

FORWARD EXPONENTIAL PERFORMANCES: PRICING AND OPTIMAL RISK SHARING

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Abstract. In an incomplete market model, we consider financial agents whose investment criteria are modelled by forward exponential performance processes. The problem of contingent claim indifference valuation is first addressed and a number of properties are proved and discussed. Special attention is given to the comparison between the forward exponential and the backward exponential utility indifference valuation. In addition, we construct the problem of optimal risk sharing in this forward setting and solve it when the agents' forward performance criteria are exponential.

Key words. Forward performance criteria, stochastic utility, stochastic risk aversion, indifference price, optimal risk sharing, exponential utility, contingent claim pricing

AMS subject classification. 91G20, 91G10, 91G99

INTRODUCTION

Contingent claim pricing in incomplete markets is one of the most challenging problems in mathematical finance. In incomplete markets, there exist contingent claims for which there is no dynamic self-financing portfolio that perfectly replicates their payoffs. A consequence of this is that the non-arbitrage arguments provide only an interval of prices consistent with the non-arbitrage

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assumption. The answer to the question which is the “*correct*” price within this interval requires a model of agents’ risk preferences and perhaps their endowments or/and their beliefs.

One of the most fruitful literature on financial agents’ risk preferences is the one on utility function. Based on the works of von Neumann & Morgenstein [34] and Herstein & Milnor [17], this theory suggests that an agent, who models her risk preferences through a utility function, is going to invest in the financial market with the aim to maximize the expectation of her utility function. Starting with the seminal works [27], [28], [36], [40], the *utility maximization* has been extensively studied and developed in a variety of market models and utility functions (see for instance, [14], [21], [24] and [38] for an overview). If an agent’s investment criterion is the utility maximization, it is reasonable to assume that she evaluates each contingent claim by comparing the following two situations: maximization of the expected utility after buying (selling) the claim and maximization of the utility without any transaction on the claim. The price that makes these situations indifferent for the agent’s perspective is the so-called *indifference price*. This pricing mechanism was introduced in mathematical finance literature in [18] and then further developed by a number of authors (see among others [12], [26], [30] and [16] for an overview). We should highlight at this point that the indifference pricing mechanism is *subjective*, in the sense that a utility maximizer quotes prices at which she is willing to buy or sell a given contingent claim. However, there is no guarantee that these prices are the ones at which any kind of transaction actually takes place. Therefore, throughout this paper we prefer to call these prices *values* to emphasize their subjective nature.

One of the main flaws of the utility maximization (and of the induced indifference valuation) is the dependence on the time horizon at which the utility function stands. Although for investment goals and single claim pricing, fixing a certain investment/pricing time horizon may not be problematic, it creates consistency concerns. In particular, this theory does not provide a way to set another time horizon and the continuation of the investments to be consistent. Similarly, the valuing of contingent claims with maturity later than the chosen time horizon can not be addressed with the available tools. This is because there is no forward shifting of a utility function. Even more inconvenient is that fixing a utility at some time in the future leaves no room for updating the utility function (and by extension the investment goals) until the terminal horizon. It looks like an agent is stuck with her utility function and a given subjective probability measure, no matter what happens to the market, her endowment or her beliefs.

The problem of time horizon dependence of investment choices has been recently studied by a number of authors ([6], [9], [15], [41] and [42]). A common concept of these works is that agents aim to maximize, instead of a utility function, a family of state-dependent utility functions in a time-consistent way. In this paper, we work on the notion of *forward performance* or *forward utility*, which has been introduced in the works of M. Musiela and T. Zariphopoulou [31] and [32] (see also [41] for an overview). In words, this concept suggests that in contrast to the backward utility

function maximization, the agents choose a family of state-dependent utility functions and their investment goal is to find the admissible trading strategy that keeps the expectation of this family at the same level (see Definition 1.1 for the exact definition and [31] for an extended discussion). If the family of utility functions is of certain exponential type, the forward performance is called *exponential*. Explicit formulas for the optimal strategy and the optimal wealth process under this type of forward performance criteria has been provided in [32] under a Markovian market model (see also [42] for some related discussion). More recently, G. Žitković in [43] establishes the characterization of the forward exponential performance process in a general semimartingale market model and as a special case in a diffusion stochastic volatility model, similar to the one we shall impose in the present paper. One of the important part of the characterization of the forward exponential utility functions is that the risk aversion becomes stochastic process instead of constant (as in the classic exponential utility function). Furthermore, for the stochastic risk aversion process, usually denoted by γ_t , it holds that the quantity $1/\gamma_t$, which can be thought as agent's (stochastic) risk tolerance, is replicable.

The first aim of this paper (Section 2) is to contribute to the theory of exponential forward performance by investigating how agents value contingent claims under such investment criteria. Valuation in a forward manner has several differences in comparison with the backward valuation (which is induced by classic utility functions) both in financial and technical sense. In Section 2, we state and prove a number of properties of the forward indifference valuation and we point out the differences to the corresponding backward valuation. Namely, based on the characterization results in [43], we are able to prove that the dynamic version of the forward indifference valuation solves a certain type of backward stochastic differential equation (the corresponding BSDE that is solved by the backward exponential indifference price is provided in [26]). Furthermore, as pricing functional in an incomplete market, the forward indifference value can be seen as a convex risk measure in the sense of [11]. In subsection 2.2, we give the exact form of the *robust representation* of the forward exponential indifference valuation (at the same manner as the related literature on dynamic convex risk measure, see e.g. [7] and [23]). This representation is useful for proving several properties of the indifference valuation such as continuity, differentiability with respect to the units of the claim and monotonicity with respect to risk aversion (see Propositions 2.2 and 2.3).

The second part of the paper (Section 3) is dedicated to the *optimal risk sharing* between two agents whose investment criteria are based on forward exponential performances. Optimal risk sharing problem is about two agents who design the mutually beneficially sharing of their endowments. This problem is well studied under several models from classic utility functions (see [4] and [5]) to convex risk measures (see among others [2], [3], [10] and [19]). All of these models are set in a backward fashion, that is the optimization criteria have a fixed time horizon, which in fact equals to the maturity of the agents' endowments. In this paper, we initiate this problem in the

forward setting and establish its solution in three different cases regarding the model parameters: (a) when both agents have constant risk aversions, (b) when agents have common but stochastic risk aversions and (c) when agents have different and stochastic risk aversions. In cases (a) and (b), closed form solution of the contracts that optimally share the agents' random endowments are provided and compared with the classic entropic risk measure case (studied among others in [3]).

For the more general case of different and stochastic risk aversions we need to look at the time evolution of the inf-convolution risk measure induced by agents' forward performance criteria. We first establish the necessary and sufficient conditions under which this measure can be seen as one induced by some other forward exponential performance. Then, we generalize the results of [3] in the forward setting and get the stochastic differential equation satisfied by the inf-convolution measure. In the forward setting the optimal risk sharing consists of three terms, one that has to do with the sharing of the endowments (which has the same form as the backward valuation setting); one term that incorporates the sharing of agents' different beliefs (which does not depend of agents risk aversion); and a replicable term (which can be ignored since it does not transfer any risk).

The market model used in this manuscript consists of one riskless asset and one risky asset, whose drift and volatility are driven by a generalized Itô process (a special case of this model has been used in [37] and [43]). It should be pointed out that the majority of the results in this paper can be generalized in a straightforward way to models with more risky assets. We choose to work in this simplified model in order to focus on the interpretation of the results and the explanation of how agents evaluate claims and share risks under forward looking investment criteria. Furthermore, this work deals with the exponential type of forward performance, since this type is more tractable and closed form solutions can be provided. It also helps the comparison with the backward case, where there are several well-known results regarding valuation and risk sharing issues. Finally, as mentioned and illustrated in [32], exponential forward performance is quite general and captures a variety of agents' distinct characteristics.

1. MARKET SETTING

1.1. Assets and admissible strategies. The market consists of a risky and a risk-free asset. The risk-free asset is used as a numéraire and its price process evolution is given by

$$dB_t = rB_t dt$$

where $r > 0$ is a constant. The price process of the risky asset is driven by a Brownian motion written on a filtration generated by two Brownian motions. More precisely, let $(W_t^1, W_t^2)_{t \in [0, \infty)}$ be a 2-dimensional standard Brownian motion defined on a probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, where $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ is the augmented σ -algebra generated by $(W_t^1, W_t^2)_{t \in [0, \infty)}$. The price of the risky assets

$(S_t)_{t \in [0, \infty)}$ satisfies the following stochastic differential equation (SDE)

$$dS_t = \mu_t S_t dt + \sigma_t S_t dW_t^1 \quad (1)$$

where $S_0 > 0$ and $(\mu_t)_{t \in [0, \infty)}$ and $(\sigma_t)_{t \in [0, \infty)}$ are \mathbb{F} -progressively-measurable processes. It is also imposed that $\sigma_t > 0$, for every $t > 0$. As usual, we assume that the SDE (1) has a unique strong solution.

Remark 1.1. *A special case of this model is the Markovian stochastic factor model, which has been extensively used in the related literature (see among others [31], [37] and [41]). In this special case $\mu_t = \mu(Y_t)$, $\sigma_t = \sigma(Y_t)$, where $dY_t = b(Y_t)dt + a(Y_t) \left(\rho dW_t^1 + \sqrt{1 - \rho^2} dW_t^2 \right)$ and $\rho \in (-1, 1)$.*

We define the *market price of risk process* $(\lambda_t)_{t \in [0, \infty)}$, via

$$\lambda_t = \frac{\mu_t - r}{\sigma_t}, \quad t \in [0, \infty). \quad (2)$$

Throughout this paper, we impose the following technical assumption.

Assumption 1.1. *For every $T > 0$, there exists $\varepsilon > 0$ such that $\mathbb{E} \left[e^{(1/2+\varepsilon) \int_0^T \lambda_u^2 du} \right] < \infty$.*

We then define the set of admissible strategies

$$\mathcal{A} = \{ \mathbb{F}\text{-progressively measurable } \pi : \mathbb{E} \left[\int_0^t \sigma_s^2 \pi_s^2 ds \right] < \infty, \forall t > 0 \}.$$

The discounted wealth process of an admissible strategy π with initial capital x at some time $\tau \geq 0$ is denoted by $(X^{x, \pi, \tau})_{t \in [\tau, \infty)}$ and satisfies the following SDE

$$dX_t^{x, \pi, \tau} = \sigma_t \pi_t (\lambda_t dt + dW_t^1) \quad (3)$$

where $X_\tau^{x, \pi, \tau} := x$. When the initial wealth or/and the initial time are equal to zero, we simplify the notation by omitting the corresponding superscript, that is $X^{x, \pi} := X^{x, \pi, 0}$, $X^{\pi, \tau} := X^{0, \pi, \tau}$ and $X^\pi := X^{0, \pi, 0}$. We also define the set $\mathcal{A}^\infty = \{ \pi \in \mathcal{A} : X_t^\pi \in \mathbb{L}^\infty(\mathcal{F}_t), \forall t \geq 0 \}$.

For the model at hand, we introduce the following notations for any time horizon $T > 0$.

$$\mathcal{P}_T = \{ \mathbb{F}\text{-progressively measurable } \nu : \int_0^T \nu_u^2 du < \infty, \text{ a.s.} \} \quad (4)$$

$$\mathcal{P} = \bigcap_{T > 0} \mathcal{P}_T \quad (5)$$

and

$$\mathcal{N} = \{ (\beta, \nu) \in \mathcal{P} \times \mathcal{P} : Z^{\beta, \nu} \text{ is a true } \mathbb{P}\text{-martingale} \}, \quad (6)$$

where $(Z^{\beta, \nu})_{t \in [0, \infty)}$ is the solution of the equation

$$dZ_u^{\beta, \nu} = -Z_u^{\beta, \nu} (\beta_u dW_u^1 + \nu_u dW_u^2).$$

Note that under Assumption 1.1, $(\lambda, 0) \in \mathcal{N}$.

For every arbitrarily chosen time horizon T and every $(\beta, \nu) \in \mathcal{N}$, we define the probability measure $\mathbb{Q}^{\beta, \nu} \sim \mathbb{P}|_{\mathcal{F}_T}$ by its R-N derivative $\frac{d\mathbb{Q}^{\beta, \nu}}{d(\mathbb{P}|_{\mathcal{F}_T})} = Z_T^{\beta, \nu}$. We also define the set $\mathcal{P}^\lambda \subseteq \mathcal{P}$, which contains all processes $\nu \in \mathcal{P}$ such that $(\lambda, \nu) \in \mathcal{N}$ (similarly we define the set \mathcal{P}_T^λ).

A simple application of Girsanov Theorem implies that for every $\nu \in \mathcal{P}^\lambda$ the discounted stock price $\frac{S_t}{B_t}$ is a local-martingale under the measure $\mathbb{Q}^{\lambda, \nu}$. In fact, for the set of equivalent local-martingale measures $\mathcal{M}_T^e = \left\{ \mathbb{Q} \sim \mathbb{P} : \frac{S_t}{B_t} \text{ is a } \mathbb{Q}\text{-local martingale in } [0, T] \right\}$ it holds that:

$$\mathcal{M}_T^e = \left\{ \mathbb{Q}^{\lambda, \nu} : \nu \in \mathcal{P}_T^\lambda \right\}$$

(see [8] for the proof).

1.2. The forward exponential performance criteria. In this manuscript we assume that agent's investment goals are modelled by so-called forward performance criteria (also called forward or stochastic utilities and self-generating random utilities) introduced in [31] (see also [33] and [43]).

Definition 1.1. *A map $U : \Omega \times [0, \infty) \times \mathbb{R} \mapsto \mathbb{R}$ is called a forward performance process if:*

- (i) *It is measurable with respect to the product of the progressive σ -algebra on $\Omega \times [0, \infty)$ and the Borel σ -algebra on \mathbb{R} .*
- (ii) *For fixed $\omega \in \Omega$ and $t \in [0, \infty)$, the mapping $x \mapsto U_t(x)$ is strictly increasing and strictly concave.*
- (iii) *For all $s \geq t$ and $X \in \mathbb{L}^\infty(\mathcal{F}_t)$*

$$U_t(X) = \operatorname{esssup}_{\pi \in \mathcal{A}^\infty} \mathbb{E} [U_s(X_s^{X, \pi, t}) | \mathcal{F}_t] \quad (7)$$

A forward performance is called exponential if there exist adapted processes $(A_t)_{t \in [0, \infty)}$ and $(\gamma_t)_{t \in [0, \infty)}$ such that $\gamma_t > 0$ a.s. for every $t \geq 0$ and

$$U_t(x) = -e^{-\gamma_t x + A_t} \quad (8)$$

for $x \in \mathbb{R}$, and $t \geq 0$.

Remark 1.2. *We follow the definition given in [43], which does not require that the optimal strategy is attained. The reason for this choice is that we prefer to focus on pricing and risk sharing issues rather than the technicalities on the existence of the optimal strategy (see also the related discussion in [43].) An analysis of the portfolio management problem with forward exponential criteria has been provided in [32], where the authors give explicit formulas for the optimal portfolio π^* and the associated optimal wealth process X^{x, π^*} , for a variety of model parameters.*

When agent's risk preferences are modelled by a utility function $U(x)$, her investment criterion (up to some certain time horizon T) is the maximization of the expected utility function. That is

for every $t \in [0, T]$, optimal trading strategy is defined by

$$\operatorname{esssup}_{\pi \in \mathcal{A}^\infty} \mathbb{E}[U(X_T^{X, \pi, t}) | \mathcal{F}_t]. \quad (9)$$

The utility given by $U(x) = -e^{-\gamma x}$ is called *exponential*.

The forward performance process can be seen as a modification of the above utility maximization investment criterion, in the following sense: There is no terminal time horizon set and the choice of the utility is made at time 0, i.e. $U_0(x) = -e^{\gamma_0 x + A_0}$. For every future time t , the utility is updated by replacing the risk aversion coefficient γ_0 with a stochastic one γ_t , and the term A_0 with A_t . Hence, we shall call the investment criterion (9) *backward* to emphasize its main difference with the forward performance criteria.

The following characterization of the forward exponential performance processes has been proven in [43].

Theorem 1.1 (Žitković, 2009). *Suppose that Assumption 1.1 holds and let $U_t(x) = -e^{-\gamma_t x + A_t}$ be a forward exponential performance where $\frac{1}{\gamma_t} \in \mathbb{L}^\infty(\mathcal{F}_t)$ for all $t \geq 0$. Then, there exist processes $(\vartheta_t)_{t \in [0, \infty)}$, $(\phi_t)_{t \in [0, \infty)}$ and $(p_t)_{t \in [0, \infty)}$ such that $\vartheta \in \mathcal{A}^\infty$ and $\frac{1}{\gamma_t} = X_t^{\frac{1}{\gamma_0}, \vartheta}$, with $\gamma_0 \in \mathbb{R}_+$, $\phi, p \in \mathcal{P}$ and*

$$A_t = A_0 + \frac{1}{2} \int_0^t (\lambda_u - \delta_u)^2 du + \gamma_t X_t^p - \frac{1}{2} \int_0^t \phi_u^2 du - \int_0^t \phi_u dW_u^2 \quad (10)$$

where $\delta_t = \gamma_t \vartheta_t \sigma_t$.

On the other hand, if some continuous process $(A_t)_{t \in [0, \infty)}$ admits representation (10) where $\gamma_t = \left(X_t^{\frac{1}{\gamma_0}, \vartheta}\right)^{-1}$ for some process $\vartheta \in \mathcal{A}^\infty$ and for all $t > 0$ it holds that $\sup_{s \in [0, t]} (|\phi_s| + |\vartheta_s \sigma_s| + |p_s|) \in \mathbb{L}^\infty(\mathcal{F}_t)$ and $e^{n\gamma_t + A_t} \in \mathbb{L}^1(\mathcal{F}_t)$, for each $n \in \mathbb{N}$, then the random field defined as $U_t(x) = -e^{-\gamma_t x + A_t}$ is a forward exponential performance.

Example 1.1. *A simple example of a forward exponential performance is the case where $\vartheta_t = \phi_t = 0$ for all $t \geq 0$. This corresponds to constant risk aversion coefficient (γ is not a stochastic process). If we further set $p_t = -\frac{\lambda_t}{\gamma \sigma_t}$, for every time $t \geq 0$, we have that $A_t = A_0 - \frac{1}{2} \int_0^t \lambda_u^2 du - \int_0^t \lambda_u dW_u^1$ (an example that has been used in [41]).*

Remark 1.3. *Note that the boundedness assumption of $1/\gamma_t$ is not very restrictive in financial sense, since for a given time t , $1/\gamma_t$ denotes agent's the risk tolerance, which is normally a bounded quantity. In addition, in view of the definition of the forward exponential performance, we may ignore without loss of generality the initial term A_0 .*

Furthermore, under the additional assumption that $p \in \mathcal{A}^\infty$, the maximization problem (7) and hence the forward exponential performance remain the same if in equation (10) we set $p = 0$. In what follows we impose the following standing assumption.

Assumption 1.2. *Process p in representation (10) belongs in \mathcal{A}^∞ .*

Under Assumption 1.2, the characterization of forward exponential performances in Theorem 1.1 allows us to identify the exact elements of these performance criteria. More precisely, the decomposition of process A_t consists of two parts:

- The integral $\int_0^t (\lambda_u - \delta_u)^2 du$, which reflects how the agent incorporates the market's development in her investment criteria, in a way that this incorporation takes into account her stochastic risk tolerance level.
- The sum $\frac{1}{2} \int_0^t \phi_u^2 du + \int_0^t \phi_u dW_u^2$. This term does not depend on the level of risk aversion γ_t and it can be considered as the way the changes of the unhedgeable source of the market make the agent update her subjective probability measure \mathbb{P} (see also the related comment in [42]). This is because $-e^{-\gamma_t x + A_t}$ can be written as $-e^{-\gamma_t x + \frac{1}{2} \int_0^t (\lambda_u - \delta_u)^2 du} Z_t^{0, \phi}$.

In what follows we will identify a forward exponential performance process by its *characterization pair* $(\vartheta_t, \phi_t)_{t \in [0, \infty)}$, where $\vartheta \in \mathcal{A}^\infty$ and $\phi \in \mathcal{P}$.

An important difference between the forward and the standard (backward) exponential investment criteria (defined in (9)) is that in the former when the optimal strategy in (7) is attained, it *does* depend on the initial wealth. However, this dependence is quite clear (thanks to the replicability of $1/\gamma_t$). In our setting, attainment of the optimal strategy in (7) and (9) refers to the attainment of the essential supremum by same strategy π for every $s \geq t$, for every initial time $t \geq 0$ and every initial (bounded) wealth X .

Proposition 1.1. *Let $(\vartheta_t, \phi_t)_{t \in [0, \infty)}$ be the characterization pair of a forward exponential performance and set the initial time equal to zero. If the optimal trading strategy process in (7) is attained, then*

$$\pi_t^*(y) = \pi_t^*(x) + \gamma_0(y - x)\vartheta_t \quad (11)$$

and

$$X_t^{y, \pi^*(y)} - X_t^{x, \pi^*(x)} = \frac{\gamma_0}{\gamma_t}(y - x). \quad (12)$$

for all $x, y \in \mathbb{R}$ and $t \geq 0$, where $\pi^*(x)$ stands for the optimal strategy with initial wealth x .

Proof. We first fix initial wealths $x, y \in \mathbb{R}$. Since $\pi^*(x)$ is the optimal trading strategy for initial wealth x , it holds that $\forall t \geq 0$

$$\mathbb{E}[U_t(X_t^{x, \pi^*(x)})] = U_0(x)$$

or equivalently

$$\mathbb{E} \left[-e^{-\gamma_t(x + \int_0^t \pi_s^*(x) dS_s) + A_t} \right] = -e^{-\gamma_0 x + A_0}. \quad (13)$$

Similarly for initial wealth y , it holds that

$$\mathbb{E} \left[-e^{-\gamma_t(y + \int_0^t \pi_s^*(y) dS_s) + A_t} \right] = -e^{-\gamma_0 y + A_0}. \quad (14)$$

We multiply both sides of equation (13) by $e^{-\gamma_0(y-x)}$ and get

$$\mathbb{E} \left[-e^{-\gamma_t \left(x + \frac{\gamma_0}{\gamma_t} (y-x) + \int_0^t \pi_s^*(x) dS_s \right) + A_t} \right] = -e^{-\gamma_0 y + A_0}. \quad (15)$$

But,

$$\begin{aligned} \text{LHS of (15)} &= \mathbb{E} \left[-e^{-\gamma_t \left(x + \gamma_0(y-x) \left(\frac{1}{\gamma_0} + \int_0^t \vartheta_s dS_s \right) + \int_0^t \pi_s^*(x) dS_s \right) + A_t} \right] \\ &= \mathbb{E} \left[-e^{-\gamma_t \left(y + \int_0^t (\gamma_0(y-x) \vartheta_s + \pi_s^*(x)) dS_s \right) + A_t} \right]. \end{aligned}$$

Taking into account equation (14), we get the desired relation (11).

Now, for (12) we have that for every $t \geq 0$

$$\begin{aligned} X_t^{y, \pi^*(y)} &= y + \int_0^t \pi_s^*(y) dS_s \\ &= y + \int_0^t \pi_s^*(x) dS_s + \gamma_0(y-x) \int_0^t \vartheta_s dS_s \\ &= y + \int_0^t \pi_s^*(x) dS_s + \gamma_0(y-x) \left(\frac{1}{\gamma_t} - \frac{1}{\gamma_0} \right) \\ &= X_t^{x, \pi^*(x)} + \frac{\gamma_0}{\gamma_t} (y-x). \end{aligned}$$

□

In the rest of this manuscript, we assume that the agents' initial wealth is equal to zero. For any nonzero initial wealth we may apply Proposition 1.1.

2. VALUATION OF CONTINGENT CLAIMS BASED ON FORWARD INDIFFERENCE

If an agent's investment goals are determined by a forward exponential performance, it is reasonable to suppose that she uses indifference arguments in order to give values to contingent claims. The idea of indifference valuation was introduced in the finance literature in [18] and then developed and analyzed for a number of utility functions and market settings (see among others [16] and the references therein). This (subjective) valuation concept compares two situations, the one where a contingent claim is bought or sold and another where there is no transaction on this claim.

For the model at hand, for a certain time horizon $T > 0$ we consider an \mathcal{F}_T -measurable payoff C . The (buyer) indifference value is the price p that makes the agent indifferent between buying the claim at p and not buying it at all. In our forward performance setting the buyer's value of a payoff C at any time $t \in [0, T]$, denoted by $v_t(C)$ is the \mathcal{F}_t -measurable solution of the following equation

$$U_t(x) = \text{esssup}_{\pi \in \mathcal{A}^\infty} \mathbb{E} \left[U_T \left(x - v_t(C) + C + \int_t^T \pi_s dS_s \right) \middle| \mathcal{F}_t \right] \quad (16)$$

that is

$$-e^{-\gamma_t x + A_t} = \operatorname{esssup}_{\pi \in \mathcal{A}^\infty} \mathbb{E} \left[-e^{-\gamma_T (x - v_t(C) + C + \int_t^T \pi_s dS_s) + A_T} \middle| \mathcal{F}_t \right] \quad (17)$$

Due to replicability of $1/\gamma_t$, (17) can equivalently be written as

$$-e^{-\gamma_t (x + v_t(C)) + A_t} = \operatorname{esssup}_{\pi \in \mathcal{A}^\infty} \mathbb{E} \left[-e^{-\gamma_T (x + C + \int_t^T \pi_s dS_s) + A_T} \middle| \mathcal{F}_t \right] \quad (18)$$

Hence we get that the forward indifference value at time t of a claim C is given by

$$v_t(C) = -x - \frac{A_t}{\gamma_t} + \frac{1}{\gamma_t} \log \left(\operatorname{essinf}_{\pi \in \mathcal{A}^\infty} \mathbb{E} \left[e^{-\gamma_T (x + C + \int_t^T \pi_s dS_s) + A_T} \middle| \mathcal{F}_t \right] \right)$$

and again by the replicability of $1/\gamma_t$ we have that

$$v_t(C) = -\frac{A_t}{\gamma_t} + \frac{1}{\gamma_t} \log \left(\operatorname{essinf}_{\pi \in \mathcal{A}^\infty} \mathbb{E} \left[e^{-\gamma_T (C + \int_t^T \pi_s dS_s) + A_T} \middle| \mathcal{F}_t \right] \right). \quad (19)$$

Note that that the indifference value $v_t(C)$ does not depend on the initial wealth (see also [25], [29], [30] and [42]).

Results on the value that solves equation (16) have been provided in [30] and in [29] for specific types of forward exponential performances in a stochastic factor model and in a binomial-type market model respectively. The use of forward exponential performance criteria for pricing contingent claims of American-type has been studied in [25] under the assumption of constant risk aversion.

One interesting question about these pricing mechanisms is how the forward exponential indifference valuation differs with the indifference valuation induced by exponential utilities. More precisely, we want to compare the solution $v_t(C)$ of (16), with the solution $v_t^B(C)$ of the following equation

$$\operatorname{esssup}_{\pi \in \mathcal{A}^\infty} \mathbb{E} \left[-e^{-\gamma (x + v_t^B(C) + \int_t^T \pi_s dS_s)} \middle| \mathcal{F}_t \right] = \operatorname{esssup}_{\pi \in \mathcal{A}^\infty} \mathbb{E} \left[-e^{-\gamma (x + C + \int_t^T \pi_s dS_s)} \middle| \mathcal{F}_t \right] \quad (20)$$

where $\gamma \in \mathbb{R}_+$ is the risk aversion coefficient. Throughout this paper the price process $v^B(C)$ will be called *backward exponential indifference value process*.

Such a comparison has been studied in [29] and [31] (for constant risk aversion process). In the present section, we aim to extend the results on forward indifference valuation for general forward exponential performance process and by doing so to highlight the special properties of the forward valuation regarding its comparison with the backward exponential valuation.

2.1. The BSDE representation of the indifference value. It has been proved in [26] that, under continuous filtration, the indifference value process under backward exponential utility satisfies a certain type of backward stochastic differential equation (BSDE). Adapted to our market model, the (buyer) backward indifference value of a contingent claim $C \in \mathbb{L}^\infty(\mathcal{F}_T)$ satisfies the following BSDE

$$v_t^B(C) = v_T^B(C) - \frac{\gamma}{2} \int_t^T \zeta_u^2 du - \int_t^T \zeta_u d\hat{W}_u^2 - \int_t^T \theta_u dS_u \quad (21)$$

and

$$v_T^B(C) = C, \quad (22)$$

where $\gamma \in \mathbb{R}_+$ is the constant risk aversion coefficient, $\theta \in \mathcal{A}$, $(\hat{W}_t^2)_{t \in [0, T]}$ is a Brownian motion under minimal entropy martingale measure and under this measure $\left(\int_0^t \zeta_u d\hat{W}_u^2\right)_{t \in [0, T]}$ is a true martingale. We call the triple $(v_t^B(C), \zeta_t, \theta_t)_{t \in [0, T]}$ a solution of BSDE (21) with terminal condition (22). Theorem 13 in [26] guarantees that for $C \in \mathbb{L}^\infty(\mathcal{F}_T)$ there is a unique uniformly bounded solution.

The following proposition establishes that the above representation has a nice extension in the case of the forward exponential performance.

Proposition 2.1. *Impose Assumption 1.1 and let $(\vartheta_t, \phi_t)_{t \in [0, \infty)}$ be the characterization pair of a forward exponential performance and assume that there exists a constant K_γ such that $\sup_{t \in [0, T]} \|\gamma_t\|_{\mathbb{L}^\infty} \leq K_\gamma$ and that $\phi \in \mathcal{P}_T^\lambda$. The forward exponential indifference (buyer) value process of a contingent claim $C \in \mathbb{L}^\infty(\mathcal{F}_T)$ is the unique uniformly bounded solution, $(v_t(C))_{t \in [0, T]}$, of the following BSDE under the martingale measure $\mathbb{Q}^{\lambda, \phi}$*

$$v_t(C) = v_T(C) - \frac{1}{2} \int_t^T \gamma_u \zeta_u^2 du - \int_t^T \zeta_u dW_u^{2, \phi} - \int_t^T \theta_u dS_u \quad (23)$$

and

$$v_T(C) = C \quad (24)$$

for some processes $(\theta_t, \zeta_t)_{t \in [0, \infty)}$, such that

$$\mathbb{E}_{\mathbb{Q}^{\lambda, \phi}} \left[\int_0^T \theta_u^2 du \right] < \infty \quad \text{and} \quad \mathbb{E}_{\mathbb{Q}^{\lambda, \phi}} \left[\int_0^T \zeta_u^2 du \right] < \infty, \quad (25)$$

where $W_t^{2, \phi} = W_t^2 + \int_0^t \phi_u du$.

Proof. Recall the indifference valuation problem (18)

$$-e^{-\gamma_t(X+v_t(C))+A_t} = \operatorname{esssup}_{\pi \in \mathcal{A}^\infty} \mathbb{E} \left[-e^{-\gamma_T(X+\int_t^T \pi_s dS_s+C)+A_T} \mid \mathcal{F}_t \right], \quad (26)$$

for every $X \in \mathbb{L}^\infty(\mathcal{F}_t)$, where process A is given by the characterization (10), with $A_0 = 0$. We now define the process $(\tilde{A}_t)_{t \in [0, \infty)}$ as

$$\tilde{A}_t = \begin{cases} -\gamma_t v_t(C) + A_t, & t \leq T; \\ -\gamma_t C + A_t, & t > T. \end{cases} \quad (27)$$

By the replicability of $1/\gamma_t$ we get that for every $t \geq 0$

$$-e^{-\gamma_t X + \tilde{A}_t} = \operatorname{esssup}_{\pi \in \mathcal{A}^\infty} \mathbb{E} \left[-e^{-\gamma_T(X+\int_t^T \pi_s dS_s)+\tilde{A}_T} \mid \mathcal{F}_t \right]. \quad (28)$$

In the view of Definition 1.1, problem (28) leads to another forward exponential performance, where the risk aversion process remains γ_t and the characteristic process is given by \tilde{A}_t . By Theorem 1.1, there exist processes $z, \tilde{p} \in \mathcal{P}$ such that

$$\tilde{A}_t = \tilde{A}_0 + \frac{1}{2} \int_0^t (\lambda_u - \delta_u)^2 du + \gamma_t X_t^{\tilde{p}} - \frac{1}{2} \int_0^t z_u^2 du - \int_0^t z_u dW_u^2.$$

Hence, for any $t \in [0, T]$

$$\begin{aligned} v_t(C) &= \frac{A_t - \tilde{A}_t}{\gamma_t} \\ &= \frac{1}{\gamma_t} \left(A_0 - \tilde{A}_0 + \gamma_t X_t^{\tilde{p}} - \frac{1}{2} \int_0^t (\phi_u^2 - z_u^2) du - \int_0^t (\phi_u - z_u) dW_u^2 \right) \\ &= -\frac{\tilde{A}_0}{\gamma_0} + X_t^{\tilde{p} - A_0 \vartheta} - \frac{1}{\gamma_t} \left(\frac{1}{2} \int_0^t (\phi_u^2 - z_u^2) du + \int_0^t (\phi_u - z_u) dW_u^2 \right). \end{aligned}$$

Note that with the above notation $v_0(C) = -\frac{\tilde{A}_0}{\gamma_0}$ and $W^{2,\phi}$ is a Brownian motion under the measure $\mathbb{Q}^{\lambda,\phi}$, strongly orthogonal to S .

Hence, the indifference value process satisfies the following equation

$$v_t(C) = v_0(C) + X_t^{\tilde{p}} + \frac{1}{\gamma_t} \left(\frac{1}{2} \int_0^t (\phi_u - z_u)^2 du + \int_0^t (\phi_u - z_u) dW_u^{2,\phi} \right).$$

Let $\Lambda_t = \frac{1}{2} \int_0^t (\phi_u - z_u)^2 du + \int_0^t (\phi_u - z_u) dW_u^{2,\phi}$, for every $t \in [0, \infty)$. A simple application of Ito's formula implies that

$$\frac{\Lambda_t}{\gamma_t} = \frac{1}{2} \int_0^t \frac{(\phi_u - z_u)^2}{\gamma_u} du + \int_0^t \frac{(\phi_u - z_u)}{\gamma_u} dW_u^{2,\phi} + \int_0^t \Lambda_u \vartheta_u \sigma_u dW_u^{1,\lambda},$$

where $W_t^{1,\lambda} = W_t^1 + \int_0^t \lambda_u du$. Note that the process $\zeta = \frac{\phi - z}{\gamma}$ belongs in \mathcal{P}_T , thanks to uniform boundedness of γ_t for all $t \in [0, T]$ and the fact that $\phi, z \in \mathcal{P}_T$. It is left to set $\theta_t = (\Lambda_t \vartheta_u + \tilde{p}_t) \sigma_t \in \mathcal{P}$ and apply Lemma 2.1 below to get the integrability of $\int_0^T \theta_u^2 du$ and $\int_0^T \zeta_u^2 du$ under the measure $\mathbb{Q}^{\lambda,\phi}$.

Finally, the uniqueness of the solution follows from Proposition B.1 for $\gamma = g = G$ and $\phi = \psi$. \square

Lemma 2.1. *Impose the condition of Proposition 2.1 and let $(v_t(C), \zeta_t, \theta_t)$ be the solution of (23) and terminal condition (24) for some contingent claim $C \in \mathbb{L}^\infty(\mathcal{F}_T)$, where $(v_t(C))_{t \in [0, T]}$ is uniformly bounded. Then, there exists a constant $K > 0$ such that*

$$\sup_\tau \mathbb{E}_{\mathbb{Q}^{\lambda,\phi}} \left[\int_\tau^T \sigma_t^2 \theta_t^2 dt \middle| \mathcal{F}_\tau \right] + \sup_\tau \mathbb{E}_{\mathbb{Q}^{\lambda,\phi}} \left[\int_\tau^T \zeta_t^2 dt \middle| \mathcal{F}_\tau \right] < K$$

where the supremum is taken under any stopping time $\tau \in [0, T]$.

Proof. We first apply the Itô's formula for the process $e^{v_t(C)}$ which implies that

$$d(e^{v_t(C)}) = e^{v_t(C)} \left(\frac{\zeta_t^2}{2} + \frac{\gamma_t \dot{\zeta}_t^2}{2} + \frac{\theta_t^2 \sigma_t^2}{2} \right) dt + e^{v_t(C)} \zeta_t d\tilde{W}_t^2 + e^{v_t(C)} \theta_t \sigma_t d\tilde{W}_t^1 \quad (29)$$

where, $W_t^{1,\lambda} = W_t^1 + \int_0^t \lambda_u du$. There is a sequence of stopping times τ_n , with $\tau_n \nearrow T$ such that $\int_0^{t \wedge \tau_n} \zeta_u dW_u^{2,\phi}$ and $\int_0^{t \wedge \tau_n} \theta_u \sigma_u dW_u^{1,\lambda}$ are $\mathbb{Q}^{\lambda,\phi}$ -martingales. The boundness of C implies that for any stopping time $\tau \in [0, T]$

$$\begin{aligned} e^{\|C\|_\infty} &\geq \mathbb{E}_{\mathbb{Q}^{\lambda,\phi}} \left[e^{v_{\tau_n}(C)} - e^{v_{\tau \wedge \tau_n}(C)} \middle| \mathcal{F}_{\tau \wedge \tau_n} \right] = \mathbb{E}_{\mathbb{Q}^{\lambda,\phi}} \left[\int_{\tau \wedge \tau_n}^{\tau_n} e^{v_t(C)} \left(\frac{\zeta_t^2 + \sigma_t^2 \theta_t^2 + \gamma_t \zeta_t^2}{2} \right) dt \middle| \mathcal{F}_{\tau \wedge \tau_n} \right] \\ &\geq \mathbb{E}_{\mathbb{Q}^{\lambda,\phi}} \left[\int_{\tau_n}^{\tau \wedge \tau_n} e^{v_t(C)} \left(\frac{\zeta_t^2 + \sigma_t^2 \theta_t^2}{2} \right) dt \middle| \mathcal{F}_{\tau \wedge \tau_n} \right] \\ &\geq \frac{e^{-\|C\|_\infty}}{2} \mathbb{E}_{\mathbb{Q}^{\lambda,\phi}} \left[\int_{\tau \wedge \tau_n}^{\tau_n} \zeta_t^2 dt \middle| \mathcal{F}_{\tau \wedge \tau_n} \right] + \frac{e^{-\|C\|_\infty}}{2} \mathbb{E}_{\mathbb{Q}^{\lambda,\phi}} \left[\int_{\tau \wedge \tau_n}^{\tau_n} \sigma_t^2 \theta_t^2 dt \middle| \mathcal{F}_{\tau \wedge \tau_n} \right] \end{aligned}$$

Hence, $\mathbb{E}_{\mathbb{Q}^{\lambda,\phi}} \left[\int_{\tau \wedge \tau_n}^{\tau_n} \zeta_t^2 dt \middle| \mathcal{F}_{\tau \wedge \tau_n} \right] + \mathbb{E}_{\mathbb{Q}^{\lambda,\phi}} \left[\int_{\tau \wedge \tau_n}^{\tau_n} \sigma_t^2 \theta_t^2 dt \middle| \mathcal{F}_{\tau \wedge \tau_n} \right] \leq 2e^{\|C\|_\infty}$. Letting $n \rightarrow \infty$ completes the proof. \square

An agent with characteristic pair (ϑ, ϕ) is indifferent of buying the claim C at price $v_0(C)$ or not buying it at all. However, in case she buys the claim, she gets a random endowment at time T equal at the payoff C . Under this random endowment the characteristic pair of her forward exponential performance criterion changes. In other words, we are asking how a given forward exponential performance changes when a random endowment is taking into account. Using the arguments in the proof of Proposition 2.1, we get the exact form of the characteristic pair when the agent has endowment C delivered at time T .

Corollary 2.1. *Impose the condition of Proposition 2.1 and let $(v_t(C), \zeta_t, \theta_t)$ be the solution of (23) and terminal condition (24) for some contingent claim $C \in \mathbb{L}^\infty(\mathcal{F}_T)$, where $(v_t(C))_{t \in [0, T]}$ is uniformly bounded. If an agent with characteristic pair (ϑ, ϕ) buys claim C at price $v_0(C)$, the characteristic pair becomes (ϑ, ϕ^C) where*

$$\phi_t^C = \begin{cases} \phi_t - \gamma_t \zeta_t, & t \leq T; \\ \phi_t, & t > T, \end{cases} \quad (30)$$

and the forward performance process becomes $U_t^C(x) = -e^{-\gamma_t x + A_t^C}$, where

$$A_t^C = -\frac{v_0(C)}{\gamma_0} + \frac{1}{2} \int_0^t (\lambda_u - \delta_u)^2 du + \gamma_t X_t^\theta - \frac{1}{2} \int_0^t (\phi_u^C)^2 du - \int_0^t \phi_u^C dW_u^2$$

and θ is a process such that $\theta_t = 0$, for every $t > T$.

Remark 2.1. *Note that the constant risk aversion of equation (21) becomes stochastic in (23) in a mild manner. In addition, the minimal entropy martingale measure is replaced by the measure $\mathbb{Q}^{\lambda,\phi}$.*

2.2. The robust representation. As in the backward indifference valuation, the forward valuation can be considered as a dynamic (convex) risk measure in the sense of [7] (see also [42]). For a fixed time horizon T , the map $-v_t(\cdot) : \mathbb{L}^\infty(\mathcal{F}_T) \mapsto \mathbb{L}^\infty(\mathcal{F}_t)$ is convex, cash invariant and decreasing. In the following theorem, we state its robust representation. For this we need to define the set of martingale measures with finite entropy

$$\mathcal{P}_T^H = \{\nu \in \mathcal{P}_T^\lambda : \mathbb{E}_{\mathbb{Q}^{\lambda,\nu}}[\log Z_T^{\lambda,\nu}] < \infty\}. \quad (31)$$

Theorem 2.1. *Impose Assumptions 1.1 and 1.2, let $(\vartheta_t, \phi_t)_{t \in [0, \infty)}$ be the characterization of a forward exponential performance and $T > 0$ some time horizon. If we assume that there exist constants $K_\gamma, \epsilon > 0$ such that*

- (i) $\mathbb{E}[e^{(1+\epsilon)\int_0^T \phi_u^2 du}] < \infty$ and
- (ii) $\sup_{t \in [0, T]} \|\gamma_t\|_{\mathbb{L}^\infty} < K_\gamma$,

where $\gamma_t = \left(X_t^{\frac{1}{\gamma_0}, \vartheta}\right)^{-1}$, the forward indifference (buyer) valuation process $(v_t(\cdot))_{t \in [0, T]}$ defined in (18) has the following representation

$$v_t(C) = \operatorname{ess\,inf}_{\nu \in \mathcal{P}_T^H} \{\mathbb{E}_{\mathbb{Q}^{\lambda,\nu}}[C | \mathcal{F}_t] + \alpha_{t,T}(\mathbb{Q}^{\lambda,\nu})\} \quad (32)$$

for every $C \in \mathbb{L}^\infty(\mathcal{F}_T)$, where

$$\alpha_{t,T}(\mathbb{Q}^{\lambda,\nu}) = \frac{1}{2} \mathbb{E}_{\mathbb{Q}^{\lambda,\nu}} \left[\int_t^T \frac{(\nu_s - \phi_s)^2}{\gamma_s} ds \middle| \mathcal{F}_t \right]. \quad (33)$$

Furthermore, the infimum in (32) is attained by the process $\nu_t^* = \phi_t + \zeta_t \gamma_t$, for $t \in [0, T]$, where ζ is the corresponding part of the solution of BSDE (23).

The proof of Theorem 2.1 is relatively lengthy and technical and for reader's convenient is placed in the Appendix A.

Remark 2.2. *The martingale measure which minimizes the penalty process $\alpha_{t,T}$, is the measure $\mathbb{Q}^{\lambda,\phi}$, which does not depend on the time horizon T . This means that the agent's marginal utility valuation (the so-called Davis price) is the (conditional) expectation of the payoff under the same martingale measure regardless the maturity of the claim. This is in contrast with the backward exponential valuation, where the corresponding measure is the minimal entropy martingale measure, which depends on the time horizon the utility lives in.*

We can now take advantage of representation (32) and prove some properties of the valuation under forward exponential performance criteria. Throughout the rest of this section, we fix the process ϕ in the characterization pair of the forward exponential performance and use the notation $v_t(C; \gamma)$ to emphasize (when needed) the dependence of the indifference valuation on the risk aversion process.

Proposition 2.2. *Impose the conditions of Theorem 2.1 and fix a time horizon $T > 0$. For every contingent claim $C \in \mathbb{L}^\infty(\mathcal{F}_T)$ the following statements hold true.*

- (i) *The forward (buyer) indifference value is decreasing with respect to risk aversion in the following sense: If $(g_t)_{t \in [0, T]}$ is another risk aversion process with $\gamma_T \geq g_T$, a.s., then*

$$v_t(C; \gamma) \leq v_t(C; g) \quad \text{a.s.},$$

for any $t \in [0, T]$.

- (ii) *Let $((\gamma_t(n))_{t \in [0, T]})_{n \in \mathbb{N}}$ be a sequence of risk aversion processes, such that $\gamma_T(n) \nearrow \infty$ in \mathbb{P} , as $n \rightarrow \infty$. Then*

$$v_t(C; \gamma(n)) \longrightarrow \inf_{\nu \in \mathcal{P}_T^H} \mathbb{E}_{\mathbb{Q}^{\lambda, \nu}}[C | \mathcal{F}_t] \quad \text{a.s.},$$

for any $t \in [0, T]$.

Also if $\gamma_T(n) \searrow 0$ in \mathbb{P} , as $n \rightarrow \infty$ then,

$$v_t(C; \gamma(n)) \longrightarrow \mathbb{E}_{\mathbb{Q}^{\lambda, \phi}}[C | \mathcal{F}_t] \quad \text{a.s.},$$

for any $t \in [0, T]$.

- (iii) *For each risk aversion coefficient, the forward indifference valuation is time consistent in the sense that*

$$v_\tau(v_s(C)) = v_\tau(C) \quad \text{a.s.},$$

for any stopping times τ, s with $\tau \leq s \leq T$.

- (iv) *The indifference valuation is replication invariance, i.e.*

$$v_t(C + X_T^\theta) = v_t(C) + X_t^\theta \quad \text{a.s.},$$

for every $\theta \in \mathcal{A}$.

Proof. Part (i) follows directly from the robust representation of the forward indifference valuation given in (32). Again from (32) and the monotone convergence theorem we get the first item of part (ii). For the limit of the indifference value when $\gamma_T(n) \searrow 0$, we use similar arguments as the ones in [26]. Thanks to the robust representation of the $v_t(C; \gamma(n))$, it is enough to show that

$$\liminf_{n \rightarrow \infty} v_t(C; \gamma(n)) \geq \mathbb{E}_{\mathbb{Q}^{\lambda, \phi}}[C | \mathcal{F}_t].$$

We shall show the above inequality for $t = 0$, since the more general case is proved similarly. By Fenchel-Young inequality $xp \geq \frac{p-e^{-ax}}{a} - \frac{p \log(p)}{a}$, we get that for every $\nu \in \mathcal{P}_T^H$

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}^{\lambda, \nu}}[C] &= \mathbb{E}_{\mathbb{Q}^{\lambda, \phi}} \left[C \frac{Z_T^{0, \nu}}{Z_T^{0, \phi}} \right] \geq \mathbb{E}_{\mathbb{Q}^{\lambda, \phi}} \left[\frac{\frac{Z_T^{0, \nu}}{Z_T^{0, \phi}} - e^{-\gamma_T(n)C}}{\gamma_T(n)} - \frac{1}{\gamma_T(n)} \frac{Z_T^{0, \nu}}{Z_T^{0, \phi}} \log \left(\frac{Z_T^{0, \nu}}{Z_T^{0, \phi}} \right) \right] \\
&= \mathbb{E}_{\mathbb{Q}^{\lambda, \phi}} \left[\frac{\frac{Z_T^{0, \nu}}{Z_T^{0, \phi}} - e^{-\gamma_T(n)C}}{\gamma_T(n)} \right] - \mathbb{E}_{\mathbb{Q}^{\lambda, \nu}} \left[\frac{1}{\gamma_T(n)} \frac{Z_T^{0, \nu}}{Z_T^{0, \phi}} \log \left(\frac{Z_T^{0, \nu}}{Z_T^{0, \phi}} \right) \right] \\
&= \frac{1}{\gamma_0(n)} - \mathbb{E}_{\mathbb{Q}^{\lambda, \phi}} \left[\frac{e^{-\gamma_T(n)C}}{\gamma_T(n)} \right] - \frac{1}{2} \mathbb{E}_{\mathbb{Q}^{\lambda, \nu}} \left[\frac{1}{\gamma_T(n)} \int_0^T (\phi_s - \nu_s)^2 ds \right] \\
&= \mathbb{E}_{\mathbb{Q}^{\lambda, \phi}} \left[\frac{1 - e^{-\gamma_T(n)C}}{\gamma_T(n)} \right] - \frac{1}{2} \mathbb{E}_{\mathbb{Q}^{\lambda, \nu}} \left[\frac{1}{\gamma_T(n)} \int_0^T (\phi_s - \nu_s)^2 ds \right]
\end{aligned}$$

Thus,

$$\liminf_{n \rightarrow \infty} v_0(C; \gamma(n)) \geq \liminf_{n \rightarrow \infty} \mathbb{E}_{\mathbb{Q}^{\lambda, \phi}} \left[\frac{1 - e^{-\gamma_T(n)C}}{\gamma_T(n)} \right],$$

which gives the desired result.

From the definition of the forward indifference valuation (18) and representation (19), we get that time consistent property (iii) is in fact equivalent to equation

$$\operatorname{ess\,inf}_{\pi \in \mathcal{A}^\infty} \mathbb{E} \left[e^{-\gamma_T(C + \int_\tau^T \pi_u dS_u) + A_T} \middle| \mathcal{F}_\tau \right] = \operatorname{ess\,inf}_{\theta \in \mathcal{A}^\infty} \mathbb{E} \left[e^{-\gamma_s(-v_s(C) + \int_\tau^s \theta_u dS_u) + A_s} \middle| \mathcal{F}_\tau \right], \quad (34)$$

where τ, s are stopping times such that $\tau \leq s \leq T$. Again by using representation (19) and under slight abuse of notation we can write the RHS of (34) as

$$\operatorname{ess\,inf}_{\theta \in \mathcal{A}^\infty} \mathbb{E} \left[\operatorname{ess\,inf}_{\pi \in \mathcal{A}^\infty} \mathbb{E} \left[e^{-\gamma_T(C + \int_s^T \pi_u dS_u) + A_T - \gamma_s \int_\tau^s \theta_u dS_u} \middle| \mathcal{F}_s \right] \middle| \mathcal{F}_\tau \right].$$

Due to the reblicability of the process $1/\gamma$, the latter term equals to

$$\operatorname{ess\,inf}_{\theta \in \mathcal{A}^\infty} \mathbb{E} \left[\operatorname{ess\,inf}_{\pi \in \mathcal{A}^\infty} \mathbb{E} \left[e^{-\gamma_T(C + \int_s^T \pi_u dS_u + \int_\tau^s \theta_u dS_u) + A_T} \middle| \mathcal{F}_s \right] \middle| \mathcal{F}_\tau \right].$$

Then, (34) follows by the dynamic programming principle (see among others Theorem 3.1 of [39]).

Finally, part (iv) is a consequence of the indifference valuation robust representation (32). \square

Remark 2.3. *All items of Proposition 2.2 can be considered as extensions of the properties of the backward indifference value $v_t^B(C)$, defined through exponential utility function in (20). For example, although the forward performance criterion has stochastic risk aversion, the monotonicity of the indifference value is preserved, something that is consistent with the financial intuition: the higher the risk aversion at the time of maturity is, the lower price the buyer is going to bid.*

The main difference between forward and backward valuation is that the so-called marginal martingale probability measure, i.e. the measure that minimizes penalty function, is $\mathbb{Q}^{\lambda, \phi}$ in forward

valuation and the minimal entropy martingale measure in the backward case. The important difference between these measures is that $\mathbb{Q}^{\lambda, \phi}$ does not depend on a time horizon.

Remark 2.4. A special case of the forward exponential performance is when the risk aversion is constant. Then, the problem of the indifference valuation has an immediate relation with the associated problem of the backward valuation. This is because, given a maturity T of a contingent claim, the forward utility function at T can be written as $U_T(x) = -e^{-\gamma(x - \frac{AT}{\gamma})}$, and the term $-\frac{AT}{\gamma}$ can be thought as an \mathcal{F}_T -measurable random endowment. Therefore, we may consider the forward performance indifference valuation as a backward indifference valuation under this random endowment. A number of properties of this value, called conditional indifference price, have been provided in the Appendix of [1].

One further property of the indifference value that can be proved using the robust representation of the price is the following.

Proposition 2.3. *Impose the conditions of Theorem 2.1 and let $C \in \mathbb{L}^\infty(\mathcal{F}_T)$ be a contingent claim for some time horizon $T > 0$. Then, the function*

$$\mathbb{R} \ni a \mapsto f(a) = v_0(aC)$$

is differentiable and

$$f'(a) = \mathbb{E}_{\mathbb{Q}^{\lambda, \phi(aC)}}[C] \tag{35}$$

where $\phi_t^{(aC)} = \phi_t - \gamma_t \zeta_t(aC)$, for $t \in [0, T]$ and $\zeta(aC)$ is the corresponding part of the solution of (23) for boundary condition aC .

Proof. We will show the result when $a = 0$. Thanks to item (ii) of Proposition 2.2, we have

$$\begin{aligned} \lim_{\epsilon \searrow 0} \frac{v_0(\epsilon C)}{\epsilon} &= \lim_{\epsilon \searrow 0} \min_{\nu \in \mathcal{P}_T^H} \mathbb{E}_{\mathbb{Q}^{\lambda, \nu}} \left[C + \frac{1}{2\epsilon\gamma_T} \int_0^T (\nu_u - \phi_u)^2 du \right] \\ &= \mathbb{E}_{\mathbb{Q}^{\lambda, \phi}}[C] \end{aligned}$$

We also observe that $\lim_{\epsilon \nearrow 0} \frac{v_0(\epsilon C)}{\epsilon} = -\lim_{\epsilon \searrow 0} \frac{v_0(-\epsilon C)}{\epsilon} = \mathbb{E}_{\mathbb{Q}^{\lambda, \phi}}[C]$, which means that $f'(0) = \mathbb{E}_{\mathbb{Q}^{\lambda, \phi}}[C]$.

The more general case of $a \neq 0$ follows from the same arguments and Remark 2.1. \square

Note that in the case of backward valuation the situation is similar. Namely, the function $g(a) = v^B(aC)$ is also differentiable and its derivative is given as $\mathbb{E}_{\mathbb{Q}(aC)}[C]$, where the $\mathbb{Q}(aC)$ is the martingale measure that minimizes the relative entropy with risk to the measure $\mathbb{P}(aC)$ defined by its R-N derivative $\frac{d\mathbb{P}(aC)}{d\mathbb{P}} = ce^{-\gamma aC}$, for the appropriate constant c (see [20] for details on this result).

Remark 2.5. *The derivative of the indifference valuation with respect to the units of a given claim can be used in the determination of the number of units that the agent is willing to sell/buy when*

the price of the contingent claim is given. In other words, it leads to the agent's demand function on this claim in the same manner as in [1]. This differentiation result can be applied for a vector of claims in straightforward way.

The arguments of the proposition below follow similar lines as those in Proposition 14 of [26].

Proposition 2.4. *Impose the conditions of Theorem 2.1 and let C^n be a bounded sequence in $\mathbb{L}^\infty(\mathcal{F}_T)$ such that $C^n \rightarrow C$ in probability for some $C \in \mathbb{L}^\infty(\mathcal{F}_T)$. Then*

$$\sup_{0 \leq t \leq T} |v_t(C^n) - v_t(C)| \rightarrow 0 \quad (36)$$

in probability.

Proof. Thanks to Proposition 2.1, the indifference values of C and C^n satisfy the following relations

$$\begin{aligned} C &= v_t(C) + \frac{1}{2} \int_t^T \gamma_u \zeta_u^2 du + \int_t^T \zeta_u dW_u^{2,\phi} + \int_t^T \theta_u \sigma_u dW_u^{1,\lambda} \\ C^n &= v_t(C^n) + \frac{1}{2} \int_t^T \gamma_u (\zeta_u^n)^2 du + \int_t^T \zeta_u^n dW_u^{2,\phi} + \int_t^T \theta_u^n \sigma_u dW_u^{1,\lambda} \end{aligned}$$

for some processes ζ, ζ^n, θ and θ^n . Hence,

$$v_t(C^n) - v_t(C) = C^n - C + \int_t^T (\theta_u - \theta_u^n) \sigma_u dW_u^{1,\lambda} + \frac{1}{2} \int_t^T \gamma_u (\zeta_u^2 - (\zeta_u^n)^2) du + \int_t^T (\zeta_u - \zeta_u^n) dW_u^{2,\phi}. \quad (37)$$

We then define for each $n \in \mathbb{N}$, the sequence of processes $\nu_t(n) = -\frac{1}{2} \gamma_t (\zeta_t + \zeta_t^n)$ and by Lemma 2.1 $\nu(n) \in \mathcal{P}_T^\lambda$ for each n . Under the probability measure $\mathbb{Q}^{(n)}$ defined through its R-N derivative

$$\frac{d\mathbb{Q}^{(n)}}{d\mathbb{Q}^{\lambda,\phi}} = \mathcal{E}_T \left(\int_0^\cdot \nu_u(n) dW_u^{2,\phi} \right)$$

we have that

$$v_t(C^n) - v_t(C) = \mathbb{E}_{\mathbb{Q}^{(n)}}[C^n - C | \mathcal{F}_t]. \quad (38)$$

The next step is to show that the process $\left(\int_0^t \gamma_u (\zeta_u + \zeta_u(n)) dW_u^{2,\phi} \right)_{t \in [0, T]}$ is a $BMO(\mathbb{Q}^{\lambda,\phi})$ -martingale.

Indeed, thanks to the Lemma 2.1 we have that

$$\begin{aligned} & \left\| \int_0^\cdot \gamma_u (\zeta_u + \zeta_u(n)) dW_u^{2,\phi} \right\|_{BMO}^2 = \sup_\tau \left\| \mathbb{E}_{\mathbb{Q}^{\lambda,\phi}} \left[\int_\tau^T \gamma_u^2 (\zeta_u + \zeta_u(n))^2 du \middle| \mathcal{F}_\tau \right] \right\|_{\mathbb{L}^\infty} \\ & \leq K_\gamma^2 \sup_\tau \left\| \mathbb{E}_{\mathbb{Q}^{\lambda,\phi}} \left[\int_\tau^T (\zeta_u(n))^2 du \middle| \mathcal{F}_\tau \right] \right\|_{\mathbb{L}^\infty} + K_\gamma^2 \sup_\tau \left\| \mathbb{E}_{\mathbb{Q}^{\lambda,\phi}} \left[\int_\tau^T (\zeta_u)^2 du \middle| \mathcal{F}_\tau \right] \right\|_{\mathbb{L}^\infty} < \infty \end{aligned}$$

where supremum is taken under all stopping times τ in $[0, T]$. Theorem 3.1 of [22] guarantees the existence of two constants $p > 1$ and $c > 0$ such that

$$\sup_{0 \leq t \leq T} \mathbb{E}_{\mathbb{Q}^{\lambda,\phi}} \left[e^{-\frac{p}{2} \int_t^T \nu_u(n)^2 du - p \int_t^T \nu_u(n) dW_u^{2,\phi}} \middle| \mathcal{F}_t \right] < c$$

for each $n \in \mathbb{N}$. Then, (36) follows by applying the Hölder's and the Doob's maximal inequalities. \square

3. OPTIMAL RISK SHARING

In the present section, we consider two financial agents whose investment criteria are modelled by forward exponential performances and we address the problem of *optimal risk sharing*. We denote the characterization pairs of agents' performance criteria by $(\theta_t, \phi_t)_{t \in [0, \infty)}$ and $(\vartheta_t, \psi_t)_{t \in [0, \infty)}$, with the risk aversion processes γ, g defined by the equations $\frac{1}{\gamma} = X^{\frac{1}{\gamma_0}, \theta}$ and $\frac{1}{g} = X^{\frac{1}{g_0}, \vartheta}$. Also, U_t^i stands for the corresponding agent's forward utility at time t and A^i is the associated process in the representation (10), for $i = 1, 2$. In addition, we assume that each agent has some initial (non-replicable) endowment in her portfolio, denoted by $\mathcal{E}^i \in \mathbb{L}^\infty(\mathcal{F}_T)$, for some time horizon $T > 0$. The sum $\mathcal{E} = \mathcal{E}^1 + \mathcal{E}^2$ is the so-called *aggregated random endowment*.

The problem of optimal risk sharing (as formed in the mathematical finance literature in [2] and [3]) is finding a contract C^* and a price p^* which solve the following problem

$$\operatorname{argmax}_{C \in \mathbb{L}^\infty(\mathcal{F}), p \in \mathbb{R}} \sup_{\pi \in \mathcal{A}^\infty} \mathbb{E} \left[U_T^1(\mathcal{E}^1 + \int_0^T \pi_s dS_u + C - p) \right]$$

Given that

$$\sup_{\pi \in \mathcal{A}^\infty} \mathbb{E} \left[U_T^2(\mathcal{E}^2 + \int_0^T \pi_s dS_u) \right] \leq \sup_{\pi \in \mathcal{A}^\infty} \mathbb{E} \left[U_T^2(\mathcal{E}^2 + \int_0^T \pi_s dS_u - C + p) \right]$$

As shown in Section 2 of [2], from the definition of the indifference valuation (18) and its replication invariance property (part (iv) of Proposition 2.2), we get the following more tractable equivalent problem

$$\operatorname{argmax}_{C \in \mathbb{L}^\infty(\mathcal{F})} \{v_0^1(\mathcal{E}^1 + C) + v_0^2(\mathcal{E}^2 - C)\} \quad (39)$$

where, $v_0^i(\cdot)$ denotes the (buyer) indifference valuation of the corresponding agent at time $t = 0$, where $i = 1, 2$.

Definition 3.1. *We say that agents are in Pareto optimal situation if the set of the solutions of problem (39) consists of replicable claims or equivalently if*

$$v_0^1(\mathcal{E}^1 + C) + v_0^2(\mathcal{E}^2 - C) < v_0^1(\mathcal{E}^1) + v_0^2(\mathcal{E}^2). \quad (40)$$

for every C which is not replicable.

Problem (39) introduces the so-called *inf-convolution measure*

$$\rho(\mathcal{E}) = \inf_{C \in \mathbb{L}^\infty(\mathcal{F})} \{\rho^1(\mathcal{E} - C) + \rho^2(C)\} \quad (41)$$

where $\rho^i(C) = -v_0^i(C)$ for $i = 1, 2$ is the convex risk measure induced by the associated forward exponential performance criteria. Note that ρ maps payoffs in $\mathbb{L}^\infty(\mathcal{F}_T)$ to $\mathbb{R} \cup \{-\infty\}$, where the time horizon T is the maturity of the agents' endowments.

Assumption 3.1. *There exist constants $\epsilon, K > 0$ such that $\sup_{t \in [0, T]} \|\gamma_t\|_{\mathbb{L}^\infty}, \sup_{t \in [0, T]} \|g_t\|_{\mathbb{L}^\infty} < K$, $\mathbb{E}[e^{(1+\epsilon) \int_0^T \phi_u^2 du}]$, $\mathbb{E}[e^{(1+\epsilon) \int_0^T \psi_u^2 du}] < \infty$ and Assumption 1.2 holds for both processes A^1 and A^2 .*

A first result, the proof of which is based on Theorem 2.1, is that the inf-convolution measure of two forward exponential performance processes is *not* a risk measure that is induced by another forward exponential performance criterion (something which is in contrast with the inf-convolution measure induced by exponential utility functions).

Proposition 3.1. *Impose Assumptions 1.1 and 3.1. The inf-convolution risk measure ρ defined in (41) is induced by a forward exponential performance process if and only if $\phi_t = \psi_t$ for all $t \in [0, T]$, a.s. In this case, the characteristic pair of the forward exponential performance is $(\theta_t + \vartheta_t, \phi_t)_{t \in [0, T]}$ and the risk aversion process is given by*

$$\Gamma_t = \frac{\gamma_t g_t}{\gamma_t + g_t}. \quad (42)$$

Hence, the forward exponential performance can be written as $U_t(x) = -e^{\Gamma_t x + A_t}$, where

$$A_t = \frac{1}{2} \int_0^t (\lambda_u - \delta_u)^2 du - \frac{1}{2} \int_0^t \phi_u^2 du - \int_0^t \phi_u dW_u^2, \quad (43)$$

and $\delta_t = \Gamma_t(\theta_t + \vartheta_t)\sigma_t$.

Proof. Theorem 3.6 in [3] states that the penalty function of the inf-convolution measure is the sum of the penalty function of the involved risk measures. In our setting, the penalty function of ρ can be written as

$$\alpha_{t, T}(\mathbb{Q}^{\lambda, \nu}) = \frac{1}{2} \mathbb{E}_{\mathbb{Q}^{\lambda, \nu}} \left[\int_t^T \frac{(\nu_s - \phi_s)^2}{\gamma_s} + \frac{(\nu_s - \psi_s)^2}{g_s} ds \middle| \mathcal{F}_t \right] \quad (44)$$

for every $t \in [0, T]$.

Assume that ρ is induced by a forward exponential performance. A necessary condition for this is the existence of a risk aversion process Γ and a process $\tilde{\phi}$ such that

$$\mathbb{E}_{\mathbb{Q}^{\lambda, \nu}} \left[\int_t^T \frac{(\nu_s - \tilde{\phi}_s)^2}{\Gamma_s} ds \middle| \mathcal{F}_t \right] = \mathbb{E}_{\mathbb{Q}^{\lambda, \nu}} \left[\int_t^T \frac{(\nu_s - \phi_s)^2}{\gamma_s} + \frac{(\nu_s - \psi_s)^2}{g_s} ds \middle| \mathcal{F}_t \right]$$

for every $t \in [0, T]$ and for every process $\nu \in \mathcal{P}_T^H$. Setting $\nu = \tilde{\phi}$ and taking into account the positivity of γ and g , we get that $\phi_t = \psi_t$ for all $t \in [0, T]$, a.s. But $\frac{\gamma+g}{\gamma g}$ is replicable and bounded, therefore

$$\mathbb{E}_{\mathbb{Q}^{\lambda, \nu}} \left[\frac{1}{\Gamma_T} \int_t^T (\nu_s - \tilde{\phi}_s)^2 ds \middle| \mathcal{F}_t \right] = \mathbb{E}_{\mathbb{Q}^{\lambda, \nu}} \left[\frac{\gamma_T + g_T}{\gamma_T g_T} \int_t^T (\nu_s - \phi_s)^2 ds \middle| \mathcal{F}_t \right]$$

for every $t \in [0, T]$ and for every process $\nu \in \mathcal{P}_T^H$, which first implies that $\tilde{\phi} = \phi$ and then $\Gamma_t = \frac{\gamma_t g_t}{\gamma_t + g_t}$, $\forall t \in [0, T]$.

For the inverse part, we assume that $\phi_t = \psi_t$ for all $t \in [0, T]$, a.s. Equation (44) implies that

$$\alpha_{t,T}(\mathbb{Q}^{\lambda,\nu}) = \frac{1}{2} \mathbb{E}_{\mathbb{Q}^{\lambda,\nu}} \left[\int_t^T \frac{\gamma_s + g_s}{\gamma_s g_s} (\nu_s - \phi_s)^2 ds \middle| \mathcal{F}_t \right]$$

for every t and ν . Letting $\Gamma_t = \frac{\gamma_t g_t}{\gamma_t + g_t}$ completes the proof. \square

Remark 3.1. *Proposition 3.1 states that only in the case where agents adapt their subjective probability measure up to the maturity of their endowments in the same manner, the representative agent can be considered as behaving under forward exponential performance criteria. In other words, when $\phi = \psi$ Theorem 2.3 in [3] has a direct extension in the case of stochastic risk aversion.*

3.1. The special case of replicable endowments. A special case is when agents do not have any endowment in their initial portfolios or when both endowments are replicable, that is $\exists \pi^1, \pi^2 \in \mathcal{A}^\infty$ and $c^1, c^2 \in \mathbb{R}$ such that $\mathcal{E}^i = X_T^{c^i, \pi^i}$, for $i = 1, 2$.

Proposition 3.2. *Let Assumptions 1.1 and 3.1 hold true and assume that \mathcal{E}^1 and \mathcal{E}^2 are replicable. Then, agents are in Pareto optimal situation if and only if $\phi_t = \psi_t$ for every $t \in [0, \infty)$ a.s.*

Proof. The result follows from the dual representation of the indifference valuation (64) and the Theorem 3.6 of [3], which states that Pareto optimality is equivalent to the equality of the agents' marginal martingale measures, that is $\mathbb{Q}^{\lambda, \phi} = \mathbb{Q}^{\lambda, \psi}$. \square

Proposition 3.2 implies that agents are willing to trade some non-replicable claims if and only if the way they adapt their subjective probability measure is not the same at all times. Note that this statement is independent on the agents' risk aversion processes γ and g . Another way to interpret this result is that if agents do not include in their utilities the unhedgeable part of the market (i.e. when $\phi = \psi = 0$), they are unwilling to make any non-replicable transaction, no matter how their risk aversion processes differ to each other. This can be seen as a generalization of the corresponding result in the case of backward exponential utility (see Proposition 3.8 in [1]).

3.2. The case of constant risk aversions. In the simplified case where the agents' risk aversion are constant, we may exploit robust representation (32) and explicitly solve the sharing problem (39). This is in fact because under constant risk aversion, the contingent claim valuation problem can be written as conditional indifference valuation under classical exponential utility (see Remark 2.4). Hence, we are able to establish not only the way that agents optimally share their endowments but also how they can trade the difference in their beliefs (i.e. the difference between processes ϕ and ψ).

Proposition 3.3. *Impose Assumptions 1.1 and 3.1 and suppose in addition that agents' risk aversions are constants, i.e. $\gamma, g \in \mathbb{R}_+$. Any claim of the form*

$$\frac{1}{\gamma + g} \left(\int_0^T \frac{\psi_t^2 - \phi_t^2}{2} dt + \int_0^T (\psi_t - \phi_t) dW_t^2 \right) + \frac{g\mathcal{E}^2 - \gamma\mathcal{E}^1}{\gamma + g} + X_T^{c, \pi} \quad (45)$$

for some $c \in \mathbb{R}$ and $\pi \in \mathcal{A}^\infty$, solves the optimal sharing problem (39).

Proof. By robust representation (32) (see also its proof), we get that for every claim C it holds that

$$v_0^1(\mathcal{E}^1 + C) + v_0^2(\mathcal{E}^2 - C) \leq \inf_{\nu \in \mathcal{P}_T^H} \mathbb{E}_{\mathbb{Q}^{\lambda, \nu}} \left[\mathcal{E} + \frac{1}{2\gamma} \int_0^T (\nu_s - \phi_s)^2 ds + \frac{1}{2g} \int_0^T (\nu_s - \psi_s)^2 ds \right]. \quad (46)$$

For very claim C^* of the form (45), we have that

$$\begin{aligned} v_0^1(\mathcal{E}^1 + C^*) &= \inf_{\nu \in \mathcal{P}_T^H} \mathbb{E}_{\mathbb{Q}^{\lambda, \nu}} \left[\frac{g}{\gamma + g} \mathcal{E} + \frac{1}{2} \int_0^T \left(\frac{(\nu_s - \phi_s)^2}{\gamma} + \frac{\psi_s^2 - \phi_s^2}{\gamma + g} - \frac{2(\psi_s - \phi_s)}{\gamma + g} \right) ds \right] + c \\ &= \frac{g}{\gamma + g} \inf_{\nu \in \mathcal{P}_T^H} \mathbb{E}_{\mathbb{Q}^{\lambda, \nu}} \left[\mathcal{E} + \frac{1}{2\gamma} \int_0^T (\nu_s - \phi_s)^2 ds + \frac{1}{2g} \int_0^T (\nu_s - \psi_s)^2 ds \right] + c \end{aligned}$$

Similarly, we get that

$$\begin{aligned} v_0^2(\mathcal{E}^2 - C^*) &= \inf_{\nu \in \mathcal{P}_T^H} \mathbb{E}_{\mathbb{Q}^{\lambda, \nu}} \left[\frac{\gamma}{\gamma + g} \mathcal{E} + \frac{1}{2} \int_0^T \left(\frac{(\nu_s - \psi_s)^2}{g} - \frac{\psi_s^2 - \phi_s^2}{\gamma + g} + \frac{2(\psi_s - \phi_s)}{\gamma + g} \right) ds \right] - c \\ &= \frac{\gamma}{\gamma + g} \inf_{\nu \in \mathcal{P}_T^H} \mathbb{E}_{\mathbb{Q}^{\lambda, \nu}} \left[\mathcal{E} + \frac{1}{2\gamma} \int_0^T (\nu_s - \phi_s)^2 ds + \frac{1}{2g} \int_0^T (\nu_s - \psi_s)^2 ds \right] - c \end{aligned}$$

Therefore,

$$v_0^1(\mathcal{E}^1 + C^*) + v_0^2(\mathcal{E}^2 - C^*) = \inf_{\nu \in \mathcal{P}_T^H} \mathbb{E}_{\mathbb{Q}^{\lambda, \nu}} \left[\mathcal{E} + \frac{1}{2\gamma} \int_0^T (\nu_s - \phi_s)^2 ds + \frac{1}{2g} \int_0^T (\nu_s - \psi_s)^2 ds \right]$$

which together with (46) completes the proof. \square

In other words, the optimal risk sharing part consists of three elements: the optimal sharing of the agents' random endowments which is exactly the same as in the backward exponential utility case (see [2]), the sharing of the agents' perspectives about the probability measure (in the way they are incorporated on the agents' forward performances) and a replicable part (which can essentially be ignored since it does not transfer any risk).

If there are no endowments, the agents share their difference of beliefs regarding the evolution of the probability measure through the contract with payoff $\frac{1}{\gamma + g} \left(\int_0^T \frac{\psi_t^2 - \phi_t^2}{2} dt + \int_0^T (\psi_t - \phi_t) dW_t^2 \right)$. Note that the expectation of this payoff increases (in absolute terms) as the difference of the processes ϕ and ψ increases (this means that intense difference in beliefs implies high volume of transaction).

3.3. The case of stochastic risk aversions. In the case where the agents' risk aversion coefficients are stochastic, the optimal sharing problem is more involved, since the methods used in the backward exponential utility case can not be applied. Recall that problem (39) is equivalent to

finding a claim C^* that maximizes the sum $v_0^1(\mathcal{E}^1 + C) + v_0^2(\mathcal{E}^2 - C)$, for which we have

$$\begin{aligned} & v_0^1(\mathcal{E}^1 + C) + v_0^2(\mathcal{E}^2 - C) = \\ &= \inf_{\nu \in \mathcal{P}_T^H} \mathbb{E}_{\mathbb{Q}^{\lambda, \nu}} \left[\mathcal{E}^1 + C + \frac{1}{2} \int_0^T \frac{(\nu_s - \phi_s)^2}{\gamma_s} ds \right] + \inf_{\nu \in \mathcal{P}_T^H} \mathbb{E}_{\mathbb{Q}^{\lambda, \nu}} \left[\mathcal{E}^2 - C + \frac{1}{2} \int_0^T \frac{(\nu_s - \psi_s)^2}{g_s} ds \right] \\ &= \inf_{\nu \in \mathcal{P}_T^H} \mathbb{E}_{\mathbb{Q}^{\lambda, \nu}} \left[\mathcal{E} + C + \frac{1}{2} \int_0^T \frac{(\nu_s - \phi_s)^2}{\gamma_s} ds \right] + \inf_{\nu \in \mathcal{P}_T^H} \mathbb{E}_{\mathbb{Q}^{\lambda, \nu}} \left[-C + \frac{1}{2} \int_0^T \frac{(\nu_s - \psi_s)^2}{g_s} ds \right]. \end{aligned}$$

The way that the optimal risk sharing problem can be solved is through the construction of the dynamics of the inf-convolution measure, which will give the existence and an implicit form of the optimal risk sharing contract.

We first note that under Assumptions 1.1 and 3.1, the agents' dynamic indifference value for any bounded contingent claim C solve the following BSDE's under the minimal martingale measure $\mathbb{Q}^{\lambda, 0}$ (see Proposition 2.1)

$$-d\rho_t^1(C) = dX_t^{\pi^1} + \frac{1}{2} (\gamma_t(\zeta_t^1)^2 - 2\zeta_t^1\phi_t) dt - \zeta_t^1 dW_t^2, \quad \rho_T^1(C) = -C \quad (47)$$

$$-d\rho_t^2(C) = dX_t^{\pi^2} + \frac{1}{2} (g_t(\zeta_t^2)^2 - 2\zeta_t^2\psi_t) dt - \zeta_t^2 dW_t^2, \quad \rho_T^2(C) = -C \quad (48)$$

where $\pi^1, \pi^2 \in \mathcal{A}$ and $\zeta^1, \zeta^2 \in \mathcal{P}_T$. Adapting the argument lines of Section 3 of [2], we consider a claim $C \in \mathbb{L}^\infty(\mathcal{F}_T)$ and introduce the BSDE

$$-dF_t = f(t, \zeta_t) dt - \zeta_t dW_t^2 + dX_t^\theta, \quad F_T = -C, \quad (49)$$

where for every $t \in [0, T]$

$$f(t, \zeta_t) = \frac{1}{2} \frac{\gamma_t g_t}{\gamma_t + g_t} (\zeta_t^2 + (\psi_t - \phi_t)^2) - \frac{\zeta_t (\gamma_t \psi_t + g_t \phi_t) + (\psi_t - \phi_t)^2}{\gamma_t + g_t}. \quad (50)$$

The solution of (49) is given by a triple $(F_t(C), \zeta_t, \theta_t)$.

The following theorem solves the optimal risk sharing problem under forward exponential performance criteria.

Theorem 3.1. *Impose Assumptions 1.1 and 3.1, assume furthermore that $\int_0^T (\psi_u - \phi_u)^2 du \in \mathbb{L}^\infty(\mathcal{F}_T)$ and let $(\rho_t^{1,2}(\mathcal{E}), \zeta_t, \theta_t^{1,2})$ be the solution of (49). Then, for every $t \in [0, T]$*

$$\rho_t^{1,2}(\mathcal{E}) = \inf_{C \in \mathbb{L}^\infty(\mathcal{F})} \{\rho_t^1(\mathcal{E} - C) + \rho_t^2(C)\}$$

and the optimal risk sharing claims are of the form

$$C^* = \mathcal{E}^2 - \int_0^T \left(\frac{g_t}{2} \left(\frac{\gamma_t \zeta_t + (\psi_t - \phi_t)}{\gamma_t + g_t} \right)^2 + \frac{\gamma_t \zeta_t + (\psi_t - \phi_t)}{\gamma_t + g_t} \psi_t \right) dt - \int_0^T \frac{\gamma_t \zeta_t + (\psi_t - \phi_t)}{\gamma_t + g_t} dW_t^2 + X_T^{c, \pi} \quad (51)$$

for some $c \in \mathbb{R}$ and $\pi \in \mathcal{A}^\infty$.

Proof. First we get from the following Lemma 3.1 that BSDE (49) admits a unique uniformly bounded solution. After simple calculations we get that for every processes z and y and for every $t \in [0, T]$

$$f(t, z_t) \leq \frac{1}{2} (\gamma_t(z_t - y_t)^2 - 2(z_t - y_t)\phi_t) + \frac{1}{2} (g_t y_t^2 - 2y_t \psi_t). \quad (52)$$

Let $C \in \mathbb{L}^\infty(\mathcal{F}_T)$ be an arbitrarily chosen claim and $(\rho_t^1(\mathcal{E} - C), \zeta_t^1, \pi_t^1, \cdot)$ and $(\rho_t^2(C), \zeta_t^2, \pi_t^2, \cdot)$ be the solutions of the BSDE's (47) and (48) with boundary conditions $C - \mathcal{E}$ and $-C$ respectively. This implies that if we set $\tilde{\pi}_t = \pi_t^1 + \pi_t^2$ and $\tilde{\zeta}_t = \zeta_t^1 + \zeta_t^2$, the triple $(\rho_t^1(\mathcal{E} - C) + \rho_t^2(C), \tilde{\zeta}_t, \tilde{\pi}_t)$ is a solution of the following BSDE

$$-dC_t = \frac{1}{2} (\gamma_t(z_t - \zeta_t^2)^2 - 2(z_t - \zeta_t^2)\phi_t) dt + \frac{1}{2} (g_t(\zeta_t^2)^2 - 2\zeta_t^2\psi_t) dt - z_t dW_t^2 + dX_t^\theta, \quad C_T = -\mathcal{E} \quad (53)$$

for ζ^2 given. Then by Proposition B.1 and inequality (52), we get that $\rho_t^{1,2}(\mathcal{E}) \leq \rho_t^1(\mathcal{E} - C) + \rho_t^2(C)$ for every claim C and every time $t \in [0, T]$. Also for process ζ

$$f(t, \zeta_t) = \frac{1}{2} (\gamma_t(\zeta_t - \hat{z}_t)^2 - 2(\zeta_t - \hat{z}_t)\phi_t) + \frac{1}{2} (g_t \hat{z}_t^2 - 2\hat{z}_t \psi_t)$$

and for every $t \in [0, T]$, where $\hat{z}_t = \frac{\zeta_t \gamma_t + (\psi_t - \phi_t)}{\gamma_t + g_t} \in \mathcal{P}_T^\lambda$.

The next step is to observe that for the process

$$\hat{C}_t = - \int_0^t \left(\frac{g_s}{2} \hat{z}_s^2 - \hat{z}_s \psi_s \right) ds + \int_0^t \hat{z}_s dW_s^2$$

the triple $(\hat{C}_t, \hat{z}_t, 0)$ is the unique solution of (48) with boundary condition $\hat{C} = \hat{C}_T$, i.e., $\hat{C}_t = \rho_t^2(\hat{C})$.

We also have that if $(\rho_t^1(\mathcal{E} - \hat{C}), z_t^1, \pi_t^1)$ is the solution of (47) with boundary condition $\hat{C} - \mathcal{E}$, then $(\rho_t^1(\mathcal{E} - \hat{C}) + \rho_t^2(\hat{C}), z_t^1 + \hat{z}_t, \pi_t^1)$ is the solution of (49) with boundary condition $-\mathcal{E}$. Thanks to Lemma 3.1, we have that $\rho_t^1(\mathcal{E} - \hat{C}) + \rho_t^2(\hat{C}) = \rho_t^{1,2}(\mathcal{E})$, which in turn means that $C^* = \mathcal{E} - \hat{C}$ is an optimal risk sharing contract. The fact that we can add/subtract any replicable claim on the optimal risk sharing contract is a consequence of the replication invariance property of the indifference valuation (see Proposition 2.2, item (iv)).

The final part of the proof is to show that every optimal risk sharing claim admits the form (51). If \hat{C} is another optimal sharing claim then

$$\min_{C \in \mathbb{L}^\infty(\mathcal{F})} \{\rho_0^1(\mathcal{E} - C) + \rho_0^2(C)\} = \rho_0^1(\mathcal{E} - \hat{C}) + \rho_0^2(\hat{C}) = \rho_0^1(\mathcal{E} - C^*) + \rho_0^2(C^*).$$

Due to the convexity of the risk measures, this means that for every $l \in [0, 1]$ the contract $lC^* + (1-l)\hat{C}$ is also an optimal risk sharing contract. This in particular implies that

$$\rho_0^2(lC^* + (1-l)\hat{C}) = l\rho_0^2(C^*) + (1-l)\rho_0^2(\hat{C})$$

or equivalently,

$$v_0^2(lC^* + (1-l)\hat{C}) = lv_0^2(C^*) + (1-l)v_0^2(\hat{C}). \quad (54)$$

In the view of the robust representation (32), equality (54) implies the existence of a process $\nu^* \in \mathbb{P}_T^H$ such that

$$\nu^* = \operatorname{argmin}_{\nu \in \mathcal{P}_T^H} \{\mathbb{E}_{\mathbb{Q}^{\lambda, \nu}}[C^*] + \alpha_{0, T}(\mathbb{Q}^{\lambda, \nu})\} = \operatorname{argmin}_{\nu \in \mathcal{P}_T^H} \{\mathbb{E}_{\mathbb{Q}^{\lambda, \nu}}[\hat{C}] + \alpha_{0, T}(\mathbb{Q}^{\lambda, \nu})\}. \quad (55)$$

Now, Proposition 2.1 guarantees the existence of processes $\xi^*, \hat{\xi}$ that satisfy condition (25) and processes $\theta^*, \hat{\theta} \in \mathcal{A}^\infty$ such that

$$C^* = v_0^2(C^*) + \frac{1}{2} \int_0^T g_u(\xi_u^*)^2 du + \int_0^T \xi_u^* dW_u^{2, \psi} + \int_0^T \theta_u^* dS_u \quad (56)$$

$$\hat{C} = v_0^2(\hat{C}) + \frac{1}{2} \int_0^T g_u(\hat{\xi}_u)^2 du + \int_0^T \hat{\xi}_u dW_u^{2, \psi} + \int_0^T \hat{\theta}_u dS_u. \quad (57)$$

But from Theorem 2.1, $\nu_t^* = \psi_t + \xi_t^* g_t$ and $\nu_t^* = \psi_t + \hat{\xi}_t g_t$ for all $t \in [0, T]$. Hence, $\xi_t^* = \hat{\xi}_t$ for all $t \in [0, T]$, which combined with (56) and (57) implies the existence of some constant c' and process $\theta' \in \mathcal{A}^\infty$ such that $C^* = \hat{C} + X_T^{c', \theta'}$. The latter equality completes the proof. \square

Lemma 3.1. *If we impose Assumptions 1.1 and 3.1 and assume furthermore that $\int_0^T (\psi_u - \phi_u)^2 du \in \mathbb{L}^\infty(\mathcal{F}_T)$, the BSDE (49) admits a unique solution $(F_t(C), z_t, \theta_t)$ for every $C \in \mathbb{L}^\infty(\mathcal{F}_T)$, with $(C_t)_{t \in [0, T]}$ being uniformly bounded. In addition, $F_t(C)$ is for any time $t \in [0, T]$ a convex risk measure.*

Proof. First we define the process $k_t = \frac{\gamma_t \psi_t + g_t \phi_t}{\gamma_t + g_t}$, for $t \in [0, T]$, and note that $k \in \mathcal{P}_T^\lambda$. Under the martingale measure $\mathbb{Q}^{\lambda, k}$ the BSDE (49) is written as

$$-dF_t = \frac{1}{2}(\Gamma_t z_t^2 dt - 2z_t dW_t^{2, k}) + dX_t^\theta + dL_t, \quad F_T = -C \quad (58)$$

where $\Gamma_t = \frac{\gamma_t g_t}{\gamma_t + g_t}$, $W^{2, k} = W_t^2 + \int_0^t k_u du$ is a standard Brownian motion under $\mathbb{Q}^{\lambda, k}$, orthogonal to W^1 and $L_t = \frac{1}{2} \int_0^t \left(\Gamma_u (\psi_u - \phi_u)^2 - \frac{(\psi_u - \phi_u)^2}{\gamma_u + g_u} \right) du$. We then observe that the BDSE

$$-d\hat{F}_t = \frac{1}{2}(\Gamma_t z_t^2 dt - 2z_t d\hat{W}_t^2) + dX_t^\theta \quad \hat{F}_T = -C + L_T \quad (59)$$

admits a solution $(\hat{F}_t, \hat{z}_t, \hat{\theta}_t)$, which is in fact the dynamic risk measure $\hat{\rho}_t(C - L_T)$ induced by the exponential performance criteria with characterization pair $(\theta_t + \vartheta_t, k_t)$. This means that $(\hat{F}_t - L_t, \hat{z}_t, \hat{\theta}_t)$ is a solution of (58), where $(\hat{F}_t - L_t)_{t \in [0, T]}$ is uniformly bounded. The uniqueness of the solution follows by Proposition B.1.

Finally, the fact that $F_t(C)$ is a dynamic convex risk measure is a consequence of Proposition B.1 and the convexity of $f(t, \zeta_t)$ with respect to ζ_t (see also Proposition 5.1 of [13] and Proposition 5.1 of [35]). \square

Remark 3.2. *We note that when $\phi_t = \psi_t$ for every $t \in [0, T]$, equation (49) becomes similar to (47) and (48), where the risk aversion process is given by Γ_t (see (42)). This means that the*

inf-convolution measure is a dynamic risk measures that is induced by an agent with exponential forward performance criteria with characteristic pair $(\theta_t + \vartheta_t, \phi_t)$ (see also Proposition 3.1).

Before we end this subsection, it worths to check what happens to the optimal risk sharing contract when the agents have the same risk aversion process. As it is shown below, in this case the optimal risk sharing problem (39) can be solved explicitly.

Proposition 3.4. *Impose Assumptions 1.1 and 3.1 and suppose that $\gamma_t = g_t$ for every $t \in [0, T]$. Then, the optimal risk sharing claims are of the form*

$$\frac{1}{2} \int_0^T \frac{\psi_t^2 - \phi_t^2}{\gamma_t} dt + \int_0^T \frac{\psi_t - \phi_t}{\gamma_t} dW_t^2 + \frac{\mathcal{E}^2 - \mathcal{E}^1}{2} + X_T^{c, \pi} \quad (60)$$

for some $c \in \mathbb{R}$ and $\pi \in \mathcal{A}^\infty$.

Proof. It is enough to show that for every claim C^* of the form (60)

$$v_0^1(\mathcal{E}^1 + C^*) + v_0^2(\mathcal{E}^2 - C^*) = \inf_{\nu \in \mathcal{P}_T^H} \mathbb{E}_{\mathbb{Q}^{\lambda, \nu}} \left[\mathcal{E} + \frac{1}{2\gamma_T} \int_0^T ((\nu_s - \phi_s)^2 + (\nu_s - \psi_s)^2) ds \right]. \quad (61)$$

We first observe that for every such C^* and every $\nu \in \mathcal{P}_T^H$, $\mathbb{E}_{\mathbb{Q}^{\lambda, \nu}} \left[\mathcal{E}^1 + C^* + \frac{1}{2} \int_0^T \frac{(\nu_s - \phi_s)^2}{\gamma_s} ds \right]$ equals to

$$\begin{aligned} & \mathbb{E}_{\mathbb{Q}^{\lambda, \nu}} \left[\frac{\mathcal{E}}{2} + \frac{1}{2\gamma_T} \left(\int_0^T ((\nu_s - \phi_s)^2 + \psi_s^2 - \phi_s^2) ds + \int_0^T (\psi_s - \phi_s) dW_s^2 \right) \right] \\ &= \mathbb{E}_{\mathbb{Q}^{\lambda, \nu}} \left[\frac{\mathcal{E}}{2} + \frac{1}{4\gamma_T} \left(\int_0^T ((\nu_s - \phi_s)^2 + (\nu_s - \psi_s)^2) ds \right) \right] \end{aligned}$$

Applying the same calculations to $\mathbb{E}_{\mathbb{Q}^{\lambda, \nu}} \left[\mathcal{E}^2 - C^* + \frac{1}{2} \int_0^T \frac{(\nu_s - \psi_s)^2}{\gamma_s} ds \right]$, we get that

$$v_0^1(\mathcal{E}^1 + C^*) + v_0^2(\mathcal{E}^2 - C^*) = 2 \inf_{\nu \in \mathcal{P}_T^H} \mathbb{E}_{\mathbb{Q}^{\lambda, \nu}} \left[\frac{\mathcal{E}}{2} + \frac{1}{4\gamma_T} \left(\int_0^T ((\nu_s - \phi_s)^2 + (\nu_s - \psi_s)^2) ds \right) \right]$$

which is equivalent to (61).

Uniqueness of the form (60) is proved by the same arguments as the ones in Theorem 3.1. \square

If the agents have the same risk aversion process, the sharing of their endowments is exactly the same as the corresponding situation of entropic risk measures (i.e., when agents have the same risk aversion coefficient). This implies that after the transaction both agents have the same random endowment, $\mathcal{E}/2$. Note also the similarity of the other terms of the optimal contract with those of the case analyzed in subsection 3.2.

APPENDIX A

Proof of Theorem 2.1.

We fix an arbitrary chosen contingent claim $C \in \mathbb{L}^\infty(\mathcal{F}_T)$ and following the steps in the proof of Proposition 2.1 we set $\tilde{A}_t = -\gamma_t v_t(C) + A_t$. A first form of the robust representation of the indifference valuation may be provided by applying Theorem 4.4 in [43]. In particular, we have that

$$\frac{1}{\gamma_t} \left(\log \left(\frac{1}{\gamma_t} \right) - 1 - \tilde{A}_t \right) = \operatorname{essinf}_{\mathbb{Q} \in \mathcal{M}_T^\varepsilon} \mathbb{E}_{\mathbb{Q}} \left[\frac{1}{\gamma_T} \left(\log \left(\frac{1}{\gamma_T} \frac{Z_T^{\mathbb{Q}}}{Z_t^{\mathbb{Q}}} \right) - 1 - \tilde{A}_T \right) \middle| \mathcal{F}_t \right], \quad (62)$$

where $(Z_t^{\mathbb{Q}})_{t \in [0, T]}$ is the density process of the probability measure \mathbb{Q} with respect to measure \mathbb{P} and is defined by $Z_t = \mathbb{E}_{\mathbb{P}} \left[\frac{d\mathbb{Q}}{d(\mathbb{P}|_{\mathcal{F}_T})} \middle| \mathcal{F}_t \right]$, for $t \in [0, T]$. Equation (62) implies that

$$\frac{1}{\gamma_t} \left(\log \left(\frac{1}{\gamma_t} \right) - 1 - A_t \right) + v_t(C) = \operatorname{essinf}_{\mathbb{Q} \in \mathcal{M}_T^\varepsilon} \mathbb{E}_{\mathbb{Q}} \left[\frac{1}{\gamma_T} \left(\log \left(\frac{1}{\gamma_T} \frac{Z_T^{\mathbb{Q}}}{Z_t^{\mathbb{Q}}} \right) - 1 - A_T \right) + C \middle| \mathcal{F}_t \right]. \quad (63)$$

A simple rearrangement of the terms gives that indifference value process has the following dual representation

$$v_t(C) = \operatorname{essinf}_{\mathbb{Q} \in \mathcal{M}_T^\varepsilon} \{ \mathbb{E}_{\mathbb{Q}}[C | \mathcal{F}_t] + H_t(\mathbb{Q}, T) \} - H_t^{(0)}(T), \quad (64)$$

where

$$H_t(\mathbb{Q}, T) = \mathbb{E} \left[h \left(\frac{1}{\gamma_T} \frac{Z_T^{\mathbb{Q}}}{Z_t^{\mathbb{Q}}} \right) - \frac{1}{\gamma_T} \frac{Z_T^{\mathbb{Q}}}{Z_t^{\mathbb{Q}}} A_T \middle| \mathcal{F}_t \right],$$

with $h(y) = y \log(y) - y$ for $y > 0$, and $H_t^{(0)}(T) = \operatorname{essinf}_{\mathbb{Q} \in \mathcal{M}_T^\varepsilon} H_t(\mathbb{Q}, T)$. In other words the penalty function of the dual representation of the v_t is equal to $H_t(\mathbb{Q}, T) - H_t^{(0)}(T)$, for $t \in [0, T]$.

The next step is to show that the non-constant term of the penalty function, namely the process $(H_t(\mathbb{Q}, T))_{t \in [0, T]}$, can be written in the following way

$$H_t(\mathbb{Q}^{\lambda, \nu}, T) = h \left(\frac{1}{\gamma_t} \right) - \frac{A_t}{\gamma_t} + \frac{1}{2} \mathbb{E}_{\mathbb{Q}^{\lambda, \nu}} \left[\int_t^T \frac{(\nu_s - \phi_s)^2}{\gamma_s} ds \middle| \mathcal{F}_t \right], \quad (65)$$

for every $\nu \in \mathcal{P}_T^H$. Indeed, taking into account Theorem 1.1, we have that

$$\begin{aligned}
H_t(\mathbb{Q}^{\lambda,\nu}, T) &= \mathbb{E} \left[h \left(\frac{1}{\gamma_T} \frac{Z_T^{\lambda,\nu}}{Z_t^{\lambda,\nu}} \right) - \frac{1}{\gamma_T} \frac{Z_T^{\lambda,\nu}}{Z_t^{\lambda,\nu}} A_T \middle| \mathcal{F}_t \right] = \mathbb{E} \left[h \left(\frac{1}{\gamma_t} \frac{Z_T^{\lambda-\delta,\nu}}{Z_t^{\lambda-\delta,\nu}} \right) - \frac{1}{\gamma_t} \frac{Z_T^{\lambda-\delta,\nu}}{Z_t^{\lambda-\delta,\nu}} A_T \middle| \mathcal{F}_t \right] \\
&= \frac{1}{\gamma_t} \mathbb{E}_{\mathbb{Q}^{\lambda-\delta,\nu}} \left[\log \left(\frac{1}{\gamma_t} \frac{Z_T^{\lambda-\delta,\nu}}{Z_t^{\lambda-\delta,\nu}} \right) - A_T \middle| \mathcal{F}_t \right] - \frac{1}{\gamma_t} \\
&= h \left(\frac{1}{\gamma_t} \right) + \frac{1}{\gamma_t} \mathbb{E}_{\mathbb{Q}^{\lambda-\delta,\nu}} \left[\log \left(\frac{Z_T^{\lambda-\delta,\nu}}{Z_t^{\lambda-\delta,\nu}} \right) - A_T \middle| \mathcal{F}_t \right] \\
&= h \left(\frac{1}{\gamma_t} \right) - \frac{A_t}{\gamma_t} + \frac{1}{2\gamma_t} \mathbb{E}_{\mathbb{Q}^{\lambda-\delta,\nu}} \left[\int_t^T (\nu_s - \phi_s)^2 ds \middle| \mathcal{F}_t \right] \\
&\quad + \frac{1}{\gamma_t} \mathbb{E}_{\mathbb{Q}^{\lambda-\delta,\nu}} \left[- \int_t^T (\lambda_s - \delta_s) dW_s^{1,\lambda-\delta} - \int_t^T (\nu_s - \phi_s) dW_s^{2,\nu} \middle| \mathcal{F}_t \right],
\end{aligned}$$

where $W_t^{1,\lambda-\delta} = W_t^1 + \int_0^t (\lambda_u - \delta_u) du$ and $W_t^{2,\nu} = W_t^2 + \int_0^t \nu_u du$, for $t \in [0, T]$. In order to show (65), it is left to prove that for every $\nu \in \mathcal{P}_T^H$, the expectation of the stochastic integrals is zero.

For this, thanks to the uniform boundedness of process γ , it is enough to show that

$$\mathbb{E}_{\mathbb{Q}^{\lambda,\nu}} \left[\int_0^T (\lambda_s - \delta_s)^2 ds + \int_0^T (\nu_s - \phi_s)^2 ds \right] < \infty.$$

By item (i) and the inequality $y \log y - y + e^x \leq xy$, which holds for every $y > 0$ and $x \in \mathbb{R}$, it follows that $\int_0^T \lambda_s^2 ds$ and $\int_0^T \phi_s^2 ds$ belong in $\mathbb{L}^1(\mathbb{Q}^{\lambda,\nu}, \mathcal{F}_T)$. Also, finite entropy of $\mathbb{Q}^{\lambda,\nu}$ implies that $\mathbb{E}_{\mathbb{Q}^{\lambda,\nu}} \left[\int_0^T \nu_s^2 ds \right] < \infty$ (see the proof of Theorem 1.9 in [22]), whereas the same condition for process δ is guaranteed by the uniform boundedness of γ (see also Lemma 2.1). It is left to observe that replicability of $\left(\frac{1}{\gamma_t} \right)_{t \in [0, T]}$ implies that

$$\mathbb{E}_{\mathbb{Q}^{\lambda,\nu}} \left[\frac{1}{\gamma_T} \int_t^T (\nu_s - \phi_s)^2 ds \middle| \mathcal{F}_t \right] = \mathbb{E}_{\mathbb{Q}^{\lambda,\nu}} \left[\int_t^T \frac{(\nu_s - \phi_s)^2}{\gamma_s} ds \middle| \mathcal{F}_t \right],$$

which finishes the proof of equation (65).

The next step is to find the martingale measure that minimizes the term $H_t(\mathbb{Q}, T)$ and show that it belongs in the family of measures with $\nu \in \mathcal{P}_T^H$. Note that item (i) and an application of Hölder's inequality guarantees that

$$\exists \tilde{p} > 1 \quad \text{such that} \quad \mathbb{E} \left[\exp \left(\frac{\tilde{p}}{2} \int_0^T (\phi_u^2 + \lambda_u^2) du \right) \right] < \infty, \tag{66}$$

which implies that $\phi \in \mathcal{P}_T^\lambda$. In the view of representation (65), this means that $\phi = \underset{\nu \in \mathcal{P}_T^\lambda}{\operatorname{argmin}} H_t(\mathbb{Q}^{\lambda,\nu}, T)$.

Hence, this minimization can be restricted to the set \mathcal{P}_T^H if the measure $\mathbb{Q}^{\lambda,\phi}$ has finite entropy with respect to \mathbb{P} . The latter however is guaranteed by (66) (see also Remark 1.2 in [22]). Therefore,

$$H_t^{(0)}(T) = \operatorname{essinf}_{\nu \in \mathcal{P}_T^H} H_t(\mathbb{Q}^{\lambda,\nu}, T) = h \left(\frac{1}{\gamma_t} \right) - \frac{A_t}{\gamma_t}.$$

The final step of the proof is to show that the essential infimum in (32) is attained by the measure $\mathbb{Q}^{\nu^*, \lambda}$ and that $\nu^* \in \mathcal{P}_T^H$, where we recall that $\nu_t^* = \phi_t + \zeta_t \gamma_t$. From (64), we have that for every $\nu \in \mathcal{P}_T^\lambda$ and for all $t \in [0, T]$

$$v_t(C) \leq \mathbb{E}_{\mathbb{Q}^{\lambda, \nu}}[C | \mathcal{F}_t] + H_t(\mathbb{Q}^{\lambda, \nu}, T) - H_t^{(0)}(T).$$

Proposition 2.1 guarantees that there is a triple $(v_t(C), \zeta_t, \theta_t)_{t \in [0, T]}$ that solves the BSDE (23). Thus, for every $\nu \in \mathcal{P}_T^\lambda$

$$v_t(C) = \mathbb{E}_{\mathbb{Q}^{\lambda, \nu}} \left[C + \frac{1}{2} \int_t^T (2\zeta_u(\nu_u - \phi_u) - \gamma_u \zeta_u^2) du - \int_t^T \zeta_u dW_u^{2, \nu} - \int_t^T \delta_u dW_u^{1, \lambda} \middle| \mathcal{F}_t \right]. \quad (67)$$

Note also that for process ν^* , it holds that $2\zeta(\nu^* - \phi) - \gamma\zeta^2 = \frac{(\nu^* - \phi)^2}{\gamma}$. From Lemma 2.1 it follows that $\left(\int_0^t \zeta_u dW_u^{2, \phi} \right)_{t \in [0, T]}$ is a $BMO(\mathbb{Q}^{\lambda, \phi})$ -martingale and by Theorem 3.1 of [22] and the Hölder's inequality we get that $\nu^* \in \mathcal{P}_T^\lambda$. It is left to show that this minimizer belongs in \mathcal{P}_T^H too. For this, we first show that $\left(\int_0^t \zeta_u dW_u^{2, \nu^*} \right)_{t \in [0, T]}$ and $\left(\int_0^t \delta_u dW_u^{1, \lambda} \right)_{t \in [0, T]}$ are $BMO(\mathbb{Q}^{\lambda, \nu^*})$ -martingales. Following the lines of the proof of Lemma 2.1, we apply Itô's formula for the process $(e^{\beta v_t(C)})_{t \in [0, T]}$ (where $\beta \in \mathbb{R}_+$ and $(v_t(C))_{t \in [0, T]}$ is the solution of (23)) to get that for any stopping time $\tau \in [0, T]$

$$e^{\beta v_\tau(C)} - e^{\beta v_\tau(C)} = \int_\tau^T e^{\beta v_t(C)} \left(\frac{\zeta_t^2}{2} \beta(\beta - \gamma_t) + \frac{\beta^2}{2} \delta_t^2 \right) dt + \int_\tau^T e^{\beta v_t(C)} \beta \zeta_t dW_t^{2, \nu^*} + \int_\tau^T e^{\beta v_t(C)} \beta \delta_t dW_t^{1, \lambda}.$$

Note that there is a sequence of stopping times $\tau_n \nearrow T$ such that $\int_0^{t \wedge \tau_n} \beta e^{\beta v_s(C)} \zeta_s dW_s^{2, \nu^*}$ and $\int_0^{t \wedge \tau_n} \beta e^{\beta v_s(C)} \delta_s dW_s^{1, \lambda}$ are true $\mathbb{Q}^{\lambda, \nu^*}$ -martingales. Boundedness of the claim C implies that for every $n \in \mathbb{N}$

$$2e^{\beta \|C\|_{L^\infty}} \geq \mathbb{E}_{\mathbb{Q}^{\lambda, \nu^*}} \left[\int_{t \wedge \tau_n}^T e^{\beta v_t(C)} \frac{\zeta_t^2}{2} \beta(\beta - \gamma_t) dt \middle| \mathcal{F}_{\tau \wedge \tau_n} \right] + \mathbb{E}_{\mathbb{Q}^{\lambda, \nu^*}} \left[\int_{t \wedge \tau_n}^T e^{\beta v_t(C)} \frac{\beta^2}{2} \delta_t^2 dt \middle| \mathcal{F}_{\tau \wedge \tau_n} \right].$$

Hence for $\beta > K_\gamma$,

$$2e^{\beta \|C\|_{L^\infty}} \geq e^{-\beta \|C\|_{L^\infty}} \frac{\beta(\beta - K_\gamma)}{2} \mathbb{E}_{\mathbb{Q}^{\lambda, \nu^*}} \left[\int_{t \wedge \tau_n}^T \zeta_t^2 dt \middle| \mathcal{F}_{\tau \wedge \tau_n} \right] + e^{-\beta \|C\|_{L^\infty}} \frac{\beta^2}{2} \mathbb{E}_{\mathbb{Q}^{\lambda, \nu^*}} \left[\int_{t \wedge \tau_n}^T \delta_t^2 dt \middle| \mathcal{F}_{\tau \wedge \tau_n} \right].$$

Letting $n \rightarrow +\infty$ and taking supremum with respect to stopping times implies that $\left(\int_0^t \zeta_u dW_u^{2, \nu^*} \right)_{t \in [0, T]}$ and $\left(\int_0^t \delta_u dW_u^{1, \lambda} \right)_{t \in [0, T]}$ are $BMO(\mathbb{Q}^{\lambda, \nu^*})$ -martingales.

To show then that measure $\mathbb{Q}^{\lambda, \nu^*}$ has finite entropy, it remains to prove that $\mathbb{E}_{\mathbb{Q}^{\lambda, \nu^*}} \left[\int_0^T \phi_t^2 dt \right]$ and $\mathbb{E}_{\mathbb{Q}^{\lambda, \nu^*}} \left[\int_0^T \lambda_t^2 dt \right]$ are finite. For this, note that

$$\mathbb{E}_{\mathbb{Q}^{\lambda, \nu^*}} \left[\int_0^T \phi_t^2 dt \right] = \mathbb{E}_{\mathbb{Q}^{\lambda, \phi}} \left[e^{-\frac{1}{2} \int_0^T \gamma_t^2 \zeta_t^2 dt - \int_0^T \gamma_t \zeta_t dW_t^{2, \phi}} \int_0^T \phi_t^2 dt \right]. \quad (68)$$

Thanks to Lemma 2.1 and the uniform boundedness of $(\gamma_t)_{t \in [0, T]}$, $\left(\int_0^t \gamma_u \zeta_u dW_u^{2, \phi} \right)_{t \in [0, T]}$ is $BMO(\mathbb{Q}^{\lambda, \phi})$ -martingale. By Theorem 3.1 of [22], there exists a constant $\hat{p} > 1$ such that

$$\mathbb{E}_{\mathbb{Q}^{\lambda, \phi}} \left[e^{-\frac{\hat{p}}{2} \int_0^T \gamma_t^2 \zeta_t^2 dt - \hat{p} \int_0^T \gamma_t \zeta_t dW_t^{2, \phi}} \right] < \infty. \quad (69)$$

Another application of Hölder's inequality in (68) gives that its terms are finite if and only if $\mathbb{E}_{\mathbb{Q}^{\lambda, \phi}} \left[\left(\int_0^T \phi_t^2 dt \right)^{\hat{q}} \right]$ is finite, where $\frac{1}{\hat{p}} + \frac{1}{\hat{q}} = 1$. However, this is guaranteed by Assumption 1.1 and item (i).

The same arguments prove that $\mathbb{E}_{\mathbb{Q}^{\lambda, \nu^*}} \left[\int_0^T \lambda_t^2 dt \right] < \infty$.

□

APPENDIX B

Proposition B.1. *Impose Assumptions 1.1 and 3.1 and assume that (C_t, z_t, θ_t) and (C'_t, z'_t, θ'_t) are solutions of the following BSDE's*

$$\begin{aligned} dC_t &= f(t, z_t)dt + \theta_t dW_t^1 + z_t dW_t^2, & C_T &= C \\ dC'_t &= f'(t, z'_t)dt + \theta'_t dW_t^1 + z'_t dW_t^2, & C'_T &= C' \end{aligned}$$

where $\sup_{t \in [0, T]} |C_t|, \sup_{t \in [0, T]} |C'_t| \in \mathbb{L}^\infty(\mathcal{F}_T)$, with $C \leq C'$ a.s., (W_t^1, W_t^2) is a 2-dimensional Brownian motion, $f : \Omega \times [0, T] \times \mathcal{P} \rightarrow \mathbb{R}$ is given by

$$f(t, z_t) = \frac{G_t}{2} z_t^2 - z_t \frac{\gamma_t \psi_t + g_t \phi_t}{\gamma_t + g_t} + \frac{(\psi_t - \phi_t)^2}{2} \left(G_t - \frac{1}{\gamma_t + g_t} \right)$$

where $\sup_{t \in [0, T]} |G_t| \in \mathbb{L}^\infty(\mathcal{F}_T)$ and $f' : \Omega \times [0, T] \times \mathcal{P} \rightarrow \mathbb{R}$ is a smooth random function for which

$$f(t, z_t) \leq f'(t, z_t), \quad \text{a.s.} \tag{70}$$

Then, $C_t \leq C'_t$, a.s. for every $t \in [0, T]$.

Proof.

$$\begin{aligned} C_t - C'_t - (C_0 - C'_0) &= \int_0^t (f(s, z'_s) - f'(s, z'_s)) ds + \int_0^t (z_t - z'_t) dW_s^2 \\ &+ \int_0^t (f(s, z_s) - f(s, z'_s)) ds + \int_0^t (\theta_t - \theta'_t) dW_s^1 \end{aligned}$$

We then observe that

$$\begin{aligned} f(t, z_t) - f(t, z'_t) &= \frac{G_t}{2} (z_t^2 - (z'_t)^2) - (z_t - z'_t) \frac{\gamma_t \psi_t + g_t \phi_t}{\gamma_t + g_t} \\ &= (z_t - z'_t) K_t \end{aligned}$$

where $K_t = \frac{G_t}{2} (z_t + z'_t) - \frac{\gamma_t \psi_t + g_t \phi_t}{\gamma_t + g_t}$. Now $K \in \mathcal{P}_T^\lambda$ by Lemma 2.1 and the uniform boundness of γ and g . Hence, $\mathbb{Q}^{\lambda, K}$ is a martingale measure and therefore $C_t - C'_t - (C_0 - C'_0) - \int_0^t (f(s, z'_s) - f'(s, z'_s)) ds$ is a true $\mathbb{Q}^{\lambda, K}$ -martingale (see also the proof of Theorem 2.1). Taking expectation under $\mathbb{Q}^{\lambda, K}$ and exploiting the assumed inequality (70) gives the intended inequality of the solutions. □

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