptance set, then*

(1) *$\rho_{\mathcal{A}}$ admits a robust representation of the following form*

$$\rho_{\mathcal{A}}(B) = \sup_{\mathbb{Q} \in \mathcal{M}_a} \{\mathbb{E}^{\mathbb{Q}}[-B] - \alpha_{\mathcal{A}}(\mathbb{Q})\} \tag{2.9}$$

for every $B \in \mathbb{L}^\infty$, *where* $\alpha_{\mathcal{A}}(\mathbb{Q}) = \sup_{B\in\mathcal{A}}\{\mathbb{E}^{\mathbb{Q}}[-B]\}$, *i.e.,* $\mathcal{M}_{\mathcal{A}} \subseteq \mathcal{M}_a$, *where* $\mathcal{M}_{\mathcal{A}}$ *is the effective domain of* α *(the set of all probability measures* $\mathbb{Q}$ *such that* $\alpha(\mathbb{Q}) < \infty$*).*

(2) *The set of measures, denoted by* $\partial\rho_{\mathcal{A}}(B)$, *at which the supremum in (2.9) is attained, is non-empty.*

Proof. Thanks to the results in [27] and [36], it is enough to show that for every $\mathbb{Q} \notin \mathcal{M}_a$, $\alpha_{\mathcal{A}}(\mathbb{Q}) = +\infty$. For every such $\mathbb{Q}$, there exists an admissible terminal wealth $X \in \mathcal{X}(x)$, such that $\mathbb{E}^{\mathbb{Q}}[X] > x$, i.e., there exists a portfolio $\boldsymbol{\vartheta} \in \boldsymbol{\Theta}$, such that $(\boldsymbol{\vartheta}\cdot\mathbf{S})_t$ is uniformly bounded from below and $\mathbb{E}^{\mathbb{Q}}[(\boldsymbol{\vartheta}\cdot\mathbf{S})_T] > 0$ (see Theorem 5.6 in [20]). Then, for every $k \in \mathbb{N}$, we define $B_k = -((\boldsymbol{\vartheta}\cdot\mathbf{S})_T \wedge k)$, which belongs to $\mathbb{L}^\infty$. Hence, $B_k + (\boldsymbol{\vartheta}\cdot\mathbf{S})_T = ((\boldsymbol{\vartheta}\cdot\mathbf{S})_T - k)\mathbf{1}_{\{(\boldsymbol{\vartheta}\cdot\mathbf{S})_T \geq k\}} \geq 0$, which means that $B_k + (\boldsymbol{\vartheta}\cdot\mathbf{S})_T \in \tilde{\mathcal{A}}$ for every $k \in \mathbb{N}$. Also, by (2.3) we have that $\lambda B_k \in \mathcal{A}$, for all $\lambda > 0$. Thus, $\alpha_{\mathcal{A}}(\mathbb{Q}) \geq \mathbb{E}^{\mathbb{Q}}[-\lambda B_k]$, for every $k \in \mathbb{N}$. Finally, it is enough to first let $k \to \infty$ and use the Monotone convergence theorem to get that $\alpha_{\mathcal{A}}(\mathbb{Q}) \geq \lambda\mathbb{E}^{\mathbb{Q}}[(\boldsymbol{\vartheta}\cdot\mathbf{S})_T]$, and then let $\lambda \to \infty$. □

Corollary 2.8. *If* $\mathcal{A}$ *is a weak-∗ closed acceptance set, then* $\rho_{\mathcal{A}}(\cdot)$ *satisfies the following replication invariance property: for every* $B \in \mathbb{L}^\infty$ *and every* $C \in \mathcal{R} \cap \mathcal{X}(x)$, *we have* $\rho_{\mathcal{A}}(B + C) = \rho_{\mathcal{A}}(B) - x$.

Remark 2.9. The technical notion of weak-∗-closedness used above is often better understood if replaced by a weaker, but easier to check, notions related to pointwise convergence coupled with appropriate one-sided boundedness requirements. We refer the reader to Chapter 4. of [27] and the references therein.

2.6. **Risk-equivalence.** The following definition (see also [2]) will be used extensively in the sequel:

Definition 2.10. Two random variables $B, C \in \mathbb{L}^\infty$ are said to be *risk-equivalent* (or *equivalent with respect to risk*), denoted by $B \sim C$, if $B - C \in \mathcal{R}$.

It is straightforward to check that the relation $\sim$ is indeed an equivalence relation in $\mathbb{L}^\infty$. The condition $B \sim C$ means that the claims with payoffs B and C carry the same unhedgeable risk. Moreover, it is easy to see that the condition $B \sim C$ implies that

$$\forall\,\lambda \in [0,1],\ \rho_{\mathcal{A}}(\lambda B + (1-\lambda)C) = \lambda\rho_{\mathcal{A}}(B) + (1-\lambda)\rho_{\mathcal{A}}(C). \tag{2.10}$$

On the other hand, if $B \nsim C$, convex combinations of the payoffs B and C may lead to reduction of risk. The following proposition characterizes those acceptance sets $\mathcal{A}$ for which this is always the case.

Proposition 2.11. *Let* $\mathcal{A}$ *be a weak-∗ closed acceptance set. Then, the following two statements are equivalent:*

(1) *For all* $B, C \in \mathcal{A}$ *with* $B \nsim C$ *and every* $\lambda \in (0,1)$, *there exists a random variable* $E \in \mathbb{L}^\infty_+$ *and* $\mathbb{Q} \in \partial\rho_{\mathcal{A}}(\lambda B + (1-\lambda)C)$ *such that,* $\mathbb{Q}(E > 0) > 0$ *and*
$$\lambda B + (1-\lambda)C - E \in \mathcal{A}.$$
(2) *For all* $B, C \in \mathbb{L}^\infty$ *the condition (2.10) implies* $B \sim C$.

Proof. We first assume that (1) holds and that $B, C \in \mathbb{L}^\infty$ are such that $B \nsim C$. Then, we have $B + \rho_{\mathcal{A}}(B), C + \rho_{\mathcal{A}}(C) \in \mathcal{A}$, and so, for any $\lambda \in (0,1)$, there exists $E \in \mathbb{L}^\infty_+$ and $\mathbb{Q} \in \partial\rho_{\mathcal{A}}(\lambda B + (1-\lambda)C)$ such that $\mathbb{Q}(E > 0) > 0$ and
$$\lambda B + (1-\lambda)C + \lambda\rho_{\mathcal{A}}(B) + (1-\lambda)\rho_{\mathcal{A}}(C) - E \in \mathcal{A}.$$

This implies that $\rho_{\mathcal{A}}(\lambda B+(1-\lambda)C-E) \leq \lambda\rho_{\mathcal{A}}(B)+(1-\lambda)\rho_{\mathcal{A}}(C)$, and so, thanks to monotonicity of $\rho_{\mathcal{A}}$, (2.10) cannot hold because

$$\begin{aligned}\rho_{\mathcal{A}}(\lambda B+(1-\lambda)C) &= \mathbb{E}^{\mathbb{Q}}[-\lambda B-(1-\lambda)C]-\alpha_{\mathcal{A}}(\mathbb{Q})\\ &< \mathbb{E}^{\mathbb{Q}}[-\lambda B-(1-\lambda)C+E]-\alpha_{\mathcal{A}}(\mathbb{Q})\\ &\leq \sup_{\tilde{\mathbb{Q}}\in\mathcal{M}_{\mathcal{A}}}\{\mathbb{E}^{\tilde{\mathbb{Q}}}[-\lambda B-(1-\lambda)C+E]-\alpha_{\mathcal{A}}(\tilde{\mathbb{Q}})\} = \rho_{\mathcal{A}}(\lambda B+(1-\lambda)C-E)\\ &\leq \lambda\rho_{\mathcal{A}}(B)+(1-\lambda)\rho_{\mathcal{A}}(C).\end{aligned}$$

Conversely, suppose that (2) holds, i.e., that (2.10) implies $B\sim C$, for all $B,C\in\mathbb{L}^\infty$. Then for any pair $B\nsim C$ and every $\lambda\in(0,1)$ we must have that

$$\rho_{\mathcal{A}}(\lambda B+(1-\lambda)C) < \lambda\rho_{\mathcal{A}}(B)+(1-\lambda)\rho_{\mathcal{A}}(C),$$

and it is enough to use $E=-\rho_{\mathcal{A}}(\lambda B+(1-\lambda)C)$ to verify that (1) holds. □

Definition 2.12. An acceptance set $\mathcal{A}$ is called *risk-strictly convex* if it satisfies either of the two equivalent statements in Proposition 2.11.

Under the assumption that the acceptance set $\mathcal{A}$ is risk-strictly convex, we can say a bit more about the effective domain of the penalty function of the induced risk measure, $\mathcal{M}_{\mathcal{A}}$.

Proposition 2.13. *If the acceptance set $\mathcal{A}$ is weak-$*$ closed and risk-strictly convex, then $\mathcal{M}_{\mathcal{A}}\subseteq\mathcal{M}_e$.*

Proof. Suppose, to the contrary, that there exists $\mathbb{Q}\in\mathcal{M}_{\mathcal{A}}\setminus\mathcal{M}_e$. Since $\mathcal{M}_{\mathcal{A}}\subseteq\mathcal{M}_a$, there exists $A\in\mathcal{F}$ be such that $\mathbb{Q}[A]=0$, but $\mathbb{P}[A]>0$. Then, since $\alpha_{\mathcal{A}}(\mathbb{Q})=\sup_{B\in\mathbb{L}^\infty}\left(\mathbb{E}^{\mathbb{Q}}[-B]-\rho_{\mathcal{A}}(B)\right)$, the choice $B=n\mathbf{1}_A-C$, for $C\in\mathbb{L}^\infty$ yields

$$\rho_{\mathcal{A}}(n\mathbf{1}_A-C)\geq\mathbb{E}^{\mathbb{Q}}[C-n\mathbf{1}_A]-\alpha_{\mathcal{A}}(\mathbb{Q})=\mathbb{E}^{\mathbb{Q}}[C]-\alpha_{\mathcal{A}}(\mathbb{Q}).$$

By convexity, $\rho(C-n\mathbf{1}_A)\leq-\rho(n\mathbf{1}_A-C)$ which implies that

$$\rho_{\mathcal{A}}(C-n\mathbf{1}_A)\leq\mathbb{E}^{\mathbb{Q}}[-C]+\alpha_{\mathcal{A}}(\mathbb{Q})<\infty,$$

for all $n\in\mathbb{N}$, and all $C\in\mathbb{L}^\infty$. Using the convexity of ρ one more time, we obtain

$$\rho(C-\mathbf{1}_A)\leq\tfrac{1}{n}\rho(C-n\mathbf{1}_A)+(1-\tfrac{1}{n})\rho(C)\leq\tfrac{1}{n}(\mathbb{E}^{\mathbb{Q}}[-C]+\alpha_{\mathcal{A}}(\mathbb{Q}))+(1-\tfrac{1}{n})\rho(C),$$

for each $n\in\mathbb{N}$, so $\rho(C-\mathbf{1}_A)=\rho(C)$. It follows now from Proposition 2.11 that $\mathbf{1}_A\in\mathcal{R}$, which is in contradiction with the assumption of No Free Lunch with Vanishing Risk. □

Remark 2.14. In the terminology of [27] (see page 173), a risk measure ρ is called *sensitive* if $\rho(-B)>0$ for every $B\in\mathbb{L}^\infty_+\setminus\{0\}$. The fact, as stated in Proposition 2.13, that the minimizers of the penalty function $\alpha_{\mathcal{A}}(\cdot)$ are equivalent to $\mathbb{P}$ when $\mathcal{A}$ is risk-strictly convex, implies that a risk-strictly convex risk measures are sensitive. Indeed, by (2.9), $\rho_{\mathcal{A}}(-B)=\mathbb{E}^{\mathbb{Q}}[B]-\alpha_{\mathcal{A}}(\mathbb{Q})$, for any $\mathbb{Q}\in\partial\rho_{\mathcal{A}}(0)$. Clearly $\mathbb{E}^{\mathbb{Q}}[B]>0$ and $\alpha_{\mathcal{A}}(\mathbb{Q})(B)\leq 0$, so $\rho_{\mathcal{A}}(-B)>0$.

Another important property of risk-strictly convex acceptance sets is the following:

Proposition 2.15. *Let $\rho_{\mathcal{A}}$ be the risk measure corresponding to a weak-$*$ closed, risk-strictly convex acceptance set $\mathcal{A}$. Then,*

$$\rho_{\mathcal{A}}(B)+\rho_{\mathcal{A}}(-B)>0,\ \textit{for } B\in\mathbb{L}^\infty\setminus\mathcal{R}.$$

In particular, $\mathcal{A}\cap(-\mathcal{A})=\mathcal{X}(0)\cap\mathcal{R}$.

Proof. For any such $B \in \mathbb{L}^\infty \setminus \mathcal{R}$, $\rho_{\mathcal{A}}(B) + B \in \mathcal{A}$ and $\rho_{\mathcal{A}}(-B) - B \in \mathcal{A}$. By assumption (since $2B \notin \mathcal{R}$), there exists $E \in \mathbb{L}^\infty_+ \backslash \{0\}$ such that

$$\frac{1}{2}(\rho_{\mathcal{A}}(B) + \rho_{\mathcal{A}}(-B)) - E \in \mathcal{A}.$$

Hence, $\frac{1}{2}(\rho_{\mathcal{A}}(B) + \rho_{\mathcal{A}}(-B)) - \rho_{\mathcal{A}}(-E) \geq 0$ and by monotonicity of $\rho_{\mathcal{A}}$, we get that $\rho_{\mathcal{A}}(B) + \rho_{\mathcal{A}}(-B) > 0$. The last statement follows from Corollary 2.8. □

3. Mutually-Agreeable Bundles

3.1. **The Agents.** We consider $I \geq 2$ financial agents and suppose that each agent i has access to a sub-market $\mathbf{S}_i$ of $\mathbf{S}$, i.e., she is allowed to invest only in $(S^{(0)}_t; S^{(j^i_1)}_t, \dots, S^{(j^i_{d_i})}_t)_{t\in[0,T]}$, where $1 \leq j^i_1 < \cdots < j^i_{d_i} \leq d$. Note that the numéraire $S^{(0)} \equiv 1$ is accessible to each agent. In order to take the whole market into account and avoid trivialities, we also assume that each component of $\mathbf{S}$ is accessible to at least one agent. Note that the Assumption 2.1 implies that $\mathcal{M}^i_a \neq \emptyset$, for all i, where

$$\mathcal{M}^i_a = \{\mathbb{Q} \ll \mathbb{P} : \mathbf{S}_i \text{ is a local martingale under } \mathbb{Q}\}, \; i = 1, 2, \dots, I.$$

We define the sets $\mathcal{X}_i$, $\mathcal{X}_i(x)$, $\mathbf{\Theta}_i$ and $\mathcal{R}_i$ exactly as in section 2, with $\mathbf{S}_i$ used in lieu of $\mathbf{S}$. Moreover, each agent is assumed to have an acceptance set $\tilde{\mathcal{A}}_i$ which satisfies the axioms *Ax1*-*Ax4*. The induced risk measure $\rho_{\mathcal{A}_i}$ on $\mathbb{L}^\infty$ will be denoted by ρ_i, and $\mathcal{M}_i$ will be the shortcut for $\mathcal{M}_{\mathcal{A}_i}$, i.e., it will stand for the effective domain of the corresponding penalty function α_i, $i = 1, 2, \dots, I$. We further assume that the intersection $\tilde{\mathcal{A}}_i \cap \mathbb{L}^\infty$, denoted by $\mathcal{A}_i$, is weak-$*$ closed and hence the induced risk measure $\rho_i = \rho_{\mathcal{A}_i}$ admits the following robust representation

$$\rho_i(B) = \sup_{\mathbb{Q}\in\mathcal{M}_i} \{\mathbb{E}^{\mathbb{Q}}[-B] - \alpha_i(\mathbb{Q})\}, \tag{3.1}$$

where $\mathcal{M}_i \subseteq \mathcal{M}^i_a$ for all i. As above, $\partial\rho_i(B)$ denotes the set of all maximizers in (3.1), for $B \in \mathbb{L}^\infty$ and $i = 1, 2, ..., I$.

3.2. **Bundles, Allocations and Agreement.** For bundle of claims $\boldsymbol{B} = (B_1, B_2, ..., B_n) \in (\mathbb{L}^\infty)^n$, $n \in \mathbb{N}$, a matrix $(a_{i,k}) = \boldsymbol{a} \in \mathbb{R}^{I\times n}$ is called a *feasible allocation* or simply an *allocation*, if $\sum_{i=1}^I a_{i,k} = 0$ for all $k = 1, 2, ..., n$. For convenience, the i-th row $(a_{i,k})_{k=1}^n$ of $\boldsymbol{a}$ will be denoted by $\boldsymbol{a}_i$; it counts the quantities of each of the n components of $\boldsymbol{B}$ held by the agent i. The set of all feasible allocations is denoted by $\mathbf{F}$, i.e.,

$$\mathbf{F} = \{\boldsymbol{a} \in \mathbb{R}^{I\times n} \text{ such that } \sum_{i=1}^{I} \boldsymbol{a}_i = (0, 0, ..., 0)\}. \tag{3.2}$$

We usually think of the elements of $\boldsymbol{B}$ as the claims (typically not replicable in the liquid market $\mathbf{S}$) the agents are trading among themselves. These claims are in zero net supply, i.e., some of the agents will be taking positive and some negative positions in them. Clearly, the agents will be willing to share the bundle of claims $\boldsymbol{B}$ according to an allocation $\boldsymbol{a} \in \mathbf{F}$ only if there exists a price vector $\mathbf{p} \in \mathbb{R}^n$, for which the position $\boldsymbol{a}_i \cdot \boldsymbol{B} - \boldsymbol{a}_i \cdot \mathbf{p}$ is acceptable for each agent i. More precisely, we give the following definition:

Definition 3.1. The pair $(\boldsymbol{B}, \boldsymbol{a}) \in (\mathbb{L}^\infty)^n \times \mathbf{F}$ of a bundle of claims and an allocation is called *mutually agreeable* if there exists a (price) vector $\mathbf{p} \in \mathbb{R}^n$ such that $\boldsymbol{a}_i \cdot \boldsymbol{B} - \boldsymbol{a}_i \cdot \mathbf{p} \in \mathcal{A}_i$, for all $i = 1, 2, ..., I$.

For a bundle $\boldsymbol{B} \in (\mathbb{L}^\infty)^n$, we define the subset $\mathcal{G}^{\boldsymbol{B}}$ of $\mathbb{R}^{I\times n}$ by

$$\mathcal{G}^{\boldsymbol{B}} = \{\boldsymbol{a} \in \mathbf{F} : (\boldsymbol{B}, \boldsymbol{a}) \text{ is mutually agreeable}\}.$$

Similarly, for an allocation $\boldsymbol{a}$, let $\mathcal{G}^{\boldsymbol{a}}$ denote the set of all feasible allocations of $\boldsymbol{B}$, acceptable for every agent:

$$\mathcal{G}^{\boldsymbol{a}} = \{\boldsymbol{B} \in (\mathbb{L}^\infty)^n : \ (\boldsymbol{B}, \boldsymbol{a}) \text{ is mutually agreeable}\}.$$

We also define $\hat{\mathcal{R}}_{\boldsymbol{a}} = \{\boldsymbol{B} \in (\mathbb{L}^\infty)^n : \boldsymbol{a}_i \cdot \boldsymbol{B} \in \mathcal{R}_i, \ \forall i = 1, 2, ..., I\}$, for $\boldsymbol{a} \in \mathbf{F}$.

While $\mathcal{G}^{\boldsymbol{B}}$ has a clear economic importance, that of its dual counterpart $\mathcal{G}^{\boldsymbol{a}}$ is purely technical, as our next proposition shows.

Proposition 3.2.

(1) *For every $\boldsymbol{B} \in (\mathbb{L}^\infty)^n$, the set $\mathcal{G}^{\boldsymbol{B}}$ is convex, closed in $\mathbb{R}^{n \times I}$ and $\boldsymbol{a} \in \mathcal{G}^{\boldsymbol{B}} \cap \mathcal{G}^{-\boldsymbol{B}}$ implies that $\boldsymbol{B} \in \hat{\mathcal{R}}_{\boldsymbol{a}}$.*

(2) *For every allocation $\boldsymbol{a} \in \mathbf{F}$, $\mathcal{G}^{\boldsymbol{a}}$ is convex. If, additionally, $\mathcal{A}_i$ is weak*-closed and risk-strictly convex for every i, then*

$$\mathcal{G}^{\boldsymbol{a}} \cap (-\mathcal{G}^{\boldsymbol{a}}) = \hat{\mathcal{R}}_{\boldsymbol{a}}, \ \textit{for all } \boldsymbol{a} \in \mathbf{F}.$$

Proof. Since the proofs of the two statements are similar, we only prove (2), which is marginally more involved.

Convexity follows directly from the convexity of $\mathcal{A}_i$'s. For the second statement, suppose first that $\boldsymbol{B} \in \mathcal{G}^{\boldsymbol{a}} \cap (-\mathcal{G}^{\boldsymbol{a}})$, i.e., there exist $\mathbf{p}, \hat{\mathbf{p}} \in \mathbb{R}^n$ such that $\boldsymbol{a}_i \cdot \boldsymbol{B} - \boldsymbol{a}_i \cdot \mathbf{p} \in \mathcal{A}_i$ and $-\boldsymbol{a}_i \cdot \boldsymbol{B} + \boldsymbol{a}_i \cdot \hat{\mathbf{p}} \in \mathcal{A}_i$, for all i. The convexity of $\mathcal{A}_i$ then implies that $\frac{1}{2}\boldsymbol{a}_i \cdot (\hat{\mathbf{p}} - \mathbf{p}) \in \mathcal{A}_i$, which yields that $\boldsymbol{a}_i \cdot (\hat{\mathbf{p}} - \mathbf{p}) \geq 0$, for all $i = 1, 2, ..., I$. Since $\sum_{i=1}^{I} \boldsymbol{a}_i = 0$, we conclude that for every agent $\boldsymbol{a}_i \cdot \mathbf{p} = \boldsymbol{a}_i \cdot \hat{\mathbf{p}}$. It follows that $\rho_i(\boldsymbol{a}_i \cdot \boldsymbol{B}) \leq -\boldsymbol{a}_i \cdot \mathbf{p}$ and also $\rho_i(-\boldsymbol{a}_i \cdot \boldsymbol{B}) \leq \boldsymbol{a}_i \cdot \mathbf{p}$ for all i. Consequently, $\rho_i(\boldsymbol{a}_i \cdot \boldsymbol{B}) + \rho_i(-\boldsymbol{a}_i \cdot \boldsymbol{B}) \leq 0$, which means (thanks to the risk-strict convexity of ρ_i) that $\boldsymbol{a}_i \cdot \boldsymbol{B} \in \mathcal{R}_i$ for all i, i.e., $\boldsymbol{B} \in \hat{\mathcal{R}}_{\boldsymbol{a}}$.

Conversely, let $\boldsymbol{B} \in \hat{\mathcal{R}}_{\boldsymbol{a}}$ so that $\boldsymbol{a}_i \cdot \boldsymbol{B} \in \mathcal{R}_i$, for all i. We pick an arbitrary $\mathbb{Q} \in \mathcal{M}_a$ and define $\mathbf{p} = \mathbb{E}^{\mathbb{Q}}[\boldsymbol{B}] = (\mathbb{E}^{\mathbb{Q}}[\boldsymbol{B}_1], \ldots, \mathbb{E}^{\mathbb{Q}}[\boldsymbol{B}_n]) \in \mathbb{R}^n$. Since $\mathbb{Q} \in \mathcal{M}_i^a$ for all i, we have $\boldsymbol{a}_i \cdot \boldsymbol{B} \in \mathcal{X}_i(\boldsymbol{a}_i \cdot \mathbf{p})$. By Corollary 2.8, $\boldsymbol{a}_i \cdot \boldsymbol{B} - \boldsymbol{a}_i \cdot \mathbf{p} + \in \mathcal{A}_i$, for all i, so that $\hat{\mathcal{R}}_{\boldsymbol{a}} \subseteq \mathcal{G}^{\boldsymbol{a}}$. It remains to observe that $\hat{\mathcal{R}}_{-\boldsymbol{a}} = \hat{\mathcal{R}}_{\boldsymbol{a}}$ and that $\mathcal{G}^{-\boldsymbol{a}} = -\mathcal{G}^{\boldsymbol{a}}$. ☐

Another important notion in this setting is the inf-convolution of risk measures, first introduced in [4].

Definition 3.3. The *inf-convolution* of the risk measures $\rho_1, \rho_2, ..., \rho_I$ is the map $\rho_1 \Diamond \ldots \Diamond \rho_I : \mathbb{L}^\infty \to \mathbb{R} \cup \{-\infty\}$, defined for $C \in \mathbb{L}^\infty$ by

$$(\rho_1 \Diamond \ldots \Diamond \rho_I)(C) = \inf \left\{ \sum_{i=1}^{I} \rho_i(B_i) \ : \ B_1, \ldots, B_I \in \mathbb{L}^\infty, \ \sum_{i=1}^{I} B_i = C \right\}. \tag{3.3}$$

Let $\mathcal{M}$ denote the intersection $\bigcap_{i=1}^{I} \mathcal{M}_i$. The following assumption is equivalent to $(\rho_1 \Diamond \ldots \Diamond \rho_I)(0) > -\infty$ (see [5]):

Assumption 3.4. $\mathcal{M} \neq \emptyset$.

Remark 3.5. Thanks to the assumption that each component of $\mathbf{S}$ is available to at least one agent, we have that $\bigcap_{i=1}^{I} \mathcal{M}_e^i = \mathcal{M}_e$. Hence, if $\mathcal{A}_i$'s are risk-strictly convex, it holds that $\mathcal{M} \subseteq \mathcal{M}_e$ and hence Assumption 3.4 is a strengthening of Assumption 2.1.

The following Proposition is a mild generalization of Theorem 3.6 in [5], where the case of $I = 2$ is addressed. The proof when $I \geq 2$ is similar and hence omitted.

Proposition 3.6. *If $\mathcal{A}_i$ is weak-$*$ closed for every $i = 1, 2, ..., I$, the Assumption 3.4 implies that the map $\rho_1 \Diamond \ldots \Diamond \rho_I : \mathbb{L}^\infty \to \mathbb{R}$ is a convex risk measure, with penalty function $h(\mathbb{Q}) = \sum_{i=1}^{I} \alpha_i(\mathbb{Q})$ whose effective domain is $\mathcal{M}$.*

Definition 3.7. We say that the agents are in a *Pareto-optimal configuration* if

$$(\rho_1 \Diamond \dots \Diamond \rho_I)(0) = 0.$$

In words, Pareto optimality implies that there is no wealth-preserving transaction that will be acceptable for everyone and strictly acceptable for at least one agent. The problem of Pareto-optimality is closely related to the problem of *optimal risk sharing* (sometimes called *Pareto optimal allocation*), which was recently addressed by many authors in the cases where agents use convex risk measures to value claim payoffs (see [5], [6], [13], [24], [31], [34]). Below, we state a well-known characterization of the Pareto optimality in terms of the minimizers of the penalty functions α_i. We remind the reader that $\partial\rho_i(B)$ stands for the *set* of maximizers in the robust representation (3.1) of $\rho_i(B)$ and omit the standard proof (see, e.g., Proposition 3.8 in [4]):

Proposition 3.8. *The agents are in a Pareto-optimal configuration if and only if*

$$\cap_{i=1}^I \partial\rho_i(0) \neq \emptyset.$$

The following proposition states that if the agents are in Pareto-optimal configuration, the risk-strict convexity assumption implies that transactions involving non-replicable claims result in strictly increased risk for at least one of the agents involved in this transaction.

Proposition 3.9. *Assume that $\mathcal{A}_i$ are weak-$*$ closed and risk-strictly convex for all $i = 1, 2, .., I$ and suppose that $(\rho_1 \Diamond \dots \Diamond \rho_I)(0) = 0$. For any choice of $B_1, B_2, ..., B_I$ in $\mathbb{L}^\infty$ with $\sum_{i=1}^I B_i = 0$, it holds that*

$$\sum_{i=1}^I \rho_i(B_i) = 0 \text{ if and only if } B_i \in \mathcal{R}_i, \text{ for all } i = 1, 2, ..., I.$$

Proof. Assume that there exists $k \in \{1, 2, ..., I\}$ such that $B_k \notin \mathcal{R}_k$. Then, by risk-strict convexity, for each $\lambda \in (0, 1)$ there exists $E \in \mathbb{L}^\infty_+ \backslash \{0\}$ such that $\rho_k(\lambda B_k - E) \leq \lambda\rho_k(B_k)$. This implies that $\rho_k(\lambda B_k) < \lambda\rho_k(B_k)$. Since $\rho_i(\lambda B_i) - \lambda\rho_i(B_i) \leq 0$, $\forall i = 1, 2, ..., I$, we have

$$\sum_{i=1}^I \rho_i(\lambda B_i) < \lambda \sum_{i=1}^I \rho_i(B_i) = 0,$$

which contradicts the assumption $(\rho_1 \Diamond \dots \Diamond \rho_I)(0) = 0$. The converse implication follows from the fact that when $B_i \in \mathcal{R}_i$, we have $\rho_i(B_i) = -\mathbb{E}^{\mathbb{Q}}[B_i]$, for any $\mathbb{Q} \in \mathcal{M}_a^i$. □

Corollary 3.10. *Assume that all $\mathcal{A}_i$ are weak-$*$ closed and risk-strictly convex and pick $\boldsymbol{a} \in \mathbf{F}$ and $\boldsymbol{B} \in \mathcal{G}^{\boldsymbol{a}}$. If $\cap_{i=1}^I \partial\rho_i(0) \neq \emptyset$ then, $\boldsymbol{a}_i \cdot \boldsymbol{B} \in \mathcal{R}_i$, for every $i = 1, 2, ..., I$.*

Proof. $\boldsymbol{B} \in \mathcal{G}^{\boldsymbol{a}}$ means that there exists a price vector $\mathbf{p}$, such that $\boldsymbol{a}_i \cdot \boldsymbol{B} - \boldsymbol{a}_i \cdot \mathbf{p} \in \mathcal{A}_i$, for all i. This implies that $\sum_{i=1}^I \rho_i(\boldsymbol{a}_i \cdot \boldsymbol{B}) \leq 0$, which, by the hypotheses and Proposition 3.9 yields that $\boldsymbol{a}_i \cdot \boldsymbol{B} \in \mathcal{R}_i$ for all $i = 1, 2, ..., I$. □

Example 3.11. Suppose that all I agents are exponential-utility maximizers with possibly different risk-aversion coefficients γ_i, $i = 1, 2, \dots, I$, i.e., $U_i(x) = -\exp(-\gamma_i x)$ (for details on the set of admissible strategies, we refer the reader to [19] and [39]). Let $\mathcal{E}_i$, $i = 1, 2, \dots, I$, denote the agents' random endowments. If we follow the arguments of Proposition 3.15 in [2], we can conclude that in the case $\mathbf{S}_i = \mathbf{S}$ for all $i \in \{1, 2, ..., I\}$, the agents will be in the Pareto-optimal configuration if and only if $\frac{\gamma_i}{\gamma_j}\mathcal{E}_i \sim \mathcal{E}_j$, for all $i, j = 1, 2, ..., I$. A special case of this condition occurs when the agents' random endowments are replicable. However, this is not the case when agents have access to different markets. To see that, let us consider the case $I = 2$, with $\mathbf{S}_1 \neq \mathbf{S}_2$ and $\mathcal{E}_1 = \mathcal{E}_2 = 0$. It follows from Theorem 2.2 in [19], that $\partial\rho_1(0) = \partial\rho_2(0)$ (both are singletons in this case) if and only if $\gamma_1(\boldsymbol{\vartheta}_1^{(0)} \cdot \mathbf{S}_1)_T = \gamma_2(\boldsymbol{\vartheta}_2^{(0)} \cdot \mathbf{S}_2)_T$, where $\boldsymbol{\vartheta}_i^{(0)}$ is the optimal trading strategy in the market $\mathbf{S}_i$ of the agent i, $i = 1, 2$.

Example 3.12. The case where agents use power utility function, $U(x) = \frac{1}{p}x^p$, where $p \neq 0$ is the relative risk aversion coefficient, is similar to the previous example. Without going into details, let us mention that it is known (see for instance [44]) that if all agents have access to the same market, then $\partial\rho_i(0) = \partial\rho_j(0)$, for all i, j if and only if the agents' relative risk aversion coefficients are equal, regardless of their initial wealths. However, when agents have access to different markets, counterexamples can be readily constructed.

4. The Partial Equilibrium Price Allocation

4.1. **The Definition.** Having introduced the setup consisting of I agents, their acceptance sets and accessible assets, we turn to the partial-equilibrium pricing problem for a fixed bundle of claims $\boldsymbol{B} \in (\mathbb{L}^\infty)^n$. The main task of section is to prove existence and uniqueness of a partial-equilibrium price for a given bundle $\boldsymbol{B} \in (\mathbb{L}^\infty)^n$. We follow the classical paradigm: the price vector $\mathbf{p}$ is a *partial-equilibrium price* of $\boldsymbol{B}$, if, when $\boldsymbol{B}$ trades at $\mathbf{p}$, demand and supply for each of its components offset each other, i.e., the market for $\boldsymbol{B}$ clears.

Clearly, we must first specify what we mean by "demand and supply", i.e., we need to analyze single agent's behavior. The natural assumption we make is that in a set of payoffs, an agent will choose the one which minimizes the capital requirement. For linear combinations of the components of $\boldsymbol{B}$, we have the following, more precise, definition:

Definition 4.1. For $i \in \{1, 2, ..., I\}$, the *agent i's demand correspondence* $Z_i : \mathbb{R}^n \to 2^{\mathbb{R}^n}$ is defined by

$$Z_i(\mathbf{p}) = \underset{\boldsymbol{a} \in \mathbb{R}^n}{\operatorname{argmin}}\{\rho_i(\boldsymbol{a} \cdot \boldsymbol{B} - \boldsymbol{a} \cdot \mathbf{p})\} \tag{4.1}$$

Once the notion of demand is defined, we are ready to give a precise definition of the equilibrium. The reader will observe that we do not require the agents' positions in the liquid markets to clear. Indeed, that is exactly what makes our equilibrium "partial". The liquid markets are interpreted as much bigger then the market for $\boldsymbol{B}$, and our agents are simply price-takers there, i.e., cannot affect the prices by trading.

Definition 4.2. We say that the pair $(\mathbf{p}, \boldsymbol{a}) \in \mathbb{R}^n \times \mathbf{F}$ is a *partial-equilibrium price-allocation (PEPA)*, if $\boldsymbol{a}_i \in Z_i(\mathbf{p})$ for every $1 \le i \le I$.

Remark 4.3. It is immediately clear that a PEPA of the form $(\mathbf{p}, \boldsymbol{a})$ will exists if and only if

$$\mathbf{0} \in \left\{ \sum_{i=1}^{I} \boldsymbol{a}_i \; : \; \boldsymbol{a}_i \in Z_i(\mathbf{p}),\, i = 1, \dots, I \right\}. \tag{4.2}$$

4.2. **Existence and Uniqueness of a Partial Equilibrium.** Our main result, Theorem 4.10, which asserts that a PEPA exists and is unique, rests on a few natural assumptions. The first one among them is a finite-dimensional version of the risk strict-convexity condition of Definition 2.12. In fact, the proof of next result is so similar to the proof of Proposition 2.11, that we omit it altogether.

Proposition 4.4. *Suppose that the acceptance set $\mathcal{A}_i$ of the agent i is weak-$*$ closed and that $\boldsymbol{B}$ is a bundle in $(\mathbb{L}^\infty)^n$. Then, the following two statements are equivalent:*

(1) *for every $(\boldsymbol{a}, m), (\boldsymbol{\delta}, k) \in \mathcal{A}(\boldsymbol{B}) = \{(\boldsymbol{a}, m) \in \mathbb{R}^n \times \mathbb{R} : \boldsymbol{a} \cdot \boldsymbol{B} + m \in \mathcal{A}\}$ such that $\boldsymbol{a} \neq \boldsymbol{\delta}$ and for every $\lambda \in (0, 1)$ there exists a random variable $E \in \mathbb{L}^\infty_+$ and $\mathbb{Q} \in \partial\rho_i(\lambda\boldsymbol{a} + (1 - \lambda)\boldsymbol{\delta}) \cdot \boldsymbol{B})$, such that $\mathbb{Q}(E > 0) > 0$ and*

$$\lambda(\boldsymbol{a} \cdot \boldsymbol{B} + m) + (1 - \lambda)(\boldsymbol{\delta} \cdot \boldsymbol{B} + k) - E \in \mathcal{A}.$$

(2) *The function* $\mathbb{R}^n \ni \boldsymbol{a} \mapsto \rho_i(\boldsymbol{a} \cdot \boldsymbol{B})$ *is strictly convex.*

Definition 4.5. Let $\mathcal{A}_i$ be a weak-$*$ closed acceptance set. For a bundle $\boldsymbol{B} \in (\mathbb{L}^\infty)^n$, we say that $\mathcal{A}$ is *strictly convex with respect to* $\boldsymbol{B}$ if either of the two equivalent statements in Proposition 4.4 holds.

Remark 4.6.

(1) For $\boldsymbol{B}$ such that $\boldsymbol{a} \cdot \boldsymbol{B} \notin \mathcal{R}_i$ for $\boldsymbol{a} \neq 0$, the strict convexity of $\mathcal{A}_i$ with respect to $\boldsymbol{B}$ is implied by the requirement of risk-strict convexity of Definition 2.12 above.

(2) One of the central technical benefits of introducing the notion of risk strict convexity is the functionality (single-valuedness) of the demand correspondence Z it induces. See Remark A.4 in the Appendix for details.

The following two assumptions will be in effect throughout the section:

Assumption 4.7. The acceptance set $\mathcal{A}_i$ is weak-$*$ closed and strictly convex with respect to $\boldsymbol{B}$, for all $i = 1, 2, ..., I$.

Assumption 4.8. For each $\boldsymbol{\delta} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$, $\inf_{\mathbb{Q} \in \mathcal{M}} \mathbb{E}^{\mathbb{Q}}[\boldsymbol{\delta} \cdot \boldsymbol{B}] < \sup_{\mathbb{Q} \in \mathcal{M}} \mathbb{E}^{\mathbb{Q}}[\boldsymbol{\delta} \cdot \boldsymbol{B}]$.

Remark 4.9. Assumption 4.8 implies, in particular, that for every $i = 1, 2, ..., I$, there is no $\boldsymbol{\delta} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ such that $\boldsymbol{\delta} \cdot \boldsymbol{B} \in \mathcal{X}_i^\infty$. That means, further, that there is no $\boldsymbol{\delta} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ such $\boldsymbol{\delta} \cdot \boldsymbol{B} \in \mathcal{R}_i$.

The following theorem contains the main result of the section. Its proof, preceded by two technical lemmas, is relegated to the Appendix.

Theorem 4.10. *Under Assumptions 3.4, 4.7 and 4.8, there exists a unique partial-equilibrium price-allocation* $(\hat{\mathbf{p}}, \hat{\boldsymbol{a}})$*. Moreover, for each* $i = 1, 2, \ldots, I$, $Z_i(\hat{\mathbf{p}}) = \{\hat{\boldsymbol{a}}_i\}$ *and there exists a probability measure* $\mathbb{Q}^i \in \mathcal{M}_i$ *such that* $\hat{\mathbf{p}} = \mathbb{E}^{\mathbb{Q}^i}[\boldsymbol{B}]$.

Remark 4.11.

(1) Theorem 4.10 can be viewed as a generalization of Theorem 5.8 in [2], where the situation described in Example 3.11 involving two exponential utility maximizers is extensively analyzed.

(2) The fact the demand correspondence depends on the bundle $\boldsymbol{B}$ only through its risk-equivalence class implies that equilibrium prices of two bundles whose components are pairwise risk-equivalent are the same.

(3) The second statement of Theorem 4.10 implies directly that $\hat{\mathbf{p}}$ is a non-arbitrage price of the bundle of claims $\boldsymbol{B}$. Alternatively, one could argue that no price $\mathbf{p}$ which is not in the appropriate abitrage-free interval could be an equilibrium price since it would lead to infinite demand and, thus, preclude market clearing.

(4) When the agents are in a Pareto optimal configuration then the pair $(\mathbf{p}, \boldsymbol{a}) \in \mathbb{R}^n \times \mathbf{F}$ is a PEPA if and only if $\boldsymbol{a} = \mathbf{0}$ and $\emptyset \neq Z_i(\mathbf{p})$, for all $i = 1, \ldots, I$. To see this, assume that the (unique) PEPA is of the form $(\mathbf{p}, \boldsymbol{a})$, for some $\boldsymbol{a} \neq \mathbf{0}$. Then, by the definition of Z_i, we have $\rho_i(\boldsymbol{a}_i \cdot \boldsymbol{B}) + \boldsymbol{a}_i \cdot \mathbf{p} \leq 0$ for all i, and, due to the strict convexity of the risk measures, the inequality is strict for at least two agents. This implies that $\sum_{i=1}^I \rho_i(\boldsymbol{a}_i \cdot \boldsymbol{B}) < 0$, which contradicts the assumption of Pareto optimality. The economic interpretation of the above statement is clear - the agents will not engage in trade if they are already in a configuration which cannot be improved upon.

5. The well-posedness of the equilibrium pricing

The exact shape of agents' acceptance sets, which incorporate their risk preferences, endowments and investment goals, is extremely difficult to estimate in practice. It is therefore natural to ask whether the induced equilibrium pricing is stable with respect small perturbation in the agents' acceptance sets. To be more precise, we want to check whether the equilibrium pricing scheme, presented in section 4, is a *well-posed problem* in the sense of Hadamard (see [30]), i.e., if its solution exists, is unique and stable with respect to the input data (the agents' acceptance sets in this case). Having solved the problem of existence and uniqueness (see Theorem 4.10), we turn our attention to the following question: can we specify a convergence (concept) $\xrightarrow{\circledast}$ for I-tuples of the weak-$*$ closed acceptance sets $\left(\mathcal{A}_i^{(m)}\right)_{i=1}^{I}=\left(\mathcal{A}_1^{(m)},\mathcal{A}_2^{(m)},...,\mathcal{A}_I^{(m)}\right)$, for which

$$\left(\mathcal{A}_1^{(m)},\mathcal{A}_2^{(m)},...,\mathcal{A}_I^{(m)}\right)\xrightarrow{\circledast}(\mathcal{A}_1,\mathcal{A}_2,...,\mathcal{A}_I)\Longrightarrow\left(\hat{\mathbf{p}}^{(m)},\hat{\boldsymbol{a}}^{(m)}\right)\rightarrow(\hat{\mathbf{p}},\hat{\boldsymbol{a}}), \tag{5.1}$$

where $(\hat{\mathbf{p}}^{(m)},\hat{\boldsymbol{a}}^{(m)})$ is the PEPA obtained by the acceptance sets $\left(\mathcal{A}_i^{(m)}\right)_{i=1}^{I}$ and $(\hat{\mathbf{p}},\hat{\boldsymbol{a}})$ is the corresponding to $(\mathcal{A}_i)_{i=1}^{I}$ PEPA?

As we shall explain shortly, it turns out that the right notion is related to *Kuratowski* convergence (see Chapter 8 in [38] and Chapter 7 in [43] for a further analysis):

Definition 5.1. A sequence of closed subsets $C_m\subseteq\mathbb{R}^l$, $l\in\mathbb{N}$, converges to the subset C in Kuratowski sense (and we write $C_m\xrightarrow{K}C$) if

$$\text{Ls }C_m\subseteq C\subseteq\text{Li }C_m, \tag{5.2}$$

where

$$\text{Li }C_m=\left\{c\in\mathbb{R}^l\,:\,c=\lim c_k,c_k\in C_k\text{ eventually}\right\}$$

and

$$\text{Ls }C_m=\left\{c\in\mathbb{R}^l\,:\,c=\lim c_k,c_k\in C_{n_k},n_k\text{ a subsequence of integers}\right\}.$$

We say that a sequence $\{f_m\}_{m\in\mathbb{N}}$ of lower semi-continuous functions $f_m:\mathbb{R}^l\rightarrow\mathbb{R}$ converges to a function f in the Kuratowski sense (and we write $f_m\xrightarrow{K}f$) if $\text{epi}(f_m)\xrightarrow{K}\text{epi}(f)$. We remind the reader that the epigraph of a function $f:\mathbb{R}^n\rightarrow\mathbb{R}$ is the set $\text{epi}(f)=\{(\boldsymbol{a},c)\in\mathbb{R}^n\times\mathbb{R}\,:\,f(\boldsymbol{a})\leq c\}$. A characterization of the Kuratowski convergence for sequences of functions is given by Theorem 8.6.3 in [38] (see also Proposition 7.2 in [43]); $f_m\xrightarrow{K}f$ if and only if the following two conditions hold:

(a) For every $x\in\mathbb{R}^k$ and every sequence x_n such that $x_n\rightarrow x$, $\liminf f_n(x_n)\geq f(x)$ and
(b) For every $x\in\mathbb{R}^k$ there exists a sequence x_n such that $x_n\rightarrow x$ and $\limsup f_n(x_n)\leq f(x)$.

The Kuratowski convergence and its versions for more general topological spaces have been extensively used in the study of the well-posedness of a variety of variational problems (see [43] for problems in $\mathbb{R}^n$ and [21] and [38] for general spaces).

In what follows, for each agent i, we consider a sequence of weak-$*$ closed acceptance sets $\mathcal{A}_i^{(m)}$ and a limiting weak-$*$ closed acceptance set $\mathcal{A}_i$, all of which satisfy the axioms *Ax1-Ax4*. The induced convex capital requirements are denoted by $\rho_i^{(m)}(\cdot)$ and $\rho_i(\cdot)$ respectively, and $\mathcal{M}_i^{(m)}$ and $\mathcal{M}_i$ stand for the effective domains of the corresponding penalty functions, $\alpha_i^{(m)}$ and α_i.

In the effort to show that the Kuratowski convergence allows for a positive answer to our central question, we establish the following auxiliary result. The reader is reminded that the definition of the sets $\mathcal{A}_i(\boldsymbol{B})$ is given in the statement of Proposition 4.4, part (1).

Lemma 5.2. *For a given bundle of claims* $\boldsymbol{B}$*, if* $\mathcal{A}_i^{(m)}(\boldsymbol{B})\xrightarrow{K}\mathcal{A}_i(\boldsymbol{B})$ *for every* $i\in\{1,2,...,I\}$*, then the sequence of functions* $\boldsymbol{a}\ni\mathbb{R}^n\mapsto\rho_i^{(m)}(\boldsymbol{a}\cdot\boldsymbol{B})$ *converges point-wise to the function* $\boldsymbol{a}\ni\mathbb{R}^n\mapsto\rho_i(\boldsymbol{a}\cdot\boldsymbol{B})$*.*

Proof. By (2.6), for a bundle $\boldsymbol{B} \in (\mathbb{L}^\infty)^n$, the set $\mathcal{A}(\boldsymbol{B})$ is the *epigraph* of the function $\mathbb{R}^n \ni \boldsymbol{a} \mapsto \rho_{\mathcal{A}}(\boldsymbol{a}\cdot\boldsymbol{B}) \in \mathbb{R}$. Hence, $\mathcal{A}_i^{(m)}(\boldsymbol{B}) \xrightarrow{K} \mathcal{A}_i(\boldsymbol{B})$ is equivalent to the Kuratowski convergence of the sequence of functions $\boldsymbol{a} \ni \mathbb{R}^n \mapsto \rho_i^{(m)}(\boldsymbol{a}\cdot\boldsymbol{B})$.

It is shown in [43], Theorem 7.17, that for any sequence $(f_m)_{m\in\mathbb{N}}$ of convex functions on $\mathbb{R}^n$, $f_m \xrightarrow{K} f$ implies that $f_m \to f$ point-wise in R^n, provided that f is a convex, lower semi-continuous function and its effective domain has non-empty interior. It is, therefore, enough to observe that the function $\boldsymbol{a} \ni \mathbb{R}^n \mapsto \rho_i(\boldsymbol{a}\cdot\boldsymbol{B})$ is convex and lower semi-continuous in $\mathbb{R}^n$, since $\rho_i : \mathbb{L}^\infty \to \mathbb{R}$ is convex and $\sigma(\mathbb{L}^\infty, \mathbb{L}^1)$-lower semi-continuous risk measure. □

As the reader can easily check, Kuratowski convergence will not, in general, preserve strict convexity. In order to guarantee that the limiting acceptance set $\mathcal{A}_i$ is strictly convex with respect to the fixed bundle of claims $\boldsymbol{B}$, we must assume that the strict convexity of $\mathcal{A}_i^{(m)}$ with respect to $\boldsymbol{B}$ satisfies a certain uniformity criterion.

Definition 5.3. A sequence of acceptance sets $(\mathcal{A}^{(m)})_{m\in\mathbb{N}}$ is *uniformly strictly convex* with respect to a bundle $\boldsymbol{B} \in (\mathbb{L}^\infty)^n$, if for every $(\boldsymbol{a}, c), (\boldsymbol{\delta}, k) \in \mathcal{A}^{(m)}(\boldsymbol{B})$ such that $\boldsymbol{a} \neq \boldsymbol{\delta}$, the following statement holds:

for every $\lambda \in (0,1)$ there exists a random variable $E \in \mathbb{L}^\infty_+$, such that $\mathbb{Q}[E > 0] > 0$, for some $\mathbb{Q} \in \partial\rho_i^{(m)}((\lambda\boldsymbol{a} + (1-\lambda)\boldsymbol{\delta})\cdot\boldsymbol{B})$ and

$$\lambda(\boldsymbol{a}\cdot\boldsymbol{B} + c) + (1-\lambda)(\boldsymbol{\delta}\cdot\boldsymbol{B} + k) - E \in \mathcal{A}^{(m)},$$

for all $m \in \mathbb{N}$.

It follows from the definition of Kuratowski convergence, that if $(\mathcal{A}_i^{(m)})_{m\in\mathbb{N}}$ is uniformly strictly convex with respect to $\boldsymbol{B}$ and $\mathcal{A}_i^{(m)}(\boldsymbol{B}) \xrightarrow{K} \mathcal{A}(\boldsymbol{B})$, then $\mathcal{A}_i$ is also strictly convex with respect to $\boldsymbol{B}$. This fact and Lemma A.3 imply, in particular, that the function $\boldsymbol{a} \ni \mathbb{R}^n \mapsto \rho_i(\boldsymbol{a}\cdot\boldsymbol{B})$ is strictly convex and differentiable on $\mathbb{R}^n$, if we further assume that $\boldsymbol{B}$ is not redundant, i.e., there is no $\boldsymbol{\delta} \in \mathbb{R}^n$, such that $\boldsymbol{\delta}\cdot\boldsymbol{B} \in \mathcal{R}_i$.

Assumption 5.4. The sequence $\{\mathcal{A}_i^{(m)}\}_{m\in\mathbb{N}}$ of acceptance sets is uniformly strictly convex with respect to the bundle $\boldsymbol{B}$.

Assumption 5.5. $\emptyset \neq \bigcap_{i=1}^I \mathcal{M}_i^{(m)} \subseteq \mathcal{M}$, for all $m \in \mathbb{N}$.

Assumption 5.6. For each $m \in \mathbb{N}$ and $\boldsymbol{\delta} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$, $\inf_{\mathbb{Q}\in\mathcal{M}^{(m)}} \mathbb{E}^{\mathbb{Q}}[\boldsymbol{\delta}\cdot\boldsymbol{B}] < \sup_{\mathbb{Q}\in\mathcal{M}^{(m)}} \mathbb{E}^{\mathbb{Q}}[\boldsymbol{\delta}\cdot\boldsymbol{B}]$.

It follows from Theorem 4.10 that under the Assumptions 5.4, 5.5 and 5.6 there exists a unique PEPA, $(\hat{\mathbf{p}}^{(m)}, \hat{\boldsymbol{a}}^{(m)})$, for every $m \in \mathbb{N}$. Furthermore, the induced strict convexity of $\mathcal{A}_i$ with respect to $\boldsymbol{B}$ means that the conditions for existence and uniqueness of PEPA hold even for the limiting risk measures ρ_i. Moreover, it turns out that those same conditions guarantee that the problem is well posed:

Theorem 5.7. *Under Assumptions 5.4, 5.5 and 5.6, for each* $m \in \mathbb{N}$ *there exists a unique PEPA* $(\hat{\mathbf{p}}^{(m)}, \hat{\boldsymbol{a}}^{(m)})$ *for agents with acceptance sets* $\left(\mathcal{A}_i^{(m)}\right)_{i=1}^I$. *Also, the convergence*

$$\mathcal{A}_i^{(m)}(\boldsymbol{B}) \xrightarrow{K} \mathcal{A}_i(\boldsymbol{B})$$

for every $i \in \{1, 2, ..., I\}$ *implies that*

(i) *There exists a unique PEPA* $(\hat{\mathbf{p}}, \hat{\boldsymbol{a}})$ *for agents with acceptance sets* $(\mathcal{A}_i)_{i=1}^I$ *and*
(ii) $(\hat{\mathbf{p}}^{(m)}, \hat{\boldsymbol{a}}^{(m)}) \longrightarrow (\hat{\mathbf{p}}, \hat{\boldsymbol{a}})$ *in* $\mathbb{R}^n \times \mathbb{R}^{n\times I}$.

Proof. The existence and the uniqueness of the PEPA for agents with acceptance sets $\left(\mathcal{A}_i^{(m)}\right)_{i=1}^I$ follows directly from Theorem 4.10. By Lemma 5.2, the Kuratowski convergence $\mathcal{A}_i^{(m)}(\boldsymbol{B}) \xrightarrow{K} \mathcal{A}_i(\boldsymbol{B})$ implies that

$\rho_i^{(m)}(\boldsymbol{a}\cdot\boldsymbol{B}) \to \rho_i(\boldsymbol{a}\cdot\boldsymbol{B})$, for every $\boldsymbol{a}\in\mathbb{R}^n$ and that the function $\boldsymbol{a}\ni\mathbb{R}^n\mapsto\rho_i(\boldsymbol{a}\cdot\boldsymbol{B})$ is strictly convex. Then, the existence and the uniqueness of the PEPA $(\hat{\mathbf{p}},\hat{\boldsymbol{a}})$, for agents with acceptance sets $(\mathcal{A}_i)_{i=1}^I$ is guaranteed again by Theorem 4.10.

Following the lines of the proof of Theorem 4.10, for each $m\in\mathbb{N}$ we define the strictly-convex function $f_m:\mathbb{R}^{n\times(I-1)}\to\mathbb{R}$ by

$$f_m(\boldsymbol{a}) = \rho_1^{(m)}(\boldsymbol{a}_1\cdot\boldsymbol{B}) + \rho_2^{(m)}(\boldsymbol{a}_2\cdot\boldsymbol{B}) + ... + \rho_{I-1}^{(m)}(\boldsymbol{a}_{I-1}\cdot\boldsymbol{B}) + \rho_I^{(m)}((-\sum_{i=1}^{I-1}\boldsymbol{a}_i)\cdot\boldsymbol{B}),$$

and note that it admits a unique minimizer, $\tilde{\boldsymbol{a}}^{(m)}\in\mathbb{R}^{n\times(I-1)}$ (where in fact, $\tilde{\boldsymbol{a}}_i^{(m)} = \hat{\boldsymbol{a}}_i^{(m)}$ for every $i = 1,2,...,I-1$). Similarly, we define the function

$$f(\boldsymbol{a}) = \rho_1(\boldsymbol{a}_1\cdot\boldsymbol{B}) + \rho_2(\boldsymbol{a}_2\cdot\boldsymbol{B}) + ... + \rho_{I-1}(\boldsymbol{a}_{I-1}\cdot\boldsymbol{B}) + \rho_I((-\sum_{i=1}^{I-1}\boldsymbol{a}_i)\cdot\boldsymbol{B}).$$

which is also strictly convex and has a unique minimizer $\tilde{\boldsymbol{a}}\in\mathbb{R}^{n\times(I-1)}$ (where, $\tilde{\boldsymbol{a}}_i = \hat{\boldsymbol{a}}_i$ for every $i = 1,2,...,I-1$). Note that the point-wise convergence $\rho_i^{(m)}(\boldsymbol{a}\cdot\boldsymbol{B})\to\rho_i(\boldsymbol{a}\cdot\boldsymbol{B})$, trivially implies that $f_m(\boldsymbol{a})\to f_m(\boldsymbol{a})$, for every $\boldsymbol{a}\in\mathbb{R}^{n\times(I-1)}$. In order to show that $\hat{\boldsymbol{a}}^{(m)}\to\hat{\boldsymbol{a}}$, as $m\to\infty$, we first recall a well-known result (see for instance Example I.7 in [21]) that if f is a convex, lower-semicontinuous function and has a minimizer $\tilde{\boldsymbol{a}}$, then for every sequence $\boldsymbol{\delta}^{(m)}\in\mathbb{R}^{n\times(I-1)}$ such that

$$f(\boldsymbol{\delta}^{(m)})\to f(\tilde{\boldsymbol{a}}),$$

it holds that $\boldsymbol{\delta}^{(m)}\to\tilde{\boldsymbol{a}}$. In other words, the problem of minimizing f in $\mathbb{R}^{n\times(I-1)}$ is *well posed in the sense of Tykhonov*. This implies that for every $\varepsilon>0$ there exists $b\in\mathbb{R}_+$ such that

$$\left\{\boldsymbol{a}\in\mathbb{R}^{n\times(I-1)}\,:\,f(\boldsymbol{a})\le b+f(\tilde{\boldsymbol{a}})\right\}\subseteq\left\{\boldsymbol{a}\in\mathbb{R}^{n\times(I-1)}\,:\,||\boldsymbol{a}-\tilde{\boldsymbol{a}}||<\varepsilon\right\}. \tag{5.3}$$

By Lemma II.21 in [21], for every $b\in\mathbb{R}_+$ and sufficiently large m we have

$$\left\{\boldsymbol{a}\in\mathbb{R}^{n\times(I-1)}\,:\,f_m(\boldsymbol{a})\le b+f_m(\tilde{\boldsymbol{a}}^{(m)})\right\}\subseteq\left\{\boldsymbol{a}\in\mathbb{R}^{n\times(I-1)}\,:\,f(\boldsymbol{a})\le 2b+f(\tilde{\boldsymbol{a}})\right\}. \tag{5.4}$$

Combination of (5.4) and (5.3) yields the convergence $\tilde{\boldsymbol{a}}^{(m)}\to\tilde{\boldsymbol{a}}$, which trivially implies the convergence of partial equilibrium allocations, $\hat{\boldsymbol{a}}^{(m)}\to\hat{\boldsymbol{a}}$. The definition of the equilibrium price yields that

$$\nabla\rho_i^{(m)}(\hat{\boldsymbol{a}}_i^{(m)}\cdot\boldsymbol{B}) = -\hat{\mathbf{p}}^{(m)},$$

for every $m\in\mathbb{N}$. Theorem 25.7 in [42] implies that the convergence $\rho_i^{(m)}(\boldsymbol{a}\cdot\boldsymbol{B})\to\rho_i(\boldsymbol{a}\cdot\boldsymbol{B})$ for every $\boldsymbol{a}\in\mathbb{R}^n$ and the fact that the limiting function $\boldsymbol{a}\ni\mathbb{R}^n\mapsto\rho_i(\boldsymbol{a}\cdot\boldsymbol{B})$ is differentiable in $\mathbb{R}^n$ yield that

$$\nabla\rho_i^{(m)}(\boldsymbol{a}\cdot\boldsymbol{B})\to\nabla\rho_i(\boldsymbol{a}\cdot\boldsymbol{B}),$$

for every $\boldsymbol{a}\in\mathbb{R}^n$ and every $i=\{1,2,...,I\}$. Furthermore, the same Theorem states that this convergence is uniform on compacts in $\mathbb{R}^n$, so

$$\hat{\mathbf{p}}^{(m)} = \nabla\rho_i^{(m)}(\hat{\boldsymbol{a}}_i^{(m)}\cdot\boldsymbol{B})\longrightarrow\nabla\rho_i(\hat{\boldsymbol{a}}_i\cdot\boldsymbol{B}) = \hat{\mathbf{p}}.$$

□

We conclude with an example in which we show what Kuratowski convergence looks like in a familiar setting:

Example 5.8. We consider the utility-based acceptance sets discussed in Example 2.5 for the agent i (see [32] for technical details) and we consider a sequence of utility functions $\left(U_i^{(m)}\right)_{m\in\mathbb{N}}$, a sequence of probability measures $\left(\mathbb{P}^{(m)}\right)_{m\in\mathbb{N}}$, and a sequence of initial wealths $\left(x_i^{(m)}\right)_{m\in\mathbb{N}}$. For every $B \in \mathbb{L}^\infty$, $x \in \mathbb{R}_+$ and $m \in \mathbb{N}$, we define the indirect utility

$$u_i^{(m)}(x|B) = \sup_{X\in\mathcal{X}(x)} \mathbb{E}^{\mathbb{P}^{(m)}}[U_i^{(m)}(X+B)],$$

where $\mathcal{X}(x)$ is a set of terminal wealths of admissible strategies; see page 848 in [32] for the exact definition (note the our set $\mathcal{X}(x)$ corresponds to their $\mathcal{X}(x,1)$, and that their notion of acceptability reduces to our admissibility due to boundedness of $\boldsymbol{B}$). The corresponding sequence of acceptance sets is then given by

$$\mathcal{A}_i^{(m)} = \left\{B \in \mathbb{L}^\infty \,:\, u_i^{(m)}(x_i^{(m)}|B) \geq u_i^{(m)}(x_i^{(m)}|0)\right\}.$$

for every $m \in \mathbb{N}$. It was proved in [35], Theorem 1.5, that the following convergence conditions

$$\mathbb{P}^{(m)} \to \mathbb{P} \text{ in total variation, } U_i^{(m)} \to U_i \text{ point-wise in } \mathbb{R}_+ \text{ and } x_i^{(m)} \to x_i,$$

(together with some additional technical assumptions), yield that for every non-redundant bundle $\boldsymbol{B} \in (\mathbb{L}^\infty)^n$, we have that

$$u_i^{(m)}(x_i^{(m)}|\boldsymbol{a}^{(m)}\cdot\boldsymbol{B}) \to u_i(x_i|\boldsymbol{a}\cdot\boldsymbol{B}), \tag{5.5}$$

for every sequence $\boldsymbol{a}^{(m)} \in \mathbb{R}^n$ that converges to some $\boldsymbol{a} \in \mathbb{R}^n$. It is, then, straightforward to get that (5.5) imply that $\mathcal{A}_i^{(m)}(\boldsymbol{B}) \xrightarrow{K} \mathcal{A}_i(\boldsymbol{B})$, which in turn guarantees that the equilibrium price-allocation of $\boldsymbol{B}$ is well-posed.

Appendix A. A proof of the main result

In order not to interfere with the flow of the exposition, we present two technical lemmas and the proof of our main Theorem 4.10 here in the Appendix.

We start by introducing necessary notation.

$$\mathcal{M}_i^{\boldsymbol{B}} = \bigcup_{\boldsymbol{\delta}\in\mathbb{R}^n} \partial\rho_i(\boldsymbol{\delta}\cdot\boldsymbol{B}) \subseteq \mathcal{M}_i,\ \mathcal{P}_i^{\boldsymbol{B}} = \left\{\mathbb{E}^{\mathbb{Q}}[\boldsymbol{B}] \,:\, \mathbb{Q}\in\mathcal{M}_i^{\boldsymbol{B}}\right\} \subseteq \mathbb{R}^n.$$

Remark A.1. The sets $\mathcal{M}_i^{\boldsymbol{B}}$ and $\mathcal{P}_i^{\boldsymbol{B}}$ defined above can be thought of as "restricted" versions of the sets $\mathcal{M}_i$ and $\left\{\mathbb{E}^{\mathbb{Q}}[\boldsymbol{B}] \,:\, \mathbb{Q}\in\mathcal{M}_i\right\}$, where only the martingale measures "compatible" with the acceptance set $\mathcal{A}_i$ are collected. These sets have been defined and used in the literature before (see, e.g., [19]).

In the exponential case, the set $\mathcal{M}_i^{\boldsymbol{B}}$ coincides with the set of equivalent martingale measures with finite entropy, $\mathcal{M}_{e,f}$ (see [19] for terminology and the proofs of results mentioned below). It is known that, when the stock-prices are locally bounded, $\mathcal{M}_{e,f}$ is dense in $\mathcal{M}_e$. Consequently, the set $\mathcal{P}_i^{\boldsymbol{B}}$ coincides with the full family of non-arbitrage prices of $\boldsymbol{B}$.

Lemma A.2. *Pick $i \in \{1,2,\ldots,I\}$ and assume that that $\mathcal{A}_i$ is weak-$*$ closed and strictly convex with respect to $\boldsymbol{B}$. Let $\{\boldsymbol{a}_k\}_{k\in\mathbb{N}}$ be a sequence of the form $\boldsymbol{a}_k = \boldsymbol{a}_0 + \gamma_k\boldsymbol{v} \in \mathbb{R}^n$, for some $\boldsymbol{v}\in\mathbb{R}^n\setminus\{0\}$, $\boldsymbol{a}_0\in\mathbb{R}^n$, and a sequence $\{\gamma_k\}_{k\in\mathbb{N}}$ of positive constants with $\lim_k \gamma_k = \infty$. If $\{\mathbb{Q}_k\}_{k\in\mathbb{N}}$ is an arbitrary sequence of probability measures with $\mathbb{Q}_k \in \partial\rho_i(-\boldsymbol{a}_k\cdot\boldsymbol{B})$, then*

$$\lim_{k\to\infty} \mathbb{E}^{\mathbb{Q}_k}[\boldsymbol{v}\cdot\boldsymbol{B}] = \sup_{\mathbb{Q}\in\mathcal{M}_i} \mathbb{E}^{\mathbb{Q}}[\boldsymbol{v}\cdot\boldsymbol{B}] = \sup_{\mathbf{p}\in\mathcal{P}_i^{\boldsymbol{B}}} \boldsymbol{v}\cdot\mathbf{p}. \tag{A.1}$$

Proof. Our first claim is that

$$\lim_{k\to\infty} \frac{\rho_i(-\boldsymbol{a}_k\cdot\boldsymbol{B})}{\gamma_k} \geq S,\ \text{ where } S = \sup_{\mathbb{Q}\in\mathcal{M}_i} \mathbb{E}^{\mathbb{Q}}[\boldsymbol{v}\cdot\boldsymbol{B}]. \tag{A.2}$$

For $\varepsilon > 0$ there exists $\mathbb{Q}^\varepsilon \in \mathcal{M}_i$ such that $\mathbb{E}^{\mathbb{Q}^\varepsilon}[\boldsymbol{v} \cdot \boldsymbol{B}] \geq S - \varepsilon/2$, and, for that choice of $\mathbb{Q}^\varepsilon$, there exists $K \in \mathbb{N}$ such that for $k \geq K$ we have

$$\tfrac{\alpha(\mathbb{Q}^\varepsilon)+\delta}{\gamma_k} \leq \varepsilon/2 \text{ where } \delta = |\rho_i(-\boldsymbol{a}_0 \cdot \boldsymbol{B})| + ||\boldsymbol{a}_0 \cdot \boldsymbol{B}||_{\mathbb{L}^\infty}.$$

Thanks to convexity of ρ_i, the ratio $\frac{\rho_i(-\boldsymbol{a}_n \cdot \boldsymbol{B}) - \rho_i(-\boldsymbol{a}_0 \cdot \boldsymbol{B})}{\gamma_n}$ is nondecreasing, so its limit as $n \to \infty$ exists in $(-\infty, \infty]$ and

$$\begin{aligned}
\lim_{k\to\infty} \frac{\rho_i(-\boldsymbol{a}_k \cdot \boldsymbol{B})}{\gamma_k} &= \sup_{k \geq K} \frac{\rho_i(-\boldsymbol{a}_k \cdot \boldsymbol{B}) - \rho_i(-\boldsymbol{a}_0 \cdot \boldsymbol{B})}{\gamma_k} \\
&= \sup_{k\geq K} \sup_{\mathbb{Q}\in\mathcal{M}_i} \left\{ \frac{\mathbb{E}^{\mathbb{Q}}[\boldsymbol{a}_k \cdot \boldsymbol{B}]}{\gamma_k} - \frac{\alpha_i(\mathbb{Q})}{\gamma_k} - \frac{\rho_i(-\boldsymbol{a}_0 \cdot \boldsymbol{B})}{\gamma_k} \right\} \\
&= \sup_{\mathbb{Q}\in\mathcal{M}_i} \sup_{k\geq K} \left\{ \mathbb{E}^{\mathbb{Q}}[\boldsymbol{v} \cdot \boldsymbol{B}] - \frac{\alpha_i(\mathbb{Q}) + \rho_i(-\boldsymbol{a}_0 \cdot \boldsymbol{B}) + \mathbb{E}^{\mathbb{Q}}[-\boldsymbol{a}_0 \cdot \boldsymbol{B}]}{\gamma_k} \right\} \\
&\geq \mathbb{E}^{\mathbb{Q}^\varepsilon}[\boldsymbol{v} \cdot \boldsymbol{B}] - \frac{\alpha_i(\mathbb{Q}^\varepsilon) + \rho_i(-\boldsymbol{a}_0 \cdot \boldsymbol{B}) + \mathbb{E}^{\mathbb{Q}^\varepsilon}[-\boldsymbol{a}_0 \cdot \boldsymbol{B}]}{\gamma_k} \geq S - \varepsilon,
\end{aligned}$$

and (A.2) follows.

We continue by noting that since $\mathbb{Q}_k \in \partial\rho_i(-\boldsymbol{a}_k \cdot \boldsymbol{B})$ and $\alpha_i(\cdot) \geq 0$, we have

$$\tfrac{1}{\gamma_k}\rho_i(-\boldsymbol{a}_k \cdot \boldsymbol{B}) \leq \tfrac{1}{\gamma_k}\mathbb{E}^{\mathbb{Q}_k}[\boldsymbol{a}_k \cdot \boldsymbol{B}] \leq \mathbb{E}^{\mathbb{Q}_k}[\boldsymbol{v} \cdot \boldsymbol{B}] + \tfrac{1}{\gamma_k}||a_0 \cdot \boldsymbol{B}||,$$

and so

$$\sup_{\mathbb{Q}\in\mathcal{M}_i} \mathbb{E}^{\mathbb{Q}_k}[\boldsymbol{v} \cdot \boldsymbol{B}] \leq \lim_{k\to\infty} \tfrac{\rho_i(-\boldsymbol{a}_k \cdot \boldsymbol{B})}{\gamma_k} \leq \liminf_{k\to\infty} \mathbb{E}^{\mathbb{Q}_k}[\boldsymbol{v} \cdot \boldsymbol{B}] \leq \limsup_{k\to\infty} \mathbb{E}^{\mathbb{Q}_k}[\boldsymbol{v} \cdot \boldsymbol{B}] \leq \sup_{\mathbb{Q}\in\mathcal{M}_i^{\boldsymbol{B}}} \mathbb{E}^{\mathbb{Q}_k}[\boldsymbol{v} \cdot \boldsymbol{B}].$$

We conclude the proof by noting that since $\mathcal{M}_i^{\boldsymbol{B}} \subseteq \mathcal{M}_i$, all the inequalities above are, in fact, equalities. ☐

Lemma A.3. *For $i \in \{1, 2, ..., I\}$, let $\boldsymbol{B} \in (\mathbb{L}^\infty)^n$ be a bundle of claims for which there is no $\boldsymbol{a} \in \mathbb{R}^n$ such that $\boldsymbol{a} \cdot \boldsymbol{B} \in \mathcal{R}_i$. If $\mathcal{A}_i$ is weak-$*$ closed and strictly convex with respect to $\boldsymbol{B}$, then the expectation $\mathbb{E}^{\mathbb{Q}}[\boldsymbol{a} \cdot \boldsymbol{B}]$ is the same for all $\mathbb{Q} \in \partial\rho_i(\boldsymbol{a} \cdot \boldsymbol{B})$, and the function $r_i : \mathbb{R}^n \to \mathbb{R}$, defined by*

$$r_i(\boldsymbol{a}) = \rho_i(\boldsymbol{a} \cdot \boldsymbol{B}) \tag{A.3}$$

is continuously differentiable and

$$\nabla r_i(\boldsymbol{a}) = \mathbb{E}^{\mathbb{Q}}[-\boldsymbol{B}], \textit{ for any } \mathbb{Q} \in \partial\rho_i(\boldsymbol{a} \cdot \boldsymbol{B}).$$

Proof. Thanks to convexity of r_i and Proposition I.5.3 in [22], it will be enough to show that $\mathbb{E}^{\mathbb{Q}}[-\boldsymbol{B}]$ is the unique subgradient of r_i at $\boldsymbol{a}$ for any $\mathbb{Q} \in \partial\rho_i(\boldsymbol{a} \cdot \boldsymbol{B})$. To proceed, we suppose that $\boldsymbol{a}^* \in \mathbb{R}^n$ satisfies $r_i(\boldsymbol{\delta}) \geq r_i(\boldsymbol{a}) - \boldsymbol{a}^* \cdot (\boldsymbol{a} - \boldsymbol{\delta})$, i.e.

$$\rho_i(\boldsymbol{\delta} \cdot \boldsymbol{B}) \geq \rho_i(\boldsymbol{a} \cdot \boldsymbol{B}) - \boldsymbol{a}^* \cdot (\boldsymbol{a} - \boldsymbol{\delta}), \tag{A.4}$$

for all $\boldsymbol{\delta} \in \mathbb{R}^n$, and that $\boldsymbol{a}^* \neq \mathbb{E}^{\mathbb{Q}}[-\boldsymbol{B}]$. Consider, first, the case when $-\boldsymbol{a}^* \in \mathcal{P}_i(\boldsymbol{B})$, i.e., when there exists $\hat{\boldsymbol{\delta}} \in \mathbb{R}^n$ such that $\hat{\boldsymbol{\delta}} \neq \boldsymbol{a}$ and $\boldsymbol{a}^* = \mathbb{E}^{\hat{\mathbb{Q}}}[-\boldsymbol{B}]$ for some $\hat{\mathbb{Q}} \in \partial\rho_i(\hat{\boldsymbol{\delta}})$. If we substitue $\hat{\boldsymbol{\delta}}$ for $\boldsymbol{\delta}$ in (A.4) we get

$$\rho_i(\hat{\boldsymbol{\delta}} \cdot \boldsymbol{B}) + \mathbb{E}^{\hat{\mathbb{Q}}}[\hat{\boldsymbol{\delta}} \cdot \boldsymbol{B}] \geq \rho_i(\boldsymbol{a} \cdot \boldsymbol{B}) + \mathbb{E}^{\hat{\mathbb{Q}}}[\boldsymbol{a} \cdot \boldsymbol{B}].$$

Note, however, that

$$\rho_i(\boldsymbol{a} \cdot \boldsymbol{B}) \quad \geq \quad \mathbb{E}^{\hat{\mathbb{Q}}}[-\boldsymbol{a} \cdot \boldsymbol{B}] - \alpha_i(\hat{\mathbb{Q}}) = \mathbb{E}^{\hat{\mathbb{Q}}}[-\boldsymbol{a} \cdot \boldsymbol{B}] + \rho_i(\hat{\boldsymbol{\delta}} \cdot \boldsymbol{B}) + \mathbb{E}^{\hat{\mathbb{Q}}}[\hat{\boldsymbol{\delta}} \cdot \boldsymbol{B}],$$

and the equality holds if and only if $\hat{\mathbb{Q}} \in \partial\rho_i(\boldsymbol{a} \cdot \boldsymbol{B})$. This implies that

$$\frac{1}{2}(\rho_i(\boldsymbol{a} \cdot \boldsymbol{B}) + \rho_i(\hat{\boldsymbol{\delta}} \cdot \boldsymbol{B})) = \rho_i\left(\frac{(\hat{\boldsymbol{\delta}} + \boldsymbol{a}) \cdot \boldsymbol{B}}{2}\right),$$

which contradicts the assumption of strict convexity with respect to $\boldsymbol{B}$.

It is left to show that if $\boldsymbol{a}^* = (a_1^*, \dots, a_n^*) \in \mathbb{R}^n$ satisfies (A.4), then $-\boldsymbol{a}^* \in \mathcal{P}_i(\boldsymbol{B})$. We argue by contradiction and assume that this is not the case. By Lemma A.2 this means that there exists a component $l \in \{1, 2, ..., n\}$ such that either $a_l^* \geq \sup_{\mathbb{Q}\in\mathcal{M}_i} \{\mathbb{E}^{\mathbb{Q}}[-B_l]\}$ or $a_l^* \leq \inf_{\mathbb{Q}\in\mathcal{M}_i} \{\mathbb{E}^{\mathbb{Q}}[-B_l]\}$. We can assume without loss of generality that the former holds and that $l = 1$. The inequality (A.4), in which $\boldsymbol{a}^1 = (a_1 + 1, a_2, a_3, ..., a_n)$ is substituted for $\boldsymbol{\delta}$, implies that

$$\rho_i(\boldsymbol{a}^1 \cdot \boldsymbol{B}) + \inf_{\mathbb{Q}\in\mathcal{M}_i} \mathbb{E}^{\mathbb{Q}}[B_1] \geq \rho_i(\boldsymbol{a} \cdot \boldsymbol{B}).$$

On the other hand, for $\mathbb{Q}^1 \in \partial\rho_i(\boldsymbol{a}^1 \cdot \boldsymbol{B})$, we have

$$\rho_i(\boldsymbol{a} \cdot \boldsymbol{B}) \quad \geq \quad \mathbb{E}^{\mathbb{Q}^1}[-\boldsymbol{a} \cdot \boldsymbol{B}] - \alpha_i(\mathbb{Q}^1) = \rho_i(\boldsymbol{a}^1 \cdot \boldsymbol{B}) + \mathbb{E}^{\mathbb{Q}^1}[B_1] > \rho_i(\boldsymbol{a}^1 \cdot \boldsymbol{B}) + \inf_{\mathbb{Q}\in\mathcal{M}_i} \{\mathbb{E}^{\mathbb{Q}}[B_1]\},$$

a contradiction. □

Remark A.4. If for $\boldsymbol{B} \in (\mathbb{L}^\infty)^n$ there is no $\boldsymbol{a} \in \mathbb{R}^n$ such that $\boldsymbol{a} \cdot \boldsymbol{B} \in \mathcal{R}_i$ and if $\mathcal{A}_i$ is strictly convex with respect to $\boldsymbol{B}$, the demand correspondence is $Z_i(\mathbf{p})$ is non-empty only when $\mathbf{p} \in \mathcal{P}_i(\boldsymbol{B})$. Indeed, thanks to its definition as a minimizer set of a differentiable convex function, the set $Z_i(\mathbf{p})$, consists of the solutions $\boldsymbol{a}$, of the equation

$$\nabla r_i(\boldsymbol{a}) = \mathbf{p}, \tag{A.5}$$

where r_i is defined in (A.3) above. Lemma A.3 states, however, that $\nabla r_i(\boldsymbol{a}) \in \mathcal{P}_i(\boldsymbol{B})$. Moreover, when $\mathbf{p} \in \mathcal{P}_i(\boldsymbol{B})$, $Z_i(\mathbf{p})$ is a singleton; this follows directly from strict convexity of r_i. Finally, it follows easily from (A.5) that Z_i is single-valued on $\mathcal{P}_i(\boldsymbol{B})$.

Proof of Theorem 4.10. We first define the strictly convex function $f : \mathbb{R}^{(I-1)\times n} \to \mathbb{R}$ by

$$f(\boldsymbol{a}) = \rho_1(\boldsymbol{a}_1 \cdot \boldsymbol{B}) + \rho_2(\boldsymbol{a}_2 \cdot \boldsymbol{B}) + ... + \rho_{I-1}(\boldsymbol{a}_{I-1} \cdot \boldsymbol{B}) + \rho_I((-\sum_{i=1}^{I-1} \boldsymbol{a}_i) \cdot \boldsymbol{B}). \tag{A.6}$$

If for some $\tilde{\boldsymbol{a}} = (\tilde{\boldsymbol{a}}_1, \dots, \tilde{\boldsymbol{a}}_{I-1}) \in \mathbb{R}^{(I-1)\times n}$ we have $\nabla f(\tilde{\boldsymbol{a}}) = \mathbf{0}$, then, $\tilde{\boldsymbol{a}}$ is the unique minimizer of f. Moreover, such $\tilde{\boldsymbol{a}}$ satisfies $\nabla\rho_i(\tilde{\boldsymbol{a}}_i \cdot \boldsymbol{B}) = \nabla\rho_I(-\left(\sum_{i=1}^{I-1} \tilde{\boldsymbol{a}}_i\right) \cdot \boldsymbol{B})$, for every $i = 1, 2, ..., I-1$. By Lemma A.3, the latter means that for any $\mathbb{Q}_i \in \partial\rho_i(\tilde{\boldsymbol{a}}_i \cdot \boldsymbol{B})$, $1 \leq i \leq I-1$ and any $\mathbb{Q}_I \in \partial\rho_i(-(\sum_{i=1}^{I-1} \tilde{\boldsymbol{a}}_i) \cdot \boldsymbol{B})$, we have

$$\mathbb{E}^{\mathbb{Q}_i}[\boldsymbol{B}] = \mathbb{E}^{\mathbb{Q}_I}[\boldsymbol{B}].$$

Therefore, the price vector $\hat{\mathbf{p}} = \mathbb{E}^{\mathbb{Q}_i}[\boldsymbol{B}]$ satisfies $Z_i(\hat{\mathbf{p}}) = \tilde{\boldsymbol{a}}_i$ for every $i = 1, 2, ..., I-1$ and $Z_I(\hat{\mathbf{p}}) = -\sum_{i=1}^{I-1} \tilde{\boldsymbol{a}}_i$. In other words, if $\hat{\boldsymbol{a}}$ denotes the allocation whose rows are given by $\hat{\boldsymbol{a}}_i = \tilde{\boldsymbol{a}}_i$, for $i = 1, 2, ..., I-1$ and $\hat{\boldsymbol{a}}_I = -\sum_{i=1}^{I-1} \tilde{\boldsymbol{a}}_i$, the pair $(\hat{\mathbf{p}}, \hat{\boldsymbol{a}})$ is a partial equilibrium price allocation. In fact, it is the unique one, since if we assume the existence of another PEPA $(\check{\mathbf{p}}, \check{\boldsymbol{a}}) \neq (\hat{\mathbf{p}}, \hat{\boldsymbol{a}})$, we get that $\check{\mathbf{p}} = \mathbb{E}^{\mathbb{Q}}[\boldsymbol{B}]$, for any $\mathbb{Q} \in \partial\rho_i(\check{\boldsymbol{a}}_i \cdot \boldsymbol{B})$, which, in turn, implies that $\nabla f(\check{\boldsymbol{a}}) = \mathbf{0}$. The latter equation contradicts the uniqueness of the minimizer of the function f.

We are left with the task of showing that $\nabla f(\boldsymbol{a})$ has a root, and assume, per contra, that this is not the case. Then, by the continuity of f, we deduce that for, each $m \in \mathbb{N}$, there exists $\boldsymbol{a}^{(m)} \in \mathbf{D}_m = \{\boldsymbol{a} \in \mathbb{R}^{(I-1)\times n} : ||\boldsymbol{a}||_1 = \sum_{k=1}^{(I-1)} \sum_{j=1}^{n} |a_{k,j}| \leq m\}$ such that $f(\boldsymbol{a}^{(m)}) \leq f(\boldsymbol{a})$ for all $\boldsymbol{a} \in \mathbf{D}_m$. Furthermore, by the strict convexity of f, it follows that $||\boldsymbol{a}^{(m)}||_1 = m$. Hence, thanks to the results of, e.g., Chapter 1 in [9], a contradiction would be reached if the following coercivity condition held:

$$F = \liminf_{m\to\infty} \frac{f(\boldsymbol{a}^{(m)})}{m} > 0. \tag{A.7}$$

By passing to a subsequence (if necessary), we can assume without loss of generality that the limits $F = \lim_{k\to\infty} \frac{f(\boldsymbol{a}^{(k)})}{k} \in \mathbb{R}$ and $\boldsymbol{a}_i^{(0)} = \lim \frac{\boldsymbol{a}_i^{(k)}}{k} \in \mathbb{R}^n$, $i = 1, 2, \ldots, I-1$ exist. Since

$$\left| \frac{\rho_i(\boldsymbol{a}_i^{(k)} \cdot \boldsymbol{B})}{k} - \frac{\rho_i(k\boldsymbol{a}_i^{(0)} \cdot \boldsymbol{B})}{k} \right| \leq || \tfrac{\boldsymbol{a}_i^{(k)}}{k} - \boldsymbol{a}_i^{(0)} || \; ||\boldsymbol{B}||_{(\mathbb{L}^\infty)^n} \to 0,$$

Lemma A.2 implies that

$$\lim_{k\to\infty} \frac{\rho_i(\boldsymbol{a}_i^{(k)} \cdot \boldsymbol{B})}{k} = \sup_{\mathbb{Q}\in\mathcal{M}_i} \mathbb{E}^{\mathbb{Q}}[\boldsymbol{a}_i^{(0)} \cdot \boldsymbol{B}], \text{ for } 1 \leq i \leq I-1, \text{ and}$$

$$\lim_{k\to\infty} \frac{\rho_I(-\sum_{j=1}^{I-1} \boldsymbol{a}_j^{(k)} \cdot \boldsymbol{B})}{k} = \sup_{\mathbb{Q}\in\mathcal{M}_I} \mathbb{E}^{\mathbb{Q}}[-\sum_{j=1}^{I-1} \boldsymbol{a}_j^{(0)} \cdot \boldsymbol{B}].$$

Consequently, (A.7) follows from

$$\begin{aligned} F &= \sum_{j=1}^{I-1} \sup_{\mathbb{Q}\in\mathcal{M}_j} \mathbb{E}^{\mathbb{Q}}[\boldsymbol{a}_j^{(0)} \cdot \boldsymbol{B}] + \sup_{\mathbb{Q}\in\mathcal{M}_I} \mathbb{E}^{\mathbb{Q}}[-\sum_{j=1}^{I-1} \boldsymbol{a}_j^{(0)} \cdot \boldsymbol{B}] \\ &\geq \sup_{\mathbb{Q}\in\mathcal{M}} \mathbb{E}^{\mathbb{Q}}[\sum_{j=1}^{I-1} \boldsymbol{a}_j^{(0)} \cdot \boldsymbol{B}] - \inf_{\mathbb{Q}\in\mathcal{M}} \mathbb{E}^{\mathbb{Q}}[\sum_{j=1}^{I-1} \boldsymbol{a}_j^{(0)} \cdot \boldsymbol{B}] > 0, \end{aligned}$$

where the strictness of the last inequality follows from Assumption 4.8. □

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