The Dissertation Committee for Michail Anthropelos
certifies that this is the approved version of the following dissertation:

Agents' Agreement and Partial Equilibrium Pricing in Incomplete Markets

Committee:

Gordan Žitković, Supervisor

Rafael de la Llave

Lorenzo Garlappi

Mihai Sirbu

Efstathios Tompaidis

Thaleia Zariphopoulou

Agents' Agreement and Partial Equilibrium Pricing in Incomplete Markets

by

Michail Anthropelos, B.S., M.S.

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Agents' Agreement and Partial Equilibrium Pricing in Incomplete Markets

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We consider two risk-averse financial agents who negotiate the price of an illiquid indivisible contingent claim in an incomplete semimartingale market environment. Under the assumption that the agents are exponential utility maximizers with non-traded random endowments, we provide necessary and sufficient conditions for the negotiation to be successful, i.e., for the trade to occur. We, also, study the asymptotic case where the size of the claim is small compared to the random endowments and give a full characterization in this case. We, then, study a partial-equilibrium problem for a bundle of divisible claims and establish its existence and uniqueness. A number of technical results on conditional indifference prices are provided. Finally, we generalize the notion of partial-equilibrium pricing in the case where the agents' risk preferences are modelled by convex capital requirements.

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Chapter 1

Introduction

In an ideal complete market, each contingent claim payoff can be perfectly replicated by dynamic trading in the market assets. Thus, any rational financial agent is indifferent between the (random) claim itself and its (deterministic) replication price (i.e., its unique arbitrage-free price). Abundant empirical evidence, however, shows that the real financial markets are far from complete; only a small portion of contingent claims payoffs can be replicated in the market to a satisfactory degree and some unavoidable risk is present. A non-specific abstract notion of rationality is no longer sufficient to single out a unique "fair price" of any contingent claim. Instead, any agent's valuation and hedging strategy for non-replicable claims should depend on her risk preferences and her current investment positions.

The fact that there is no single arbitrage-free price for any non-replicable claim can be exhibited in the over-the-counter transactions where, typically, two agents negotiate the price of a single, indivisible, non-replicable, contingent claim. The final outcome of such a negotiation eventually hinges upon three *idiosyncratic* factors; the agents' attitudes towards risk, their initial random endowments and their negotiation skills. This thesis focuses on the first two. In particular we ask the following question:

Under what conditions on the claim whose price is being negotiated, the liquid-market environment and the agents' risk attitudes will a mutually beneficial agreement be feasible?

In the fist part of this text (Chapters 2-4), whose main source is [4], we model agents' risk preferences by assuming that agents are *expected utility function maximizers* in the von Neumann-Morgenstern sense, with a common investment horizon $T > 0$. For simplicity and analytic tractability we suppose that both agents' utility functions are *exponential*, possibly with different risk-aversion parameters. In the second part (Chapter 5), we assume that agents' risk measurements, and hence their contingent claim valuations, are obtained using the general notion of a *convex risk measure* (also called *convex capital requirement*). Roughly speaking, a convex risk measure (defined in [36]; see also a short survey in the Appendix B), is a map from the set of possible investment payoffs to the real line, whose aim is to quantify the risk involved to any payoff.

Another important feature in our setting, which is not present in a major part of the past work on the subject, are the agents' *random endowments*. The agents are assumed to hold a portfolio -their initial risk exposure- and the risk assessment of any contingent claim will depend heavily upon its (co-)relation with this portfolio. Generally, we use the term random endowment to refer to the discounted accumulated wealth of all investments with maturity up to time T, undertaken by an agent at the initial time $t = 0$.

In addition to the illiquid random endowments, both agents have access to a liquid incomplete financial market modelled by a general *locally-bounded semimartingale*. As in the case of the random endowment, we assume that the prices of the market assets and all payoffs are already discounted; this means that we can freely compare values corresponding to different points in time. As a discount factor, we use a fixed traded asset, usually called the *numéraire security*.

Mathematical-finance literature abounds with information on the utility-maximization problem for a variety of utility concepts (see, for instance, [24], [54], [55], [58], [65], [66], [70], [77]). In locally-bounded semimartingale market models the problem of utility maximization and in particular the necessary and sufficient conditions for the existence of an optimal trading strategy for utility functions defined on positive real line have been established in [58] and further analyzed under the presence of random endowment in [24] and in [49]. The corresponding studies for utilities defined on the whole real line have been given in [70], and in [65] under random endowment. The special case of an exponential utility function in semimartingale markets has been extensively analyzed in [29], [53], [57] and [62].

Given the utility maximization problem, the notion of an *acceptable investment* is quite natural. An investment position is called acceptable if the addition of its payoff to the agent's random endowment results in higher maximized utility. This acceptance criterion induces the notion of *utility indifference pricing*. More precisely, the writer's (buyer's) indifference price of a contingent claim with payoff B is the minimum (maximum) amount of money v, such that $v - B$ $(B - v)$ is an acceptable investment position. Equivalently, the indifference price is the amount of money that makes an agent indifferent - in the sense of utility maximization - between selling (buying) the claim at price v and foregoing the transaction. This scenario induces a subjective pricing mechanism for each agent according to which she states her ask and bid prices for each contingent claim. The idea of indifference pricing under the utility maximization scheme was introduced in the mathematical finance literature in [47] and then extensively developed in a variety of directions. In mathematical terms, the problem of utility indifference pricing is similar to the study of utility maximization under random endowment, in the sense that one can always think of a claim payoff as (additional) random endowment (see among others [29], [47], [49], [57] and [66]).

If the presence of illiquid random endowment in agent's portfolio is assumed, we call the indifference price *conditional* (also called *relative* indifference price in [64] and [73]) to indicate the strong dependence of the price on the random endowment. In the exponential world, some of its properties can be obtained by a simple change of measure which, effectively, removes the conditionality. Other properties, however, cannot be dealt with in that manner. In Chapter 2, we establish certain properties of conditional indifference prices in a general semimartingale market setting. We show, for instance, a rather unexpected fact that conditional indifference prices (unlike their unconditional versions) do not have to be monotone in the risk-aversion parameter.

Under the conditions described above, the two agents meet at time 0 when one of the agents (the seller) offers a contingent claim with time-T payoff B to the other one (the buyer) in exchange for a lump-sum payment p at time $t = 0$. Our central question, posed above, can now be made more precise and split into two separate components:

1. Is there a number $p \in \mathbb{R}$ such that the exchange of the contingent claim B for a lump sum p is (strictly) acceptable for both agents?

2. If more than one such p exists, can we determine the exact outcome of the negotiation?

In the case when the answer to question 1. is positive, we say that the agents are *in agreement*. In Chapter 3, we give a fairly complete answer to question 1., in terms of random endowments, while we only touch upon the issues involved in question 2. In fact, it is not possible to give a definitive answer to this question without a precise model of the negotiation process (see, for instance, [12] for an overview on modelling background and [3] for a proposed scenario). A partial answer is possible, however, when the indivisibility assumption is dropped (see Chapter 4).

It is, perhaps, surprising that unless non-replicable random endowments are present, no contingent claims will lead to agreement, even for agents with very different risk-aversion coefficients. When the random endowments are indeed present, we give a necessary and sufficient condition for the set of mutually agreeable claims to be larger that the set of replicable ones. This characterization is closely related to the notion of *optimal risk sharing*, which was first studied in the context of insurance/reinsurance negotiation (see, for instance, [14], [17] and [26]) and in terms of the principal's problem (see, for instance, [48]) and recently developed in more general settings (see , [9], [10], [35], [37] and [52]).

An affirmative answer to the question 1. leads to the following problem: is there a criterion for an agreement about a *given* claim B? We propose two approaches: one through the notion of *residual risk* and the other based on asymptotic approximation of conditional indifference prices for small quantities.

The residual risk (introduced in [63]) of a random liability is defined as the difference between the liability's payoff and the terminal value of the optimal risk-monitoring strategy at maturity. We establish the following criterion, made precise in the body of the text (see Section 3.4): a claim is mutually agreeable if and only if it reduces the expected sum of the agents' residual risks under some equivalent martingale measure.

The other approach provides an explicit criterion in the asymptotic case when the size of the contingent claim B is small compared to the size of the agents' random endowments. It is possible to phrase the agreement problem in terms of a relationship between the buyer's and the seller's conditional indifference price for the claim, so it is not unusual that an asymptotic study of these quantities plays a major role. More precisely, we provide a rather general Taylor-type approximation of the conditional exponential indifference price for locally bounded semimartingales on continuous filtrations. These approximations are then used to give simple asymptotic criteria for agreeability, as well as the asymptotic size of the interval of mutually-agreeable prices. Since it is not possible to obtain closed-form representations of indifference prices in general market models, such asymptotic results can be very useful even beyond the agreement problem.

Asymptotic techniques are not new in utility maximization problems. Results on the second order approximation of the indifference price for the exponential and the power utility, in a market with Brownian motion dynamics, are given in [44] (see also [46]). These results are generalized in [59] for semimartingale markets and general utility functions defined on the positive real line. However, the arguments presented there do not cover the case of the exponential utility (e.g., the situation when the optimal wealth is not necessarily bounded from below by zero). For the exponential utility in a semimartingale market, the first derivative of the indifference price for a vector of claims is given in [51]. By imposing the assumption of continuity on the filtration, we generalize their result (as well as the asymptotic approximation in [72]) by providing a second order approximation of the price for a vector of claims.

In the case where the agents are allowed to negotiate not only the price of the claim, but also the number of units traded, the classic market clearing conditions can be used to compute these two quantities. More precisely, for a given vector of claims, we define the utility-based *demand function* for both agents. Roughly speaking, the demand function at a price p is the number of units of the given vector of claims that the agent is willing to buy at price p, where "willing" refers to the utility maximization criterion. Then we call a price p, *partial equilibrium price* if the sum of demand functions at p is zero. Furthermore, the (vector) quantity of the claims that is to be traded between the agents at equilibrium price p is called *partial equilibrium quantity.* In Chapter 4, we state the precise definition and we prove existence and uniqueness, as well as a formula of the partial equilibrium price.

The existence results of various types of competitive equilibria are a staple of quantitative economics literature, and have recently made their way into mathematical finance (see, among others, [18], [25], [34], [43] and [78]). The incomplete partial-equilibrium setting presented herein, however, new and not covered by any of the existing results. As we already mentioned above, it is only in the present setting that we can say something about question 2., i.e., about the realized price p of the offered contingent claim B.

In Chapter 5, we assume that the agents' attitudes towards risk are modelled by convex capital requirements which can be understood as a generalization of the acceptance criterion induced by the utility maximization. The notion of capital requirement or risk measure was introduced in mathematical finance literature in late nineties and since then it has captured a large part of the research activity in this field. For a short introduction to this field and a list of related references we refer the reader to Appendix B.

In our general semimartingale financial market, we fix a time horizon $T > 0$ and we endow each agent with an acceptance set, i.e., the set of all investment positions with maturity T that the agent is willing to undertake at time 0. These acceptance sets incorporate agents' risk preferences and satisfy certain rationality properties such as monotonicity and convexity. In the same fashion as in the case of indifference pricing, each agent's acceptance set induces a subjective pricing mechanism called convex risk measure. Having introduced these new concepts, we suppose the existence of $I \geq 2$ agents with common time horizon T, who wish to trade a vector of contingent claims with maturity at T. We then generalize the notion of *mutually agreeable claims* by considering a vector of claims instead of a single one and by including more than two agents. A number of properties of the set of agreeable claims are exhibited.

The notions of demand function and partial equilibrium pricing are similarly generalized in this setting: given a vector of claims, each agent's demand function maps price vectors to the units of claims that minimize the risk measure of the corresponding position. Then, the *partial equilibrium price* is the price vector at which the sum of agents' demand functions is zero and the *partial equilibrium allocation* is the corresponding allocation. We use the term "allocation" in this setting instead of "quantity" to emphasize the possibility of the participation of more than two agents. In Chapter 5, we impose the necessary assumptions which lead to an existence and uniqueness result and we discuss these findings.

The chapters are organized in the following way: In Chapter 2, we first describe the market model and introduce necessary notation and we state some properties of the conditional indifference prices, together with some new results on their unconditional versions. We also include the second order approximation of the conditional price. The notion of agreement is introduced and analyzed in Chapter 3, where its connection with the agents' residual risk and the price approximation is also discussed. The topics of Chapter 4 are the definition, the existence and the uniqueness of the partial equilibrium price-quantity for a vector of contingent claims. The generalized notions of agents' agreement and partial equilibrium price-allocation are introduced and developed in Chapter 5, where the corresponding result of existence and uniqueness is proven. In Appendix A, we state the definition and the main properties of the dynamic version of the conditional indifference prices. Separate sections are devoted to the residual risk process and the definition of the class of martingale, called BMO-martingales. Finally, in Appendix B, we present a short review of the theory of convex risk measures, where we focus on the definitions and results mentioned in Chapter 5.

To facilitate the exposition, we list the notation and the keywords mentioned throughout the text in the Index.

Chapter 2

Utility Conditional Indifference Prices

This chapter is dedicated to the description of the market model and the study of the conditional indifference price. Throughout the text, we adopt the general framework of a locally-bounded semimartingale market (see [31] for a detailed overview) and the exponential utility set up of [29] and [62].

In Section 2.1, we include the necessary notation, describe the utility maximization problem and recall some well-known related results. The naturally raised feature of the utility-based acceptance set is, then, introduced and a number of its properties are proven.

The main object of Section 2.2 is the conditional indifference price. As mentioned in the Introduction, the utility indifference price of a contingent claim B is the price that makes a utility maximizer indifferent between selling/buying the claim and omitting the transaction. The large majority of the existed literature on this field does not include the case where the agent has a random endowment in her portfolio. If a random endowment is present in agent's portfolio, the indifference price is called *conditional*. Results on this notion are presented in Section 2.2, where their differences with the corresponding results on the unconditional version are highlighted. For this analysis, we use an equivalence relation between payoffs; two random payoffs are equivalent (with respect to risk) if their difference is a replicable claim.

The part of the unhedgeable risk left to agent's portfolio after undertaking any non-replicable investment is usually called *residual risk*. In Section 2.3, we state its definition and we provide straightforward results in its writer's and buyer's indifference price.

Finally, in Section 2.4, under the assumption of continuous filtrations, we compute the second derivative of the conditional indifference price in the units of the claims. Then, we use this result to provide a rather general second order approximation of the price of a vector of contingent claims.

2.1 Market Model and Preliminary Results

In this section, we introduce the model of investment and the agents' characteristics, that is, their utility functions, random endowments, admissible strategies and indifference prices. A number of notations used in the following chapters is, also, introduced.

2.1.1 The traded assets and the agents

The financial market model is based on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$, $T > 0$, which satisfies the "usual conditions" of right-continuity and augmentation by $\mathbb{P}-$negligible sets. There are $d+1$ traded assets ($d \in \mathbb{N}$), whose discounted price processes are modelled by an $\mathbb{R}^{d+1}-$valued locally bounded semimartingale process $(S_t^{(0)}; \mathbf{S}_t)_{t\in[0,T]} =$

$(S_t^{(0)}; S_t^{(1)}, \dots, S_t^{(d)})_{t\in[0,T]}$. The first asset, $S_t^{(0)}$, plays the role of a numéraire security or a discount factor. Operationally, we simply set $S_t^{(0)} \equiv 1$, for all $t \in [0,T]$, $\mathbb{P}-$a.s..

We place ourselves in the von Neumann-Morgenstern framework (see [75]) and we assume that each market participant evaluates the risk of an uncertain position X at time T according to the expected utility $\mathbb{E}^{\mathbb{P}}[U(X+\mathcal{E})]$, where U is a utility function, $\mathcal{E}$ is the *random endowment* (accumulated illiquid wealth) and $\mathbb{P}$ is the subjective probability measure. For technical reasons, we restrict our attention to $\mathcal{E} \in \mathbb{L}^\infty$ (where $\mathbb{L}^\infty$ stands for the set $\mathbb{L}^\infty(\Omega, \mathcal{F}, \mathbb{P})$) and the class of exponential utilities

$$U(x) = -\exp(-\gamma x),\ x \in \mathbb{R},$$

where the constant $\gamma \in (0,\infty)$ is the (absolute) risk aversion coefficient.

An agent invests in the market by choosing a portfolio strategy $\boldsymbol{\vartheta}$ in an admissibility class $\boldsymbol{\Theta}$, to be specified below. The resulting *gains process* $(X_t^{\boldsymbol{\vartheta}})_{t\in[0,T]}$ is the stochastic integral $X_t^{\boldsymbol{\vartheta}} = (\boldsymbol{\vartheta}\cdot\mathbf{S})_t = \int_0^t \boldsymbol{\vartheta}_u\, d\mathbf{S}_u$. For the choice of set of admissible strategies, we follow the setup introduced in [62] (see also [29]).

Before we give a precise description of the aforementioned set $\boldsymbol{\Theta}$, we need to introduce several concepts related to the no-arbitrage requirement. We start with the sets $\mathcal{M}_a$ and $\mathcal{M}_e$ of absolutely continuous and equivalent *local martingale measures*, i.e.,

$$\mathcal{M}_a = \{\mathbb{Q} \ll \mathbb{P} : \mathbf{S} \text{ is a local martingale under } \mathbb{Q}\}$$

and

$$\mathcal{M}_e = \{\mathbb{Q} \approx \mathbb{P} \,:\, \mathbf{S} \text{ is a local martingale under } \mathbb{Q}\}\,.$$

For a probability measure $\mathbb{Q}$ on $(\Omega, \mathcal{F})$, we define

$$\mathcal{H}(\mathbb{Q}|\mathbb{P}) = \begin{cases} \mathbb{E}^{\mathbb{P}}\left[\frac{d\mathbb{Q}}{d\mathbb{P}} \ln\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)\right] & \mathbb{Q} \ll \mathbb{P}, \\ +\infty & \text{otherwise.} \end{cases}$$

The (extended) positive number $\mathcal{H}(\mathbb{Q}|\mathbb{P})$ is called the *relative entropy* of the probability measure $\mathbb{Q}$ with respect to probability measure $\mathbb{P}$. It is well-known that for every absolutely continuous probability measure $\mathbb{Q}$, $\mathcal{H}(\mathbb{Q}|\mathbb{P}) \geq 0$ and $\mathcal{H}(\mathbb{Q}|\mathbb{P}) = 0$ if and only if $\mathbb{P} = \mathbb{Q}$. Intuitively, the function $\mathcal{H}(\cdot|\mathbb{P})$ can be used to measure how "close" to $\mathbb{P}$ a probability measure $\mathbb{Q}$ is. For further details on the notion of relative entropy we refer the interested reader to [42] or [39].

We also set

$$\mathcal{M}_{e,f} = \{\mathbb{Q} \in \mathcal{M}_e \,:\, \mathcal{H}(\mathbb{Q}|\mathbb{P}) < \infty\}$$

and enforce the following assumption.

Assumption 2.1.1. $\mathcal{M}_{e,f} \neq \emptyset$.

Assumption 2.1.1 trivially implies that $\mathcal{M}_e \neq \emptyset$ which, in turn, guarantees that no arbitrage opportunities exist in the market. In fact, a stronger statement of "no free lunch with vanishing risk" will also hold, as proved in [30], Corollary 1.2. The additional requirement in Assumption 2.1.1 is common in the literature and ensures that the choice of the exponential function for the utility leads to a well-defined behavior for utility-maximizing agents (see, among others, [39] and [29]).

We remind the reader that $\mathbb{L}^0(\mathcal{F})$ denotes the set of all ($\mathbb{P}-$a.s. equivalent classes of) $\mathcal{F}-$measurable random variables. In the sequel, the expectation operator under a probability measure $\mathbb{Q}$ is denoted by $\mathbb{E}^{\mathbb{Q}}[\cdot]$, where the superscript $\mathbb{Q}$ is omitted in the case of the (subjective) measure $\mathbb{P}$. Also, for a random variable $B \in \mathbb{L}^0(\mathcal{F})$ with $\mathbb{E}[\exp(B)] < \infty$, we define the probability measure $\mathbb{P}_B$, through its Radon-Nikodym derivative with respect to $\mathbb{P}$ by

$$\frac{d\mathbb{P}_B}{d\mathbb{P}} = \frac{e^B}{\mathbb{E}[e^B]}.$$

Furthermore, $\mathbb{Q}^{(0)}$ denotes the probability measure in $\mathcal{M}_a$ with the minimal relative entropy with respect to $\mathbb{P}$, i.e., the probability measure for which $\mathcal{H}(\mathbb{Q}^{(0)}|\mathbb{P}) \leq \mathcal{H}(\mathbb{Q}|\mathbb{P})$, for all $\mathbb{Q} \in \mathcal{M}_a$. It is a consequence of Assumption 2.1.1 that the probability measure $\mathbb{Q}^{(0)}$ exists, is unique and belongs to $\mathcal{M}_{e,f}$ (see [39], Theorem 2.2). Similarly, for every B such that $\mathbb{E}[\exp(B)] < \infty$, there exists a unique probability measure $\mathbb{Q}^{(B)} \in \mathcal{M}_a$ such that $\mathcal{H}(\mathbb{Q}^{(B)}|\mathbb{P}_B) \leq \mathcal{H}(\mathbb{Q}|\mathbb{P}_B)$, for all $\mathbb{Q} \in \mathcal{M}_a$ (see [29], page 103).

Having introduced the required families of probability measures, we turn back to definition of the class $\mathbf{\Theta}$ of *admissible strategies*

$$\mathbf{\Theta} = \{\boldsymbol{\vartheta} \in L\,(\mathbf{S}) : (\boldsymbol{\vartheta} \cdot \mathbf{S}) \text{ is a } \mathbb{Q} - \text{martingale}, \forall\, \mathbb{Q} \in \mathcal{M}_{e,f}\} \tag{2.1.1}$$

where $L(\mathbf{S})$ is the set of all predictable $(d+1)-$dimensional $\mathbf{S}-$integrable processes on $[0, T]$. More information about the set $\mathbf{\Theta}$ of admissible strategies is given in [62] (see also remarks on the set $\mathbf{\Theta}_2$ in [29], page 104).

A random variable $B \in \mathbb{L}^0(\mathcal{F})$ is said to be *replicable* if there exists a constant c and an admissible strategy $\boldsymbol{\vartheta} \in \boldsymbol{\Theta}$ such that $B = c + (\boldsymbol{\vartheta} \cdot \mathbf{S})_T$, $\mathbb{P}-$a.s.; the set of replicable random variables will be denoted by $\mathcal{R}^0$. More generally, we introduce the following equivalence relation between random variables in $\mathbb{L}^0(\mathcal{F})$.

Definition 2.1.2. We call two random variables $B, C \in \mathbb{L}^0(\mathcal{F})$ *risk equivalent* or *equal up to replicability* and write $B \sim C$, if the difference $B - C$ is replicable.

It is clear that the relation $\sim$ is an equivalence relation on $\mathbb{L}^0(\mathcal{F})$ (since $\boldsymbol{\Theta}$ is a vector space). We note that the zero equivalence class coincides with the set $\mathcal{R}^0$ of the replicable random variables. For future reference, we let $\mathcal{R}^\infty = \mathcal{R}^0 \cap \mathbb{L}^\infty$ denote the set of all (essentially) bounded replicable random variables.

2.1.2 Utility maximization and indifference pricing

Given their risk profiles, financial agents trade in the financial market with the goal of maximizing their expected utility. More precisely, an agent with initial wealth $x \in \mathbb{R}$, risk-aversion coefficient γ and random endowment $\mathcal{E} \in \mathbb{L}^\infty$ will choose a portfolio process $\boldsymbol{\vartheta} \in \boldsymbol{\Theta}$ so as to maximize the expected utility $\mathbb{E}[-\exp(-\gamma(x + (\boldsymbol{\vartheta} \cdot \mathbf{S})_T + \mathcal{E}))]$. The value function $u_\gamma(x|\mathcal{E})$ of the corresponding optimization problem is given by

$$u_\gamma(x|\mathcal{E}) = \sup_{\boldsymbol{\vartheta} \in \boldsymbol{\Theta}} \mathbb{E}\Big[- \exp\big(-\gamma\big(x + (\boldsymbol{\vartheta} \cdot S)_T + \mathcal{E}\big)\big) \Big], \ x \in \mathbb{R}. \tag{2.1.2}$$

Overloading the notation slightly, for any random variable $B \in \mathbb{L}^\infty$ (interpreted as a contingent payoff with maturity T) we define the *indirect utility* of B by $u_\gamma(B|\mathcal{E}) = u_\gamma(0|\mathcal{E} + B)$, i.e.,

$$u_\gamma(B|\mathcal{E}) = \sup_{\boldsymbol{\vartheta}\in\boldsymbol{\Theta}} \mathbb{E}\Big[- \exp(-\gamma \left((\boldsymbol{\vartheta} \cdot S)_T + \mathcal{E} + B \right)) \Big]. \tag{2.1.3}$$

Remark 2.1.3. It was proved in [29], Theorem 2.2 (see also [53], Theorem 2.1 and Theorem 2.2.4 below) that the maximums in (2.1.2) and (2.1.3) are attained.

The indirect utility $u_\gamma(\cdot|\mathcal{E})$ induces an *acceptance set*, denoted by $\mathcal{A}_\gamma(\mathcal{E})$. More precisely, we define

$$\mathcal{A}_\gamma(\mathcal{E}) = \left\{ B \in \mathbb{L}^\infty \,:\, u_\gamma(0|\mathcal{E}) \leq u_\gamma(B|\mathcal{E}) \right\}. \tag{2.1.4}$$

Intuitively, $\mathcal{A}_\gamma(\mathcal{E})$ contains all the claims with maturity up to T that the agent with random endowment $\mathcal{E}$ and risk aversion γ accepts to undertake at time $t = 0$. Similarly, we define the set $\mathcal{A}^\circ_\gamma(\mathcal{E}) = \{B \in \mathbb{L}^\infty \,:\, u_\gamma(0|\mathcal{E}) < u_\gamma(B|\mathcal{E})\}$, which is called the agent's *strict acceptance set*.

It is a consequence of the choice of the exponential utility function that the addition of any constant initial wealth $x \in \mathbb{R}$ to the random endowment $\mathcal{E}$ does not influence the acceptance sets $\mathcal{A}_\gamma(\mathcal{E})$ and $\mathcal{A}^\circ_\gamma(\mathcal{E})$. More generally, we have the following proposition. We remind the reader that a set $\mathcal{A}$ is called *monotone* if $B \geq C$, $\mathbb{P}-$a.s. and $C \in \mathcal{A}$ imply $B \in \mathcal{A}$.

Proposition 2.1.4. *For every $\mathcal{E} \in \mathbb{L}^\infty$ and $\gamma \in (0, \infty)$, the sets $\mathcal{A}_\gamma(\mathcal{E})$ and $\mathcal{A}^\circ_\gamma(\mathcal{E})$ are convex, monotone and for every $\mathcal{E}_1, \mathcal{E}_2 \in \mathbb{L}^\infty$*

$$\mathcal{E}_1 \sim \mathcal{E}_2 \textit{ implies } \mathcal{A}_\gamma(\mathcal{E}_1) = \mathcal{A}_\gamma(\mathcal{E}_2) \textit{ and } \mathcal{A}^\circ_\gamma(\mathcal{E}_1) = \mathcal{A}^\circ_\gamma(\mathcal{E}_2) \tag{2.1.5}$$

Proof. We prove the arguments for $\mathcal{A}_\gamma(\mathcal{E})$ since the proof is the same for $\mathcal{A}^\circ_\gamma(\mathcal{E})$.

Consider $B_1, B_2 \in \mathcal{A}_\gamma(\mathcal{E})$ and $\lambda \in [0,1]$. If $\boldsymbol{\vartheta}_1, \boldsymbol{\vartheta}_2 \in \boldsymbol{\Theta}$ are the maximizers of (2.1.3) for $B = B_1$ and $B = B_2$ respectively (see Theorem 2.2.4 for the existence of $\boldsymbol{\vartheta}_1$ and $\boldsymbol{\vartheta}_2$), the concavity property of the utility function yields that

$$\begin{aligned} u_\gamma(\lambda B_1 + (1-\lambda)B_2|\mathcal{E}) \geq \mathbb{E}[U(\lambda B_1 + (1-\lambda)B_2 \quad &+ \quad \lambda\boldsymbol{\vartheta}_1 + (1-\lambda)\boldsymbol{\vartheta}_2 + \mathcal{E})] \\ \geq \lambda u_\gamma(B_1|\mathcal{E}) + (1-\lambda)u_\gamma(B_2|\mathcal{E}) \quad &\geq \quad u_\gamma(0|\mathcal{E}) \end{aligned}$$

which implies the convexity of $\mathcal{A}_\gamma(\mathcal{E})$.

Let $C \in \mathbb{L}^\infty$ and suppose that $B_1 \leq C$, $\mathbb{P}-$a.s. Then by the monotonicity of the utility function we have

$$u_\gamma(C|\mathcal{E}) \geq \mathbb{E}[U(C + \boldsymbol{\vartheta}_1 + \mathcal{E})] \geq u_\gamma(B_1|\mathcal{E}) \geq u_\gamma(0|\mathcal{E})$$

i.e., $C \in \mathcal{A}_\gamma(\mathcal{E})$.

Finally, $\mathcal{E}_1 \sim \mathcal{E}_2$ means that there exist $k \in \mathbb{R}$ and $\hat{\boldsymbol{\vartheta}} \in \boldsymbol{\Theta}$ such that $\mathcal{E}_1 - \mathcal{E}_2 = k + (\hat{\boldsymbol{\vartheta}} \cdot \mathbf{S})_T$. Hence,

$$\begin{aligned} u_\gamma(B|\mathcal{E}_1) &= \sup_{\boldsymbol{\vartheta}\in\boldsymbol{\Theta}} \mathbb{E}\Big[-\exp\left(-\gamma\left((\boldsymbol{\vartheta}\cdot\mathbf{S})_T + \mathcal{E}_2 + k + \left(\hat{\boldsymbol{\vartheta}}\cdot\mathbf{S}\right)_T + B\right)\right)\Big] \\ &= e^{-\gamma k}\sup_{\boldsymbol{\vartheta}\in\boldsymbol{\Theta}} \mathbb{E}\Big[-\exp\left(-\gamma\left((\boldsymbol{\vartheta}\cdot\mathbf{S})_T + \mathcal{E}_2 + B\right)\right)\Big] \\ &= e^{-\gamma k} u_\gamma(B|\mathcal{E}_2), \end{aligned}$$

for every $B \in \mathbb{L}^\infty$. This implies that $B \in \mathcal{A}_\gamma(\mathcal{E}_1)$ if and only if $e^{-\gamma k}u_\gamma(B|\mathcal{E}_2) \geq e^{-\gamma k}u_\gamma(0|\mathcal{E}_2)$, i.e., if and only if $B \in \mathcal{A}_\gamma(\mathcal{E}_2)$. □

The acceptance set $\mathcal{A}_\gamma(\mathcal{E})$ can be used to introduce the notion of a *conditional indifference price*. The *conditional writer's indifference price*, $\nu^{(w)}(B;\gamma|\mathcal{E})$ of the contingent claim $B \in \mathbb{L}^\infty$ is defined by

$$\nu^{(w)}(B;\gamma|\mathcal{E}) = \inf\left\{p \in \mathbb{R} \,:\, p - B \in \mathcal{A}_\gamma(\mathcal{E})\right\}, \tag{2.1.6}$$

i.e., $\nu^{(w)}(B;\gamma|\mathcal{E})$ is the minimum amount of money at which the agent with risk aversion coefficient γ and random endowment $\mathcal{E}$ is willing to sell a claim with payoff B.

It follows directly from (2.1.6) that $\nu^{(w)}(B;\gamma|\mathcal{E})$ satisfies the equation

$$u_\gamma(\nu^{(w)}(B;\gamma|\mathcal{E}) - B|\mathcal{E}) = u_\gamma(0|\mathcal{E}).$$

In analogy, the *conditional buyer's indifference price* $\nu^{(b)}(B;\gamma|\mathcal{E})$ is defined by

$$\nu^{(b)}(B;\gamma|\mathcal{E}) = \sup\left\{p \in \mathbb{R} \,:\, B - p \in \mathcal{A}_\gamma(\mathcal{E})\right\}, \tag{2.1.7}$$

i.e., $\nu^{(b)}(B;\gamma|\mathcal{E})$ is the maximum amount of money at which the agent, with risk aversion coefficient γ and random endowment $\mathcal{E}$, is willing to buy a contingent claim with payoff B. Similarly to the writer's case, $\nu^{(b)}(B;\gamma|\mathcal{E})$ satisfies the equation

$$u_\gamma(B - \nu^{(b)}(B;\gamma|\mathcal{E})|\mathcal{E}) = u_\gamma(0|\mathcal{E}).$$

In the special case where $\mathcal{E} \sim 0$, the corresponding prices are called *unconditional indifference prices* (or, simply, indifference prices) and denoted by $\nu^{(w)}\,(B;\gamma)$ and $\nu^{(b)}\,(B;\gamma)$.

The notion of the indifference price has been studied by many authors (see, among others, [44], [47], [63] and [69]). The definition of the conditional indifference price under exponential utility was given in [13] for general semimartingale model, in [64] for a binomial case model and in [73] for a diffusion model (where the price is called *relative* indifference price). A discussion of the conditional indifference price under general utility functions is given in [66].

2.2 Results on the Conditional Indifference Price

The subject of this section is the conditional indifference price and its properties. The results stated below are not only very useful for our analysis on the mutually agreeability and the partial equilibrium, but they may also be seen as interesting in their own right since they describe some of the aspects of indifference valuation under the presence of random endowment. Some new results about the unconditional indifference price (see Lemma 2.2.9, Propositions 2.2.12, 2.2.13 and 2.2.16), as well as several generalizations of existing results in the case of the conditional price (see Theorem 2.2.4, Propositions 2.2.7 and 2.2.15) are exhibited.

We start with some basic properties of the indifference prices whose proof is straightforward from the Definitions 2.1.6 and 2.1.7.

Proposition 2.2.1. *For every* $B, \mathcal{E}, \mathcal{E}' \in \mathbb{L}^\infty$ *and* $\gamma \in (0, \infty)$*, it holds that*

1. $\nu^{(b)}(B; \gamma|\mathcal{E}) = -\nu^{(w)}(-B; \gamma|\mathcal{E})$*, for* $B, \mathcal{E} \in \mathbb{L}^\infty$*.*

2. *When* $\mathcal{E} \in \mathcal{R}^\infty$ *(in particular, when* $\mathcal{E}$ *is constant)* $\nu^{(w)}(\cdot;\gamma|\mathcal{E})$ *and* $\nu^{(b)}(\cdot;\gamma|\mathcal{E})$ *coincide with their unconditional versions* $\nu^{(w)}(\cdot;\gamma)$ *and* $\nu^{(b)}(\cdot;\gamma)$*.*

3. *More generally, we have* $\nu^{(w)}(\cdot;\gamma|\mathcal{E}) = \nu^{(w)}(\cdot;\gamma|\mathcal{E}')$ *and* $\nu^{(b)}(\cdot;\gamma|\mathcal{E}) = \nu^{(b)}(\cdot;\gamma|\mathcal{E}')$ *as soon as* $\mathcal{E} \sim \mathcal{E}'$*.*

In the case where $\mathcal{E} \nsim 0$, there are two ways to write the conditional indifference price in terms of an unconditional one. This is shown in the following proposition.

Proposition 2.2.2. *For every* $B, \mathcal{E} \in \mathbb{L}^\infty$ *and* $\gamma \in (0,\infty)$*, the following statements are true*

1. *The conditional indifference prices* $\nu^{(w)}(B;\gamma|\mathcal{E})$, $\nu^{(b)}(B;\gamma|\mathcal{E})$ *can be written as the unconditional indifference prices of* B*, computed under the probability measure* $\mathbb{P}_{-\gamma\mathcal{E}}$*.*

2. *Moreover,* $\nu^{(w)}(B;\gamma|\mathcal{E}) = \nu^{(w)}(B-\mathcal{E};\gamma) - \nu^{(w)}(-\mathcal{E};\gamma)$ *and* $\nu^{(b)}(B;\gamma|\mathcal{E}) = \nu^{(b)}(B+\mathcal{E};\gamma) - \nu^{(b)}(\mathcal{E};\gamma)$*.*

Proof. It follows from the definition of the indifference price that $\nu^{(w)}(B;\gamma|\mathcal{E})$ solves the equation

$$e^{-\gamma\nu^{(w)}(B;\gamma|\mathcal{E})}u_\gamma(-B|\mathcal{E}) = u_\gamma(0|\mathcal{E}). \tag{2.2.1}$$

We also have that, for every $B \in \mathbb{L}^\infty$,

$$\begin{aligned} u_\gamma(-B|\mathcal{E}) &= \sup_{\boldsymbol{\vartheta}\in\boldsymbol{\Theta}} \mathbb{E}\Big[-\exp(-\gamma\,((\boldsymbol{\vartheta}\cdot\mathbf{S})_T + \mathcal{E} - B))\Big] \\ &= \sup_{\boldsymbol{\vartheta}\in\boldsymbol{\Theta}} \mathbb{E}\Big[\exp(-\gamma\mathcal{E})\,(-\exp(-\gamma\,((\boldsymbol{\vartheta}\cdot\mathbf{S})_T - B)))\Big] \\ &= \mathbb{E}[\exp(-\gamma\mathcal{E})] \sup_{\boldsymbol{\vartheta}\in\boldsymbol{\Theta}} \mathbb{E}^{\mathbb{P}_{-\gamma\varepsilon}}\Big[(-\exp(-\gamma\,((\boldsymbol{\vartheta}\cdot\mathbf{S})_T - B)))\Big]. \end{aligned}$$

Taking (2.2.1) into account, we get that the indifference price solves

$$\begin{aligned} e^{-\gamma\nu^{(w)}(B;\gamma|\mathcal{E})} \sup_{\boldsymbol{\vartheta}\in\boldsymbol{\Theta}} \mathbb{E}^{\mathbb{P}_{-\gamma\varepsilon}}\Big[(-\exp(-\gamma\,((\boldsymbol{\vartheta}\cdot\mathbf{S})_T - B)))\Big] \\ = \sup_{\boldsymbol{\vartheta}\in\boldsymbol{\Theta}} \mathbb{E}^{\mathbb{P}_{-\gamma\varepsilon}}\Big[(-\exp(-\gamma\,((\boldsymbol{\vartheta}\cdot\mathbf{S})_T)))\Big], \end{aligned}$$

which completes the proof of part 1 (the case of the buyer's price is similar).

For part 2, we refer the reader to [13], page 21. □

Below, we state an example of a market model that will be used in the sequel.

Example 2.2.3. This example is set in an incomplete financial market similar to the one considered in [63] (see, also, [44]). The market consists of one risky asset $S = (S_t)_{t\in[0,T]}$ with dynamics

$$dS_t = S_t\big(\mu(t)\,dt + \sigma(t)\,dW_t^{(1)}\big),$$

and an additional (non-traded) factor $Y = (Y_t)_{t\in[0,T]}$, which evolves is a unique strong solution of

$$dY_t = b(Y_t,t)\,dt + a(Y_t,t)\left(\rho dW_t^{(1)} + \rho' dW_t^{(2)}\right),$$

where $W^{(1)} = (W_t^{(1)})_{t\in[0,T]}$ and $W^{(2)} = (W_t^{(2)})_{t\in[0,T]}$ are two standard independent Brownian Motions, defined on a probability space $(\Omega, \mathbb{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$. The constant $\rho \in (-1, 1)$ is the correlation coefficient and $\rho' = \sqrt{1-\rho^2}$. We assume that the deterministic functions μ, $\sigma : [0, T] \to \mathbb{R}$ are uniformly bounded ($\sigma > 0$).

By Theorem 3 in [63] (in [63] μ and σ are constants, but the arguments carry over to this setting too), we have that

$$\nu^{(w)}(B;\gamma) = \frac{1}{\gamma(1-\rho^2)} \ln\left\{\mathbb{E}^{\mathbb{Q}^{(0)}}\left[e^{\gamma\left(1-\rho^2\right)B}\right]\right\}, \tag{2.2.2}$$

for any payoff $B \in \mathbb{L}^\infty$, such that $B = g(Y_T)$ for some bounded Borel function $g : \mathbb{R} \to \mathbb{R}$, where the Radon-Nikodym derivative of $\mathbb{Q}^{(0)}$ is given by

$$\frac{d\mathbb{Q}^{(0)}}{d\mathbb{P}} = \exp\left(-\int_0^T \frac{1}{2}\lambda^2(t)dt - \int_0^T \lambda(t)dW_t^{(1)}\right),$$

and $\lambda(t) = \frac{\mu(t)}{\sigma(t)}$ is the *Sharpe ratio* of S.

It is a direct consequence of (2.2.2) and of part 2 of Proposition 2.2.2 that, for every random endowment $\mathcal{E} \in \mathbb{L}^\infty$ such that $\mathcal{E} = q(Y_T)$ for some bounded Borel function $q : \mathbb{R} \to \mathbb{R}$, we get that

$$\nu^{(w)}(B;\gamma|\mathcal{E}) = \frac{1}{\gamma(1-\rho^2)} \ln\left\{\frac{\mathbb{E}^{\mathbb{Q}^{(0)}}\left[e^{\gamma\left(1-\rho^2\right)(B-\mathcal{E})}\right]}{\mathbb{E}^{\mathbb{Q}^{(0)}}\left[e^{-\gamma(1-\rho^2)\mathcal{E}}\right]}\right\}. \tag{2.2.3}$$

Given Proposition 2.2.2, we can generalize some well-known results on unconditional indifference prices to include the case of random endowment. Indeed, first we give the so-called *robust representation* of the indifference price, stated in the following Theorem, which is a variation of Theorem 2.2 in [29] and Theorem 2.1 in [53] (we use the notation $\mathbb{L}^1$ for the set $\mathbb{L}^1(\Omega, \mathcal{F}, \mathbb{P})$).

Theorem 2.2.4. *([29], [53])*

For $B \in \mathbb{L}^\infty$, we have

$$\nu^{(w)}(B;\gamma|\mathcal{E}) = \sup_{\mathbb{Q}\in\mathcal{M}_a} \left\{ \mathbb{E}_{\mathbb{Q}}[B] - \frac{1}{\gamma} h_{-\gamma\mathcal{E}}(\mathbb{Q}) \right\}, \tag{2.2.4}$$

where, for $C \in \mathbb{L}^\infty$, we define the map $h_C : \mathbb{L}^1 \mapsto [0, +\infty]$ as

$$h_C(\mathbb{Q}) = \begin{cases} \mathcal{H}(\mathbb{Q}|\mathbb{P}_C) - \mathcal{H}(\mathbb{Q}^{(C)}|\mathbb{P}_C) & \textit{when } \mathbb{Q} \in \mathcal{M}_a, \\ +\infty & \textit{otherwise.} \end{cases}$$

The supremum in (2.2.4) is uniquely attained by the measure $\mathbb{Q}^{(-\gamma\mathcal{E}+\gamma B)}$, which belongs in $\mathcal{M}_{e,f}$ and its Radon-Nikodym derivative with respect to $\mathbb{P}_{-\gamma\mathcal{E}+\gamma B}$ can be written as

$$\frac{d\mathbb{Q}^{(-\gamma\mathcal{E}+\gamma B)}}{d\mathbb{P}_{-\gamma\mathcal{E}+\gamma B}} = k e^{(-\gamma \boldsymbol{\vartheta}^{(-\gamma\mathcal{E}+\gamma B)}\cdot \mathbf{S})_T}, \tag{2.2.5}$$

where $\boldsymbol{\vartheta}^{(-\gamma\mathcal{E}+\gamma B)} \in \boldsymbol{\Theta}$ is the maximizer of the control problem associated with the value function $u_\gamma(-B|\mathcal{E})$ and $k \in \mathbb{R}$ is the normalization constant.

Corollary 2.2.5. *The mappings $B \mapsto \nu^{(w)}(B;\gamma|\mathcal{E})$ and $B \mapsto \nu^{(b)}(B;\gamma|\mathcal{E})$ are, respectively, lower and upper semi-continuous with respect to the weak-* topology $\sigma(\mathbb{L}^\infty, \mathbb{L}^1)$.*

Proof. It suffices to note that (2.2.4) represents $\nu^{(w)}(\cdot;\gamma|\mathcal{E})$ as a supremum of $\sigma(\mathbb{L}^\infty, \mathbb{L}^1)-$continuous and linear functionals on $\mathbb{L}^\infty$. □

Another direct consequence of the representation formula (2.2.4) is the following proposition.

Proposition 2.2.6. *The mappings* $B \mapsto \nu^{(w)}(-B;\gamma|\mathcal{E})$ *and* $B \mapsto -\nu^{(b)}(B;\gamma|\mathcal{E})$ *are convex, decreasing and replication invariant, that is*

$$\nu^{(w)}(B + (\boldsymbol{\vartheta} \cdot \mathbf{S})_T;\gamma|\mathcal{E}) = \nu^{(w)}(B;\gamma|\mathcal{E}), \text{ for all } \boldsymbol{\vartheta} \in \boldsymbol{\Theta}.$$

The function $h_{-\gamma\mathcal{E}}(\cdot)$ in Theorem 2.2.4 is sometimes called the *penalty function* for the indifference price $\nu^{(w)}(\cdot;\gamma|\mathcal{E})$, and is convex. It is in fact strictly convex on its effective domain $\mathcal{M}_{e,f}$; we remind the reader that the effective domain of a convex function f is the subset of its domain for which $f < +\infty$. It is well-known (see, e.g., [37], Lemma 3.29) that the conjugate representation,

$$\mathbb{E}[X \log X] = \sup_{Y \in \mathbb{L}^\infty} \left(\mathbb{E}[YX] - \log \mathbb{E}[e^Y]\right),$$

where we use the convention that $x\log(x) = +\infty$, for $x < 0$, is valid for all $X \in \mathbb{L}^1_+$, such that $\mathbb{E}[X] = 1$. Using this representation and the natural identification of finite measures equivalent to $\mathbb{P}$ with their Radon-Nikodym derivatives in $\mathbb{L}^1$, we can readily establish the following properties of the penalty function h.

Proposition 2.2.7. *For* $C \in \mathbb{L}^\infty$*, the function* $h_C : \mathbb{L}^1 \mapsto [0, +\infty]$ *is convex (strictly on its effective domain) and* $\sigma(\mathbb{L}^1, \mathbb{L}^\infty)-$*lower semicontinuous.*

An immediate corollary of Proposition 2.2.7 and the Hahn-Banach Theorem in the separation form (see [37] for details on convex analysis and also [52], Theorem 2.1) is the following result.

Proposition 2.2.8. *The mapping $h_{-\gamma\mathcal{E}}$ is the minimal penalty function for $\nu^{(w)}(\cdot;\gamma|\mathcal{E})$, i.e.,*

$$h_{-\gamma\mathcal{E}}(\mathbb{Q}) \leq \tilde{h}(\mathbb{Q}), \text{ for all } \mathbb{Q} \in \mathcal{M}_a,$$

whenever the function $\tilde{h}$ satisfies

$$\nu^{(w)}(B;\gamma|\mathcal{E}) = \sup_{\mathbb{Q}\in\mathcal{M}_a} \Big(\mathbb{E}_{\mathbb{Q}}[B] - \frac{1}{\gamma}\tilde{h}(\mathbb{Q})\Big), \text{ for all } B \in \mathbb{L}^\infty.$$

Moreover, the dual, conjugate representation

$$\frac{1}{\gamma}h_{-\gamma\mathcal{E}}(\mathbb{Q}) = \sup_{B\in\mathbb{L}^\infty} \left(\mathbb{E}^{\mathbb{Q}}[B] - \nu^{(w)}(B;\gamma|\mathcal{E})\right), \ \forall\, \mathbb{Q} \in \mathbb{L}^1$$

holds.

2.2.1 Conditional indifference prices and risk equivalence classes

Using the linearity of the set $\mathbf{\Theta}$ of the admissible trading strategies and the properties of the exponential utility, one can deduce that the following scaling property holds true

$$\alpha\nu^{(w)}(B;\alpha\gamma) = \nu^{(w)}(\alpha B;\gamma), \text{ for } B \in \mathbb{L}^\infty,\ \gamma,\alpha > 0. \tag{2.2.6}$$

The following Lemma states that the indifference price has a certain subadditive property, with true additivity holding only in exceptional cases.

Lemma 2.2.9. *For $B_1, B_2 \in \mathbb{L}^\infty$ and $\gamma_1, \gamma_2 > 0$, let $\tilde{\gamma} > 0$ be given by*

$$\frac{1}{\tilde{\gamma}} = \frac{1}{\gamma_1} + \frac{1}{\gamma_2}.$$

Then,

(a) $\nu^{(w)}(B_1;\gamma_1) + \nu^{(w)}(B_2;\gamma_2) \geq \nu^{(w)}(B_1+B_2;\tilde{\gamma})$, *and*

(b) the following two conditions are equivalent

(1) $\nu^{(w)}(B_1;\gamma_1) + \nu^{(w)}(B_2;\gamma_2) = \nu^{(w)}(B_1+B_2;\tilde{\gamma})$,

(2) $\frac{\gamma_1}{\tilde{\gamma}}B_1 \sim \frac{\gamma_2}{\tilde{\gamma}}B_2$.

Proof.

(a) Using the dual representation (2.2.4), the inequality in (a) above is equivalent to the following inequality

$$\sup_{\mathbb{Q}\in\mathcal{M}_a}\left(\mathbb{E}^{\mathbb{Q}}[B_1] - \frac{1}{\gamma_1}h(\mathbb{Q})\right) + \sup_{\mathbb{Q}\in\mathcal{M}_a}\left(\mathbb{E}^{\mathbb{Q}}[B_2] - \frac{1}{\gamma_1}h(\mathbb{Q})\right) \geq \sup_{\mathbb{Q}\in\mathcal{M}_a}\left(\mathbb{E}^{\mathbb{Q}}[B_1+B_2] - \frac{1}{\tilde{\gamma}}h(\mathbb{Q})\right), \quad (2.2.7)$$

which easily follows.

(b) (1) $\Rightarrow$ (2). Equation (1) implies that the equality in (2.2.7) holds. By the strict convexity of the function $h(\cdot)$ on its effective domain, i.e., on $\mathcal{M}_{e,f}$, and the scaling property (2.2.6), equality in (2.2.7) in turn implies the equality of dual minimizers

$$\mathbb{Q}^{(\frac{\gamma_1}{\tilde{\gamma}}B_1)} = \mathbb{Q}^{(\frac{\gamma_2}{\tilde{\gamma}}B_2)} = \mathbb{Q}^{(B_1+B_2)}.$$

By the representation (2.2.5) of the Radon-Nikodym derivatives of the above measures, we get

$$k_1 e^{(\boldsymbol{\vartheta}^{(\frac{\gamma_1}{\tilde{\gamma}}B_1)}\cdot\mathbf{S})_T} e^{\frac{\gamma_1}{\tilde{\gamma}}B_1} = \frac{d\mathbb{Q}^{(\frac{\gamma_1}{\tilde{\gamma}}B_1)}}{d\mathbb{P}_{\frac{\gamma_1}{\tilde{\gamma}}B_1}} \frac{d\mathbb{P}_{\frac{\gamma_1}{\tilde{\gamma}}}B_1}{d\mathbb{P}} = \frac{d\mathbb{Q}^{(\frac{\gamma_1}{\tilde{\gamma}}B_1)}}{d\mathbb{P}}$$

$$= \frac{d\mathbb{Q}^{(\frac{\gamma_2}{\tilde{\gamma}}B_2)}}{d\mathbb{P}} = \frac{d\mathbb{Q}^{(\frac{\gamma_2}{\tilde{\gamma}}B_2)}}{d\mathbb{P}_{\frac{\gamma_2}{\tilde{\gamma}}B_2}} \frac{d\mathbb{P}_{\frac{\gamma_2}{\tilde{\gamma}}}}{d\mathbb{P}} = k_2 e^{(\boldsymbol{\vartheta}^{(\frac{\gamma_2}{\tilde{\gamma}}B_2)}\cdot\mathbf{S})_T} e^{\frac{\gamma_2}{\tilde{\gamma}}B_2},$$

and so $\frac{\gamma_1}{\tilde{\gamma}}B_1 - \frac{\gamma_2}{\tilde{\gamma}}B_2 = (\boldsymbol{\vartheta}\cdot\mathbf{S})_T + k$, where $k = \log(k_2) - \log(k_1)$ and $\boldsymbol{\vartheta} = \boldsymbol{\vartheta}^{(\frac{\gamma_2}{\tilde{\gamma}}B_2)} - \boldsymbol{\vartheta}^{(\frac{\gamma_1}{\tilde{\gamma}}B_1)}$.

(2) $\Rightarrow$ (1). Conversely, suppose that

$$\frac{\gamma_1}{\tilde{\gamma}}B_1 - \frac{\gamma_2}{\tilde{\gamma}}B_2 = (\boldsymbol{\vartheta}\cdot\mathbf{S})_T + k,$$

for some $k \in \mathbb{R}$ and $\boldsymbol{\vartheta} \in \boldsymbol{\Theta}$. Using the scaling property (2.2.6), the equality in (1) is equivalent to

$$\frac{1}{\gamma_1}\nu^{(w)}\left(\frac{\gamma_1}{\tilde{\gamma}}B_1;\tilde{\gamma}\right) + \frac{1}{\gamma_2}\nu^{(w)}\left(\frac{\gamma_2}{\tilde{\gamma}}B_2;\tilde{\gamma}\right) = \frac{1}{\tilde{\gamma}}\nu^{(w)}\left(B_1+B_2;\tilde{\gamma}\right). \quad (2.2.8)$$

By the risk equivalence between $\frac{\gamma_1}{\tilde{\gamma}}B_1$ and $\frac{\gamma_2}{\tilde{\gamma}}B_2$ and the replication invariance of $v^{(w)}(\cdot;\tilde{\gamma})$, we have

$$\begin{aligned}
&\frac{1}{\gamma_1}\nu^{(w)}\left(\frac{\gamma_1}{\tilde{\gamma}}B_1;\tilde{\gamma}\right) \quad + \quad \frac{1}{\gamma_2}\nu^{(w)}\left(\frac{\gamma_2}{\tilde{\gamma}}B_2;\tilde{\gamma}\right)\\
&= \frac{1}{\gamma_1}\nu^{(w)}\left(\frac{\gamma_1}{\tilde{\gamma}B_1};\tilde{\gamma}\right) \quad + \quad \frac{1}{\gamma_2}\nu^{(w)}\left(\frac{\gamma_1}{\tilde{\gamma}}B_1 + k + (\boldsymbol{\vartheta}\cdot\mathbf{S})_T;\tilde{\gamma}\right)\\
&= \frac{1}{\tilde{\gamma}}\nu^{(w)}\left(\frac{\gamma_1}{\tilde{\gamma}}B_1;\tilde{\gamma}\right) \quad + \quad \frac{k}{\gamma_2}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\frac{1}{\tilde{\gamma}}\nu^{(w)}\left(B_1+B_2;\tilde{\gamma}\right) &= \frac{1}{\tilde{\gamma}}\nu^{(w)}\left(B_1 + \frac{\gamma_1}{\gamma_2}B_1 + \frac{\tilde{\gamma}}{\gamma_2}(k + (\boldsymbol{\vartheta}\cdot\mathbf{S})_T);\tilde{\gamma}\right)\\
&= \frac{1}{\tilde{\gamma}}\nu^{(w)}\left(\frac{\gamma_1}{\tilde{\gamma}}B_1;\tilde{\gamma}\right) + \frac{k}{\gamma_2}.
\end{aligned}$$

The equality in (2.2.8) now follows directly.

□

The conjugacy between (affine transformations of) the price functional $\nu^{(w)}(\cdot;\gamma|\mathcal{E})$ and the penalty function $h_{-\gamma\mathcal{E}}(\cdot)$, as displayed in Theorem 2.2.4 and Proposition 2.2.8, yields directly the following auxiliary result.

Lemma 2.2.10. *For $\mathcal{E}, \tilde{\mathcal{E}} \in \mathbb{L}^\infty$, $\gamma > 0$, the following two statements are equivalent*

1. $\nu^{(w)}(B;\gamma|\mathcal{E}) \geq \nu^{(w)}(B;\gamma|\tilde{\mathcal{E}})$, *for all* $B \in \mathbb{L}^\infty$

   and

2. $h_{-\gamma\mathcal{E}}(\mathbb{Q}) \leq h_{-\gamma\tilde{\mathcal{E}}}(\mathbb{Q})$, *for all* $\mathbb{Q} \in \mathcal{M}_a$.

We use Lemma 2.2.10 in the proof of the following proposition.

Proposition 2.2.11. *For $\mathcal{E} \in \mathbb{L}^\infty$ and $\gamma > 0$, the following statements are equivalent*

1. $\nu^{(w)}(B;\gamma) \geq \nu^{(w)}(B;\gamma|\mathcal{E})$, *for all* $B \in \mathbb{L}^\infty$,

2. $\nu^{(w)}(B;\gamma) = \nu^{(w)}(B;\gamma|\mathcal{E})$, *for all* $B \in \mathbb{L}^\infty$,

3. $\mathcal{E} \in \mathcal{R}^\infty$

   and

4. $\mathbb{Q}^{(0)} = \mathbb{Q}^{(-\gamma\mathcal{E})}$.

Proof. (4) ⇒ (3) As in the proof of implication (1) ⇒ (2) in Lemma 2.2.9, we can use the equation (2.2.5) in Theorem 2.2.4 to show that (4) implies (3).

(3) ⇒ (2) Follows immediately from statement (3) in Proposition 2.2.1.

(2) ⇒ (1) Clearly, (1) is weaker than (2).

(1) ⇒ (4) By Lemma 2.2.10, the equality in (2) implies that $h_{-\gamma\mathcal{E}}(\mathbb{Q}) \geq h(\mathbb{Q})$, for all $\mathbb{Q} \in \mathcal{M}_a$, i.e.,

$$\mathcal{H}(\mathbb{Q}|\mathbb{P}_{-\gamma\mathcal{E}}) - \mathcal{H}(\mathbb{Q}^{(-\gamma\mathcal{E})}|\mathbb{P}_{-\gamma_{\mathcal{E}}}) \geq \mathcal{H}(\mathbb{Q}|\mathbb{P}) - \mathcal{H}(\mathbb{Q}^{(0)}|\mathbb{P}),\ \forall\, \mathbb{Q} \in \mathcal{M}_a.$$

In particular, for $\mathbb{Q} = \mathbb{Q}^{(-\gamma\mathcal{E})}$, we get

$$\mathcal{H}(\mathbb{Q}^{(-\gamma\mathcal{E})}|\mathbb{P}) \leq \mathcal{H}(\mathbb{Q}^{(0)}|\mathbb{P}).$$

Therefore, $\mathbb{Q}^{(-\gamma\mathcal{E})} = \mathbb{Q}^{(0)}$, by the strict convexity of the relative entropy $\mathcal{H}(\cdot|\mathbb{P})$ on its effective domain. □

Considered as convex map from $\mathbb{L}^\infty$ to $\mathbb{R}$, the indifference price is not homogeneous. In fact, the homogeneity holds only for replicable claims as the following proposition states.

Proposition 2.2.12. *For $B, \mathcal{E} \in \mathbb{L}^\infty$ and $\gamma > 0$, the following statements are equivalent*

1. $\nu^{(w)}(\alpha B; \gamma|\mathcal{E}) = \alpha\nu^{(w)}(B; \gamma|\mathcal{E})$, *for some* $\alpha \in \mathbb{R} \setminus \{0, 1\}$

   and

2. $B \in \mathcal{R}^\infty$.

Proof. We assume, for simplicity, that $\mathcal{E} = 0$, otherwise, we simply change of the underlying probability to $\mathbb{P}_{-\gamma\mathcal{E}}$ (see, also, Proposition 2.2.2).

(2) $\Rightarrow$ (1) If $B \in \mathcal{R}^\infty$, then $\alpha B \in \mathcal{R}^\infty$, so (1) follows from the replication-invariance of $\nu^{(w)}(\cdot;\gamma)$.

(1) $\Rightarrow$ (2) Suppose, first, that (1) holds with $\alpha > 0$. Then

$$\sup_{\mathbb{Q}\in\mathcal{M}_a}\left(\mathbb{E}^{\mathbb{Q}}[B] - \frac{1}{\gamma}h(\mathbb{Q})\right) = \sup_{\mathbb{Q}\in\mathcal{M}_a}\left(\mathbb{E}^{\mathbb{Q}}[B] - \frac{1}{\alpha\gamma}h(\mathbb{Q})\right),$$

where $h(\cdot)$ stands for $h_0(\cdot)$. The two maximized functions are strictly concave, ordered and agree only for $\mathbb{Q}$ such that $h(\mathbb{Q}) = 0$. Therefore, the equality of their (attained) suprema forces the relation $h(\mathbb{Q}^{(\gamma B)}) = h(\mathbb{Q}^{(\alpha\gamma B)}) = 0$, which, in turn, implies that $\mathbb{Q}^{(\gamma B)} = \mathbb{Q}^{(\alpha\gamma B)} = \mathbb{Q}^{(0)}$. We can conclude that $B \in \mathcal{R}^\infty$ by using the implication (4) $\Rightarrow$ (3) in Proposition 2.2.11.

It remains to treat the case $\alpha < 0$. By considering the random variable $|\alpha| B$ instead of B, it is clear that we can assume without loss of generality of that $\alpha = -1$. This in turn yields $\nu^{(w)}(-B;\gamma) = -\nu^{(w)}(B;\gamma)$. Equivalently, we have

$$\inf_{\mathbb{Q}\in\mathcal{M}_a}\left(\mathbb{E}^{\mathbb{Q}}[B] + \frac{1}{\gamma}h(\mathbb{Q})\right) = \sup_{\mathbb{Q}\in\mathcal{M}_a}\left(\mathbb{E}^{\mathbb{Q}}[B] - \frac{1}{\gamma}h(\mathbb{Q})\right),$$

which, by positivity of $h(\cdot)$, implies that $h(\mathbb{Q}^{(\gamma B)}) = 0$. We continue as above to conclude that $B \in \mathcal{R}^\infty$. □

2.2.2 Conditional indifference price and risk aversion coefficient

It follows directly from (2.2.4) that in the case where $\mathcal{E} \sim 0$, i.e., in the case of the *unconditional* indifference prices, the mappings $\gamma \mapsto \nu^{(w)}(B;\gamma)$

and $\gamma \mapsto -\nu^{(b)}(B;\gamma)$ are non-decreasing.

Proposition 2.2.13. *For $\gamma > 0$ and $B \in \mathbb{L}^\infty$, the mapping $\gamma \mapsto \nu^{(w)}(B;\gamma)$ ($\gamma \mapsto \nu^{(b)}(B;\gamma)$) is*

1. *constant over $\mathbb{Q} \in \mathcal{M}_a$ and equal to the value $\mathbb{E}^{\mathbb{Q}}[B]$, when $B \in \mathcal{R}^\infty$ and*

2. *strictly increasing (decreasing), otherwise.*

Proof. We only establish the results for the writer's price $\nu^{(w)}(B;\gamma)$, as the case of the buyer's price follows along similar arguments.

1. By the replication invariance of the $\nu^{(w)}(\cdot;\gamma)$, the value of $\nu^{(w)}(B;\gamma)$ equals to the value $\mathbb{E}^{\mathbb{Q}}[B]$, $\mathbb{Q} \in \mathcal{M}_a$, when $B \in \mathcal{R}^\infty$.

2. Suppose now that $\nu^{(w)}(B;\gamma_1) \leq \nu^{(w)}(B;\gamma_2)$, for some $0 < \gamma_1 < \gamma_2$. By the dual representation (2.2.4), we have $\nu^{(w)}(B;\gamma_1) = \nu^{(w)}(B;\gamma_2)$, and using the scaling property (2.2.6), we get

$$\alpha\nu^{(w)}(B;\gamma_2) = \nu^{(w)}(\alpha B;\gamma_2),$$

where $\alpha = \gamma_2/\gamma_1 > 1$. By Proposition 2.2.12, $B \in \mathcal{R}^\infty$.

$\square$

A similar proposition in the conditional case *fails*. Indeed, here is a simple example. Pick $\mathcal{E} \notin \mathcal{R}^\infty$, and set $B = \mathcal{E}$, Then $\nu^{(w)}(\mathcal{E};\gamma|\mathcal{E}) = \nu^{(b)}(\mathcal{E};\gamma)$ -

a *strictly decreasing* function of γ. An even more instructive example in which the dependence of γ ceases to be monotone *at all* is given below.

Example 2.2.14. We adopt the setting of Example 2.2.3 (stated in page 22), and assume that the coefficients b and a are chosen in such a way that the distribution function of the random variable Y_T is continuous (under $\mathbb{P}$, and, therefore, under every equivalent martingale measure). Let $\mathbb{Q}^{(0)}$ be the minimal-entropy martingale measure and let $g_i : \mathbb{R} \to \mathbb{R}$, $i = 1, 2$ be two bounded Borel-measurable functions. We set

$$\mathcal{E} = -g_2(Y_T) \text{ and } B = g_1(Y_T) - g_2(Y_T)$$

and compute the conditional indifference price $\nu^{(w)}(B;\gamma|\mathcal{E})$ as a difference

$$\nu^{(w)}(B;\gamma|\mathcal{E}) = \nu^{(w)}\left(B - \mathcal{E};\gamma\right) - \nu^{(w)}\left(-\mathcal{E};\gamma\right).$$

By part 2 of Proposition 2.2.2 and (2.2.2), we have that

$$\begin{aligned}\nu^{(w)}(B;\gamma|\mathcal{E}) &= \nu^{(w)}\left(g_2(Y_T);\gamma\right) - \nu^{(w)}\left(g_1(Y_T);\gamma\right) \\ &= \frac{1}{\gamma(1-\rho^2)} \ln \mathbb{E}^{\mathbb{Q}^{(0)}}\left[\frac{e^{\gamma(1-\rho^2)g_1(Y_T)}}{e^{\gamma(1-\rho^2)g_2(Y_T)}}\right].\end{aligned}$$

The intervals of monotonicity of the mapping $\gamma \mapsto \nu^{(w)}(B;\gamma|\mathcal{E})$ therefore coincide with the intervals of monotonicity of the function $f : (0,\infty) \to \mathbb{R}$ given by

$$f(\gamma) = \frac{1}{\gamma}\left(\ln \mathbb{E}^{\mathbb{Q}^{(0)}}[X_1^\gamma] - \ln \mathbb{E}^{\mathbb{Q}^{(0)}}[X_2^\gamma]\right),$$

where the bounded and positive random variables X_i, are given by

$$X_i = \exp((1-\rho^2)g_i(Y_T)), \text{ for } i = 1, 2.$$

It is clear that, thanks to the assumption of continuity of the distribution function of of the random variable Y_T, any pair of probability distributions with compact support in $(0, \infty)$ can be chosen for X_1 and X_2 by the appropriate choice of the functions g_1 and g_2.

Thanks to the boundedness of X_1 and X_2, we can easily obtain the following asymptotic expansion for the function f around $\gamma = 0$:

$$f(\gamma) = \mathbb{E}^{\mathbb{Q}^{(0)}}[X_1] - \mathbb{E}^{\mathbb{Q}^{(0)}}[X_2] + \tfrac{1}{2}\gamma(\mathrm{Var}_{\mathbb{Q}^{(0)}}[X_1] - \mathrm{Var}_{\mathbb{Q}^{(0)}}[X_2]) + o(\gamma).$$

In a similar manner, we have

$$\lim_{\gamma\to\infty} f(\gamma) = \ln ||X_1||_{\mathbb{L}^\infty} - \ln ||X_2||_{\mathbb{L}^\infty}.$$

Therefore, if X_1 and X_2 satisfy

1. $\mathbb{E}^{\mathbb{Q}^{(0)}}[X_1] < \mathbb{E}^{\mathbb{Q}^{(0)}}[X_2]$, and
2. $\mathrm{Var}_{\mathbb{Q}^{(0)}}[X_1] < \mathrm{Var}_{\mathbb{Q}^{(0)}}[X_2]$,

the function f is strictly decreasing and negative in a neighborhood of $\gamma = 0$. If, in addition, we have

3. $||X_1||_{\mathbb{L}^\infty} > ||X_2||_{\mathbb{L}^\infty}$,

and f can not remain decreasing for all γ since

$$f(+\infty) = \ln \left(||X_1||_{\mathbb{L}^\infty}/||X_2||_{\mathbb{L}^\infty}\right) > 0 > \mathbb{E}[X_1] - \mathbb{E}[X_2] = f(0+).$$

The straightforward construction of examples of the random variables X_1 and X_2 having the above properties is left to the reader.

2.2.3 Asymptotics of the conditional indifference prices

The asymptotics of the unconditional indifference prices in the risk-aversion parameter γ are well-known (see, for instance, Corollary 5.1 in [29]), namely,

$$\begin{aligned} \lim_{\gamma\to 0} \nu^{(w)}(B;\gamma) = \mathbb{E}^{\mathbb{Q}^{(0)}}[B], \ \lim_{\gamma\to+\infty} \nu^{(w)}(B;\gamma) = \sup_{\mathbb{Q}\in\mathcal{M}_{e,f}} \mathbb{E}^{\mathbb{Q}}[B], \\ \lim_{\gamma\to 0} \nu^{(b)}(B;\gamma) = \mathbb{E}^{\mathbb{Q}^{(0)}}[B], \ \lim_{\gamma\to+\infty} \nu^{(b)}(B;\gamma) = \inf_{\mathbb{Q}\in\mathcal{M}_{e,f}} \mathbb{E}^{\mathbb{Q}}[B]. \end{aligned} \tag{2.2.9}$$

Using the conditional price decomposition stated in part 2 of Proposition 2.2.2, these asymptotics are easily extended to the conditional case.

Proposition 2.2.15. *For $B, \mathcal{E} \in \mathbb{L}^\infty$, we have*

$$\lim_{\gamma\to 0} \nu^{(w)}(B;\gamma|\mathcal{E}) = \mathbb{E}^{\mathbb{Q}^{(0)}}[B], \ \lim_{\gamma\to 0} \nu^{(b)}(B;\gamma|\mathcal{E}) = \mathbb{E}^{\mathbb{Q}^{(0)}}[B] \tag{2.2.10}$$

and

$$\lim_{\gamma\to+\infty} \nu^{(w)}(B;\gamma|\mathcal{E}) = \sup_{\mathbb{Q}\in\mathcal{M}_{e,f}} \mathbb{E}^{\mathbb{Q}}[B-\mathcal{E}] + \inf_{\mathbb{Q}\in\mathcal{M}_{e,f}} \mathbb{E}^{\mathbb{Q}}[\mathcal{E}], \tag{2.2.11}$$

$$\lim_{\gamma\to+\infty} \nu^{(b)}(B;\gamma|\mathcal{E}) = \inf_{\mathbb{Q}\in\mathcal{M}_{e,f}} \mathbb{E}^{\mathbb{Q}}[B-\mathcal{E}] + \sup_{\mathbb{Q}\in\mathcal{M}_{e,f}} \mathbb{E}^{\mathbb{Q}}[\mathcal{E}]. \tag{2.2.12}$$

One can, further, establish the continuous differentiability of the mapping $\gamma \mapsto \nu^{(w)}(B;\gamma|\mathcal{E})$, for $\gamma \in (0,\infty)$. For this, we observe that

$$\nu^{(w)}(B;\gamma|\mathcal{E}) = \frac{1}{\gamma}\left(\nu^{(w)}(\gamma(B-\mathcal{E});1) - \nu^{(w)}(-\gamma\mathcal{E};1)\right), \tag{2.2.13}$$

and we recall the result of Theorem 5.3 in [51], which states that the function $\gamma \mapsto \nu^{(w)}(\gamma C;1)$ is continuously differentiable on $(0,\infty)$ for $C \in \mathbb{L}^\infty$.

Similar results hold for the indifference price of a vector of contingent claims. Before we state the corresponding proposition, we introduce some further notation. For $n \in \mathbb{N}$, let $(\mathbb{L}^\infty)^n$ denote the set of all $n-$tuples $\boldsymbol{B} = (B_1, B_2, \dots, B_n)$ of elements of $\mathbb{L}^\infty$, with $||\boldsymbol{B}||_{(\mathbb{L}^\infty)^n} = \max_{k \le n} ||B_k||_{\mathbb{L}^\infty}$. For $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_n) \in \mathbb{R}^n$, we write $\boldsymbol{\alpha} \cdot \boldsymbol{B} = \sum_{k=1}^n \alpha_k B_k \in \mathbb{L}^\infty$ and set $|\boldsymbol{\alpha}| = \max_{k \le n} |\alpha_k|$. Also, $\mathbb{E}^{\mathbb{Q}}[\mathbf{B}]$ stands for the vector $(\mathbb{E}^{\mathbb{Q}}[B_1], \mathbb{E}^{\mathbb{Q}}[B_2], ..., \mathbb{E}^{\mathbb{Q}}[B_n]) \in \mathbb{R}^n$.

Proposition 2.2.16. *For $\mathcal{E} \in \mathbb{L}^\infty$ and $\boldsymbol{B} \in (\mathbb{L}^\infty)^n$, the function* $w : \mathbb{R}^n \times (0, \infty] \to \mathbb{R}$ *given by*

$$w(\boldsymbol{\alpha}, \gamma) = \begin{cases} \nu^{(w)}(\boldsymbol{\alpha} \cdot \boldsymbol{B}; \gamma | \mathcal{E}), & \gamma < \infty \\ \sup_{\mathbb{Q} \in \mathcal{M}_{e,f}} \mathbb{E}^{\mathbb{Q}}[\boldsymbol{\alpha} \cdot \boldsymbol{B} - \mathcal{E}] + \inf_{\mathbb{Q} \in \mathcal{M}_{e,f}} \mathbb{E}^{\mathbb{Q}}[\mathcal{E}], & \gamma = +\infty, \end{cases} \tag{2.2.14}$$

is jointly continuous, and Lipschitz continuous on every subset D of the form $D = [\gamma_0, \infty) \times \mathbb{R}^n$, $\gamma_0 > 0$.

Proof. The functional $B \mapsto \nu^{(w)}(B; \gamma | \mathcal{E})$ is positive and equal to the identity mapping for constant claims. Therefore, for $\gamma \in (0, \infty)$,

$$\begin{aligned} \left| \nu^{(w)}(\boldsymbol{\alpha}_1 \cdot \boldsymbol{B}; \gamma | \mathcal{E}) - \nu^{(w)}(\boldsymbol{\alpha}_2 \cdot \boldsymbol{B}; \gamma | \mathcal{E}) \right| &\le ||(\boldsymbol{\alpha}_1 - \boldsymbol{\alpha}_2) \cdot \boldsymbol{B}||_{\mathbb{L}^\infty} \le \\ &\le |\boldsymbol{\alpha}_1 - \boldsymbol{\alpha}_2| \, ||\boldsymbol{B}||_{(\mathbb{L}^\infty)^n}. \end{aligned} \tag{2.2.15}$$

For $\gamma = +\infty$, the validity of (2.2.15) follows by passing to the limit $\gamma \to \infty$. On the other hand, by (2.2.13), for every $B \in \mathbb{L}^\infty$ and $\gamma_0 > 0$, we have $\left| \nu^{(w)}(B; \gamma_1 | \mathcal{E}) - \nu^{(w)}(B; \gamma_2 | \mathcal{E}) \right| \le \frac{1}{\gamma_0} \left(\left| \nu^{(w)}(\gamma_1 (B - \mathcal{E}); 1) - \nu^{(w)}(\gamma_2 (B - \mathcal{E}); 1) \right| \right.$ $\left. + \left| \nu^{(w)}(-\gamma_1 \mathcal{E}; 1) - \nu^{(w)}(-\gamma_2 \mathcal{E}; 1) \right| \right) \le \frac{1}{\gamma_0} (||B - \mathcal{E}||_{\mathbb{L}^\infty} + ||\mathcal{E}||_{\mathbb{L}^\infty}) \, |\gamma_1 - \gamma_2|,$

for every $\gamma_1, \gamma_2 \in [\gamma_0, \infty)$. Therefore, for each $\gamma_0 > 0$, there exists a constant $C = C(\gamma_0) > 0$ such that

$$|w(\boldsymbol{\alpha}_1, \gamma_1) - w(\boldsymbol{\alpha}_2, \gamma_2)| \leq C\left(|\gamma_1 - \gamma_2| + |\boldsymbol{\alpha}_1 - \boldsymbol{\alpha}_2|\right),$$

for $\gamma_1, \gamma_2 \in [\gamma_0, \infty)$ and $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2 \in \mathbb{R}^n$. It is left to recall from Proposition 2.2.15 that

$$w(\boldsymbol{\alpha}, \infty) = \lim_{\gamma \to \infty} w(\boldsymbol{\alpha}, \gamma) = \sup_{\mathbb{Q} \in \mathcal{M}_{e,f}} \mathbb{E}^{\mathbb{Q}}[\boldsymbol{\alpha} \cdot \boldsymbol{B} - \mathcal{E}] + \inf_{\mathbb{Q} \in \mathcal{M}_{e,f}} \mathbb{E}^{\mathbb{Q}}[\mathcal{E}].$$

□

2.3 The Residual Risk at Maturity

The notion of *residual risk* for the indifference price valuation was defined in [63] for the setting of our Example 2.2.3., in [74] for the stochastic volatility model and in [64] for a binomial-type model. Below, we redefine this notion in the case where the agent has a random endowment in her portfolio. In Appendix A, we present a brief summary of the literature on the residual risk process.

Let $\gamma > 0$ be the agent's risk-aversion coefficient, and $B \in \mathbb{L}^\infty$ be a contingent claim. By Theorem 2.2.4, we have that the optimization problem with the value function $u_\gamma(x|\nu^{(w)}(B;\gamma) - B)$, introduced in (2.1.2), admits an essentially unique maximizer $\boldsymbol{\vartheta}^{(B)} \in \boldsymbol{\Theta}$. Following the same notation, $\boldsymbol{\vartheta}^{(0)} \in \boldsymbol{\Theta}$ stands for the optimal strategy without the claim. We then set

$$\bar{\boldsymbol{\vartheta}}_t^{(B)} = \boldsymbol{\vartheta}_t^{(B)} - \boldsymbol{\vartheta}_t^{(0)},$$

for all $t \in [0,T]$ (note that since $\mathbf{\Theta}$ is a vector space, $\bar{\boldsymbol{\vartheta}}^{(B)} \in \mathbf{\Theta}$). The corresponding wealth process

$$X_t^{(B)} = \nu^{(w)}(B;\gamma) + \int_0^t \bar{\boldsymbol{\vartheta}}_u^{(B)}\, d\mathbf{S}_u,$$

can be interpreted as the optimal risk-monitoring strategy for the writer of the claim B, compensated by $\nu^{(w)}(B;\gamma)$ at initial time. The hedging error

$$R^{(w)}(B;\gamma) = B - X_T^{(B)} \tag{2.3.1}$$

is called the *(writer's) residual risk*. $R^{(w)}(B;\gamma)$ can be interpreted as the risk remaining after the optimal hedging has been performed. Note that $R^{(w)}(B;\gamma) = 0$, $\mathbb{P}-$a.s., for all replicable claims $B \in \mathbb{L}^\infty$. In the conditional case, an analogous discussion and the decomposition formula

$$\nu^{(w)}(B;\gamma|\mathcal{E}) = \nu^{(w)}(B - \mathcal{E};\gamma) - \nu^{(w)}(-\mathcal{E};\gamma),$$

(see Proposition 2.2.2) allow us to define the *conditional residual risk* $R^{(w)}(B;\gamma|\mathcal{E})$ by

$$R^{(w)}(B;\gamma|\mathcal{E}) = R^{(w)}(B - \mathcal{E};\gamma) - R^{(w)}(-\mathcal{E};\gamma),$$

and obtain the decomposition

$$B = \nu^{(w)}(B;\gamma|\mathcal{E}) + \int_0^T \bar{\boldsymbol{\vartheta}}_t^{(B|\mathcal{E})}\, d\mathbf{S}_t + R^{(w)}(B;\gamma|\mathcal{E}), \tag{2.3.2}$$

where $\bar{\boldsymbol{\vartheta}}_t^{(B|\mathcal{E})} = \bar{\boldsymbol{\vartheta}}_t^{(B-\mathcal{E})} - \bar{\boldsymbol{\vartheta}}_t^{(-\mathcal{E})} = \boldsymbol{\vartheta}_t^{(B-\mathcal{E})} - \boldsymbol{\vartheta}_t^{(-\mathcal{E})}$, $t \in [0,T]$.

The process $(\bar{\boldsymbol{\vartheta}}_t^{(B|\mathcal{E})})_{t\in[0,T]}$, as well as the decomposition (2.3.2), could have been derived equivalently using the optimization problems used to define the conditional indifference prices.

All of the above concepts have natural analogues when seen from the buyer's side. Namely, we define the (buyer's) residual risk by $R^{(b)}(B;\gamma) = R^{(w)}(-B;\gamma)$ and by $R^{(b)}(B;\gamma|\mathcal{E}) = R^{(b)}(B+\mathcal{E};\gamma) - R^{(b)}(\mathcal{E};\gamma)$ in the conditional case.

Remark 2.3.1. Using decomposition (2.3.2) and Proposition 2.2.6, we observe that

$$\nu^{(b)}(R^{(w)}(B;\gamma|\mathcal{E});\gamma|\mathcal{E}) = \nu^{(b)}(B;\gamma|\mathcal{E}) - \nu^{(w)}(B;\gamma|\mathcal{E}).$$

In particular, $\nu^{(b)}(R^{(w)}(B;\gamma|\mathcal{E});\gamma|\mathcal{E}) < 0$ for all $\mathcal{E} \in \mathcal{R}^{\infty}$, and $B \in \mathbb{L}^{\infty} \setminus \mathcal{R}^{\infty}$. We also note that

$$\nu^{(w)}(R^{(w)}(B;\gamma|\mathcal{E});\gamma|\mathcal{E}) = 0,$$

for every $\mathcal{E}, B \in \mathbb{L}^{\infty}$.

2.4 Second Order Price Approximation

In general market models, it is not possible to obtain a closed-form representation of the indifference prices. Hence an asymptotic approximation of the prices can be very useful. In this section we provide a rather general Taylor-type approximation of the conditional exponential indifference price for our locally bounded semimartingale model under the additional assumption of continuous filtrations. In the proof of the approximation, we establish that the indifference price as a function of claim units is twice differentiable and we also give a representation of its second derivative.

Asymptotic techniques are not new in utility maximization problems.

In [60], a first order approximation of the optimal hedging strategy in a semi-martingale market for general utilities (defined on the positive real line) is provided. In [59], Theorem 10, a second order approximation is given when utility functions defined on the positive real line are considered. This generalizes the results of [44] and [46]. For exponential utility, the first derivative of the indifference price for a vector of claims is given in [51]. By imposing the assumption of continuity on the filtration, we generalize their result as well as the asymptotic approximation in [72].

2.4.1 The first and the second derivatives

We suppose that the claim under consideration has the form $\boldsymbol{\alpha} \cdot \boldsymbol{B}$ for a some vector $\boldsymbol{B} = (B_1, B_2, \ldots, B_n)$ in $(\mathbb{L}^\infty)^n$, where $\boldsymbol{\alpha} \in \mathbb{R}^n$. To facilitate the reading, we use the notation

$$w(\boldsymbol{\alpha}) = \nu^{(w)}(\boldsymbol{\alpha} \cdot \boldsymbol{B}; \gamma | \mathcal{E}),$$

for $\boldsymbol{\alpha} \in \mathbb{R}^n$.

A straightforward extension of Theorem 5.1 in [51], where we use the fact that the conditional indifference prices are just the unconditional ones under a change of measure, yields the following result.

Proposition 2.4.1. *The function w is continuously differentiable on $\mathbb{R}^n$ with*

$$\nabla w(\boldsymbol{\alpha}) = \mathbb{E}^{\mathbb{Q}^{(\gamma \boldsymbol{\alpha} \cdot \boldsymbol{B} - \gamma \mathcal{E})}}[\boldsymbol{B}] = \left(\mathbb{E}^{\mathbb{Q}^{(\gamma \boldsymbol{\alpha} \cdot \boldsymbol{B} - \gamma \mathcal{E})}}[B_1], \ldots, \mathbb{E}^{\mathbb{Q}^{(\gamma \boldsymbol{\alpha} \cdot \boldsymbol{B} - \gamma \mathcal{E})}}[B_n]\right), \ \boldsymbol{\alpha} \in \mathbb{R}^n.$$

The concept of minimized variance, defined below, is important for the study of second derivatives of the function w.

Definition 2.4.2. Let $\mathbb{Q} \in \mathcal{M}_e$ be an arbitrary equivalent martingale measure.

1. For $B \in \mathbb{L}^\infty$ we define the *projected variance* $\Delta^{\mathbb{Q}}(B)$ of B under $\mathbb{Q}$ as

$$\Delta^{\mathbb{Q}}(B) = \inf_{\boldsymbol{\vartheta} \in \boldsymbol{\Theta}^2_{\mathbb{Q}}} \mathbb{E}^{\mathbb{Q}} \left[(B - \mathbb{E}^{\mathbb{Q}}[B] - (\boldsymbol{\vartheta} \cdot \mathbf{S})_T)^2 \right], \tag{2.4.1}$$

   where, $\boldsymbol{\Theta}^2_{\mathbb{Q}} = \{\boldsymbol{\vartheta} \in L(\mathbf{S}) : (\boldsymbol{\vartheta} \cdot \mathbf{S}) \text{ is square integrable } \mathbb{Q} - \text{martingale}\}$, so that $\bigcap_{\mathbb{Q} \in \mathcal{M}_{e,f}} \boldsymbol{\Theta}^2_{\mathbb{Q}} \subset \boldsymbol{\Theta}$.

2. For $B_1, B_2 \in \mathbb{L}^\infty$ we define the *projected covariance* $\Delta^{\mathbb{Q}}(B_1, B_2)$ of B_1 and B_2 by

$$\Delta^{\mathbb{Q}}(B_1, B_2) = \tfrac{1}{2}(\Delta^{\mathbb{Q}}(B_1 + B_2) - \Delta^{\mathbb{Q}}(B_1) - \Delta^{\mathbb{Q}}(B_2)).$$

3. For a vector $\boldsymbol{B} = (B_1, \dots, B_n) \in (\mathbb{L}^\infty)^n$ and a probability measure $\mathbb{Q} \in \mathcal{M}_e$, we define the $\mathbb{Q}-$*projected variance-covariance* matrix $\boldsymbol{\Delta}^{\mathbb{Q}}(\boldsymbol{B})$ by

$$\boldsymbol{\Delta}^{\mathbb{Q}}_{ij}(\boldsymbol{B}) = \Delta^{\mathbb{Q}}(B_i, B_j),\ i, j = 1, \dots, n.$$

Remark 2.4.3.

1. The projected variance $\Delta^{\mathbb{Q}}(B)$ is the square of the $\mathbb{L}^2(\mathbb{Q})-$norm of the projection $P^{\mathbb{Q}}(B)$ of the random variable $B \in \mathbb{L}^\infty \subseteq \mathbb{L}^2(\mathbb{Q})$ onto the closed subspace $\mathbb{R} \oplus \left\{(\boldsymbol{\vartheta} \cdot \mathbf{S})_T : \boldsymbol{\vartheta} \in \boldsymbol{\Theta}^2_{\mathbb{Q}}\right\}$ of $\mathbb{L}^2(\mathbb{Q})$ (the closeness of $\left\{(\boldsymbol{\vartheta} \cdot \mathbf{S})_T : \boldsymbol{\vartheta} \in \boldsymbol{\Theta}^2_{\mathbb{Q}}\right\}$ in $\mathbb{L}^2(\mathbb{Q})$ is an immediate consequence of the $\mathbb{L}^2(d[\mathbf{S}])$-$\mathbb{L}^2(\mathbb{Q})$ isometry of stochastic integration). It follows that the projected covariance $\Delta^{\mathbb{Q}}(B_1, B_2)$ can be represented as

$$\Delta^{\mathbb{Q}}(B_1, B_2) = \mathbb{E}^{\mathbb{Q}}[P^{\mathbb{Q}}(B_1) P^{\mathbb{Q}}(B_2)].$$

In particular, $\Delta^{\mathbb{Q}}(\cdot,\cdot)$ is a bilinear functional on $\mathbb{L}^\infty \times \mathbb{L}^\infty$ and the following equality holds

$$\Delta^{\mathbb{Q}}(\boldsymbol{\alpha}\cdot\boldsymbol{B}) = \boldsymbol{\alpha}\cdot\boldsymbol{\Delta}^{\mathbb{Q}}\cdot(\boldsymbol{B})\boldsymbol{\alpha} = \sum_{i,j=1}^{n} \alpha_i \boldsymbol{\Delta}^{\mathbb{Q}}_{ij}(\boldsymbol{B})\alpha_j, \tag{2.4.2}$$

for all $\mathbb{Q}\in\mathcal{M}_e$, $\boldsymbol{B}\in(\mathbb{L}^\infty)^n$ and $\boldsymbol{\alpha}=(\alpha_1,\dots,\alpha_n)\in\mathbb{R}^n$.

2. Details on the notion of the projected variance, which is closely related to mean-variance hedging, can be found in [38] or [71]. Note that the existence of the minimizer in the Definition 2.4.2 for bounded claims can be established using the Kunita-Watanabe decomposition of the uniformly integrable $\mathbb{Q}-$martingale $(B_t)_{t\in[0,T]}$, defined as $B_t = \mathbb{E}^{\mathbb{Q}}[B|\mathcal{F}_t]$. For details on the Kunita-Watanabe decomposition we refer the reader to [2].

We recall that a filtration is called *continuous* if all adapted local martingales admit continuous versions.

Lemma 2.4.4. *Suppose that $n=1$ and that the filtration $\mathbb{F}$ is continuous. Then, for $B,\mathcal{E}\in\mathbb{L}^\infty$, the function $w:\mathbb{R}\to\mathbb{R}$ is twice differentiable at any $\alpha\in\mathbb{R}$ and its second derivative is given by*

$$w''(\alpha) = \gamma\Delta^{\mathbb{Q}^{(\gamma\alpha B-\gamma\mathcal{E})}}(B). \tag{2.4.3}$$

Proof. Without loss of generality we may suppose that $\mathcal{E}=0$ (otherwise, we work under the measure $\mathbb{P}_{-\gamma\mathcal{E}}$). We first focus on the case $\alpha=0$ and by Proposition 2.4.1, we get that $w'(0)=\mathbb{E}^{\mathbb{Q}^{(0)}}[B]$. Since Δ is clearly invariant under constant translation, we can to assume, in addition, that $\mathbb{E}^{\mathbb{Q}^{(0)}}[B] = 0$. It will, therefore, suffice to show that

$$\lim_{\alpha \to 0} \left| \frac{\nu^{(w)}(\alpha B; \gamma)}{\alpha^2} - \tfrac{\gamma}{2} \Delta^{\mathbb{Q}^{(0)}}(B) \right| = 0.$$

By the sign invariance of $\Delta^{\mathbb{Q}^{(0)}}(\cdot)$ and the scaling property (2.2.6) of the indifference prices, it suffices to consider only $\alpha > 0$, i.e., it suffices to establish that

$$\lim_{\alpha \searrow 0} \left| \frac{\nu^{(w)}(B; \alpha\gamma)}{\alpha} - \tfrac{\gamma}{2} \Delta^{\mathbb{Q}^{(0)}}(B) \right| = 0. \tag{2.4.4}$$

Theorem A.2.1 and the definition of the residual risk process in Appendix A yield that

$$\frac{\nu^{(w)}(B; \alpha\gamma)}{\alpha} = \frac{\gamma}{2} \mathbb{E}^{\mathbb{Q}^{(0)}}[\langle L^{(w)}(B; \alpha\gamma) \rangle_T] = \frac{\gamma}{2} \mathbb{E}^{\mathbb{Q}^{(0)}}[L_T^{(w)}(B; \alpha\gamma)^2],$$

where $(L_t^{(w)}(B; \alpha\gamma))_{t \in [0,T]}$ is as in Theorem A.2.1. The $BMO(\mathbb{Q}^{(0)})$−convergence of the processes $(L_t^{(w)}(B; \alpha\gamma))_{t \in [0,T]}$ from the same theorem, implies, in particular (see Theorem A.3.2), the $\mathbb{L}^2(\Omega, \mathcal{F}, \mathbb{Q}^{(0)})$−convergence of their terminal values, i.e.,

$$L_T^{(w)}(B; \alpha\gamma) \to L_T^{(w)}(B; 0) \text{ in } \mathbb{L}^2(\Omega, \mathcal{F}, \mathbb{Q}^{(0)}).$$

Therefore, it remains to prove that

$$\mathbb{E}^{\mathbb{Q}^{(0)}} \left[L_T^{(w)}(B; 0)^2 \right] = \Delta^{\mathbb{Q}^{(0)}}(B), \ \forall \, B \in \mathbb{L}^\infty.$$

Thanks to the final part of Theorem A.2.1, $L^{(w)}(B; 0)$ is strongly orthogonal to any process of the form $(\boldsymbol{\vartheta} \cdot \mathbf{S})$, for $\boldsymbol{\vartheta} \in \boldsymbol{\Theta}^2_{\mathbb{Q}^{(0)}}$. In particular, for $\boldsymbol{\vartheta} \in \boldsymbol{\Theta}^2_{\mathbb{Q}^{(0)}}$

and $\hat{\boldsymbol{\vartheta}}^{(B)}$ as in Theorem A.2.1, we have

$$B - (\boldsymbol{\vartheta} \cdot \mathbf{S})_T = \left((\hat{\boldsymbol{\vartheta}}^{(B)} - \boldsymbol{\vartheta}) \cdot \mathbf{S}\right)_T + L_T^{(w)}(B; 0),$$

Therefore,

$$\mathbb{E}^{\mathbb{Q}^{(0)}}\left[(B - (\boldsymbol{\vartheta} \cdot \mathbf{S})_T)^2\right] = \mathbb{E}^{\mathbb{Q}^{(0)}}\left[\left(\left((\hat{\boldsymbol{\vartheta}}^{(B)} - \boldsymbol{\vartheta}) \cdot \mathbf{S}\right)_T\right)^2\right] + \mathbb{E}^{\mathbb{Q}^{(0)}}\left[L_T^{(w)}(B; 0)^2\right].$$

Consequently, the minimum in the definition of $\Delta^{\mathbb{Q}^{(0)}}(B)$ is attained at $\boldsymbol{\vartheta} = \hat{\boldsymbol{\vartheta}}^{(B)}$. Therefore, $\Delta^{\mathbb{Q}}(B) = \mathbb{E}^{\mathbb{Q}^{(0)}}[L_T^{(w)}(B; 0)^2]$ and we conclude.

For $\alpha \neq 0$, we may again assume that

$$w'(\alpha) = \mathbb{E}^{\mathbb{Q}^{(\gamma\alpha B)}}[B] = 0.$$

Hence, it is enough to show that

$$\lim_{\varepsilon \to 0} \frac{w(\alpha + \varepsilon) - w(\alpha)}{\varepsilon^2} = \frac{\gamma}{2} \Delta^{\mathbb{Q}^{(\alpha\gamma B)}}(B).$$

For this, we note that $w(\alpha+\varepsilon) - w(\alpha) = v^{(w)}(\varepsilon B | -\alpha B; \gamma)$, i.e., we can rewrite the second derivative at some $\alpha \neq 0$ as the second derivative at $\alpha = 0$ with random endowment. This observation finishes the proof. □

The case $n > 1$ is covered by the following lemma.

Lemma 2.4.5. *For $\boldsymbol{\alpha}, \boldsymbol{\delta} \in \mathbb{R}^n$, $\boldsymbol{B} \in (\mathbb{L}^\infty)^n$ and $\mathcal{E} \in \mathbb{L}^\infty$ and when the filtration $\mathbb{F}$ is continuous, we have*

$$\lim_{\varepsilon \to 0} \frac{w(\boldsymbol{\alpha} + \varepsilon\boldsymbol{\delta}) - w(\boldsymbol{\alpha}) - \varepsilon \nabla w(\boldsymbol{\alpha}) \cdot \boldsymbol{\delta}}{\varepsilon^2} = \tfrac{1}{2} \sum_{i,j=1}^{n} \delta_i \boldsymbol{\Delta}_{ij}^{\mathbb{Q}^{(\gamma\boldsymbol{\alpha}\cdot\boldsymbol{B} - \gamma\mathcal{E})}}(\boldsymbol{B}) \delta_j. \quad (2.4.5)$$

Proof. We can assume that $\boldsymbol{\alpha} = (0, \ldots, 0)$ by absorbing the term $-\boldsymbol{\alpha} \cdot \boldsymbol{B}$ into the random endowment $\mathcal{E}$. The left-hand side of (2.4.5) can now be viewed as the second derivative at 0 of the function $\tilde{w} : \mathbb{R} \to \mathbb{R}$ given by

$$\tilde{w}(\varepsilon) = \nu^{(w)}(\varepsilon \boldsymbol{\delta} \cdot \boldsymbol{B}; \gamma | \mathcal{E}).$$

We finish the proof by employing Lemma 2.4.4 and using the equality (2.4.2), with $\mathbb{Q} = \mathbb{Q}^{(-\gamma \mathcal{E})}$ and $\boldsymbol{\delta}$ substituted for $\boldsymbol{\alpha}$. □

2.4.2 Approximation formula and examples

With the above results in our toolbox, we can give a second order directional Taylor-type approximation of the indifference price.

Proposition 2.4.6. *Let $\boldsymbol{B} \in (\mathbb{L}^\infty)^n$, $\boldsymbol{\alpha} \in \mathbb{R}^n$, $\gamma > 0$ and $\mathcal{E} \in \mathbb{L}^\infty$, and assume that the filtration $\mathbb{F}$ is continuous. With the notions of projected variance and covariance as in Definition 2.4.2, we have as $\varepsilon \to 0$*

$$\nu^{(w)}(\varepsilon \boldsymbol{\alpha} \cdot \boldsymbol{B}; \gamma | \mathcal{E}) = \varepsilon \boldsymbol{\alpha} \cdot \mathbb{E}^{\mathbb{Q}^{(-\gamma \mathcal{E})}}[\boldsymbol{B}] + \frac{\varepsilon^2 \gamma}{2} \boldsymbol{\alpha} \cdot \boldsymbol{\Delta}^{\mathbb{Q}^{(-\gamma \mathcal{E})}}(\boldsymbol{B}) \cdot \boldsymbol{\alpha} + o(\varepsilon^2). \quad (2.4.6)$$

Example 2.4.7. For the setup of Example 2.2.3, the corresponding approximation result is easily obtained by formula (2.2.2) (see, e.g., [44]). After changing the measure $\mathbb{P}$ to $\mathbb{P}_{-\gamma \mathcal{E}}$, it is straightforward to show that

$$\nu^{(w)}(\alpha B; \gamma | \mathcal{E}) = \alpha \mathbb{E}^{\mathbb{Q}^{(-\gamma \mathcal{E})}}[B] + \frac{\alpha^2}{2} \gamma (1 - \rho^2) \operatorname{Var}^{\mathbb{Q}^{(-\gamma \mathcal{E})}}(B) + o(\alpha^2), \; \forall B \in \mathbb{L}^\infty,$$

where $\operatorname{Var}^{\mathbb{Q}}(B)$ denotes the variance of random variable B under the probability measure $\mathbb{Q}$, i.e., $\Delta^{\mathbb{Q}^{(-\gamma \mathcal{E})}}(B) = (1 - \rho^2) \operatorname{Var}^{\mathbb{Q}^{(-\gamma \mathcal{E})}}(B)$.

Example 2.4.8. In the cases where there is no closed-form expression for the indifference price, the approximation (2.4.6) is rather useful. One of these cases is the stochastic volatility model studied in [72] and [50] (see also [45]). In this model, we suppose the existence of one risky traded asset, whose dynamics are given by

$$\begin{aligned} dS_t &= \mu(t)S_t + \sigma(Y_t,t)S_t dW_t^{(1)} \\ dY_t &= b(Y_t,t)dt + a(Y_t,t)\left(\rho dW_t^{(1)} + \rho' dW_t^{(2)}\right), \end{aligned}$$

where μ is a bounded smooth function and $W^{(1)} = \left(W^{(1)}\right)_{t\in[0,T]}$ and $W^{(2)} = \left(W^{(2)}\right)_{t\in[0,T]}$ are independent Brownian Motions (for further technical assumptions we refer the reader to [72]).

We consider claims whose payoffs are of the form $B = g(S_T, Y_T)$, where $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a bounded Borel function. For this family of claims there is no closed-form representation of the indifference price $v^{(w)}(B;\gamma)$ (in [72] a fully non-linear partial differential equation, whose solution is the dynamic version of the indifference price is stated together with results on the measure $\mathbb{Q}^{(0)}$). However, it is well-known (see [72], page 1336) that under $\mathbb{Q}^{(0)}$ the dynamics of the traded asset become

$$\begin{aligned} dS_t &= \sigma(Y_t,t)S_t d\tilde{W}_t^{(1)} \\ dY_t &= \left(K(Y_t,t)\right)dt + a(Y_t,t)\left(\rho d\tilde{W}_t^{(1)} + \rho' d\tilde{W}_t^{(2)}\right), \end{aligned}$$

where $K(Y_t,t)$ is an adapted process and $\tilde{W}^{(1)} = (\tilde{W}^{(1)})_{t\in[0,T]}$ and $\tilde{W}^{(2)} = (\tilde{W}^{(2)})_{t\in[0,T]}$ are two independent Brownian Motions under $\mathbb{Q}^{(0)}$. Setting $\tilde{\mathbf{W}}_t =$

$(\tilde{W}_t^{(1)}, \tilde{W}_t^{(2)})_{t\in[0,T]}$, we get from the martingale representation theorem that for every claim payoff $B \in \mathbb{L}^\infty(\Omega, \mathcal{F}_T^{\tilde{\mathbf{W}}}, \mathbb{Q}^{(0)})$ there exists a process $\mathbf{H}_t = (H_t^{(1)}, H_t^{(1)})_{t\in[0,T]}$, such that

$$B = \mathbb{E}^{\mathbb{Q}^{(0)}}[B] + \int_0^T H_t^{(1)} d\tilde{W}_t^{(1)} + \int_0^T H_t^{(2)} d\tilde{W}_t^{(2)}.$$

Hence, thanks to the independence between $\tilde{W}^{(1)}$ and $\tilde{W}^{(2)}$ we have

$$\Delta^{\mathbb{Q}^{(0)}}(B) = \inf_{\vartheta\in\mathbf{\Theta}^2_{\mathbb{Q}^{(0)}}} \mathbb{E}^{\mathbb{Q}^{(0)}} \left\{ \left[B - \mathbb{E}^{\mathbb{Q}^{(0)}}[B] - (\vartheta \cdot S)_T \right]^2 \right\}$$

$$= \inf_{\vartheta\in\mathbf{\Theta}^2_{\mathbb{Q}^{(0)}}} \left\{ \mathbb{E}^{\mathbb{Q}^{(0)}} \left[\left(\int_0^T H_t^{(1)} - \vartheta_t \sigma(Y_t, t) S_t \right)^2 dt \right] + \mathbb{E}^{\mathbb{Q}^{(0)}} \left[\int_0^T \left(H_t^{(2)} \right)^2 dt \right] \right\}.$$

Since the process $\hat{\vartheta}_t = \frac{H_t^{(1)}}{\sigma(Y_t,t)S_t}$ belongs to $\mathbf{\Theta}^2_{\mathbb{Q}^{(0)}}$, it follows that

$$\Delta^{\mathbb{Q}^{(0)}}(B) = \mathbb{E}^{\mathbb{Q}^{(0)}} \left[\int_0^T \left(H_t^{(2)} \right)^2 dt \right]. \tag{2.4.7}$$

Proposition 2.4.6 and assertion (2.4.7) provide a generalization of the approximation results of subsection 4.3 in [50].

Chapter 3

Agents' Agreement

The aim of this chapter is to introduce the notion of the agreement between financial agents and determine the conditions under which it is possible for two agents to agree on non-replicable claims. We first give the corresponding definition of a mutually agreeable claim and some properties of the set of such claims. Then, a non-agreement argument in the case of replicable random endowment is presented. In Section 3.3, the necessary and sufficient condition, in terms of the random endowments and the risk aversion coefficients, for agreement to hold is established and discussed. Finally, in the last two sections of this chapter, we relate the residual risk and the asymptotic approximation of the indifference price with the notion of agreement.

In what follows, we deal with the interaction of two financial agents, with risk aversion coefficients γ_1 and γ_2 and random endowments $\mathcal{E}_1, \mathcal{E}_2 \in \mathbb{L}^\infty$, respectively.

3.1 The Mutually Agreeable Claims

The indirect utility $u_\gamma(\cdot|\mathcal{E})$ induces a *preference relation* $\preceq_{\gamma,\mathcal{E}}$, on $\mathbb{L}^\infty$; for $B_1, B_2 \in \mathbb{L}^\infty$, we set

$$B_1 \preceq_{\gamma,\mathcal{E}} B_2 \ \text{ if } \ u_\gamma(B_1|\mathcal{E}) \leq u_\gamma(B_2|\mathcal{E}).$$

In words, the payoff B_2 is preferable to the payoff B_1 for the agent with random endowment $\mathcal{E}$ and risk aversion coefficient γ, if the total payoff $\mathcal{E} + B_2$ yields to her more indirect utility than the payoff $\mathcal{E} + B_1$.

Closely related to the relation $\preceq_{\gamma,\mathcal{E}}$ is its strict version $\prec_{\gamma,\mathcal{E}}$, defined by

$$B_1 \prec_{\gamma,\mathcal{E}} B_2 \ \text{ if } \ u_\gamma(B_1|\mathcal{E}) < u_\gamma(B_2|\mathcal{E}),$$

for $B_1, B_2 \in \mathbb{L}^\infty$. Note that $\mathcal{A}_\gamma(\mathcal{E}) = \{B \in \mathbb{L}^\infty : 0 \preceq_{\gamma,\mathcal{E}} B\}$ and $\mathcal{A}^\circ_\gamma(\mathcal{E}) = \{B \in \mathbb{L}^\infty : 0 \prec_{\gamma,\mathcal{E}} B\}$.

Given the agents' preference relations, we are ready to determine those claims the transaction of which benefits both agents.

Definition 3.1.1. A contingent claim $B \in \mathbb{L}^\infty$ is said to be

1. *mutually agreeable* if there exists a number $p \in \mathbb{R}$ such that $p - B \in \mathcal{A}_{\gamma_1}(\mathcal{E}_1)$ and $B - p \in \mathcal{A}_{\gamma_2}(\mathcal{E}_2)$.

2. *strictly mutually agreeable* if there exists a number $p \in \mathbb{R}$ such that $p - B \in \mathcal{A}^\circ_{\gamma_1}(\mathcal{E}_1)$ and $B - p \in \mathcal{A}^\circ_{\gamma_2}(\mathcal{E}_2)$.

If a claim B is (strictly) mutually agreeable, the set of all $p \in \mathbb{R}$ such that $p - B \in \mathcal{A}_{\gamma_1}(\mathcal{E}_1)$ and $B - p \in \mathcal{A}_{\gamma_2}(\mathcal{E}_2)$ (or $p - B \in \mathcal{A}^{\circ}_{\gamma_1}(\mathcal{E}_1)$ and $B - p \in \mathcal{A}^{\circ}_{\gamma_2}(\mathcal{E}_2)$) is called the *set of (strictly) mutually-agreeable prices* for B.

Remark 3.1.2. A discussion related to our notion of mutually agreeability is given in [52], subsection 3.6, for cash invariant monetary utility functions, but without the presence of a financial market.

Using the conditional writer's and buyer's indifference prices, $\nu^{(w)}(\cdot;\gamma_1|\mathcal{E}_1)$ and $\nu^{(b)}(\cdot;\gamma_2|\mathcal{E}_2)$, we can give a simple characterization of the set of mutually-agreeable prices.

Proposition 3.1.3. *A claim $B \in \mathbb{L}^\infty$ is mutually agreeable if and only if*

$$\nu^{(w)}(B;\gamma_1|\mathcal{E}_1) \leq \nu^{(b)}(B;\gamma_2|\mathcal{E}_2). \tag{3.1.1}$$

In that case, the set of mutually-agreeable prices for B is given by

$$[\nu^{(w)}(B;\gamma_1|\mathcal{E}_1), \nu^{(b)}(B;\gamma_2|\mathcal{E}_2)].$$

Remark 3.1.4.

1. A version of Proposition 3.1.3 for strict mutually-agreeable prices, with strict inequality in (3.1.1) and the interval $[\nu^{(w)}(B;\gamma_1|\mathcal{E}_1), \nu^{(b)}(B;\gamma_2|\mathcal{E}_2)]$ replaced by its interior $(\nu^{(w)}(B;\gamma_1|\mathcal{E}_1), \nu^{(b)}(B;\gamma_2|\mathcal{E}_2))$, holds.

2. For a contingent claim $B \in \mathbb{L}^\infty \setminus \mathcal{R}^\infty$, each (strictly) mutually agreeable price p of B satisfies $p \in (\inf_{\mathbb{Q}\in\mathcal{M}_a} \mathbb{E}^{\mathbb{Q}}[B], \sup_{\mathbb{Q}\in\mathcal{M}_a} \mathbb{E}^{\mathbb{Q}}[B])$ (see, e.g., [66],

Proposition 7.2), i.e., every mutually-agreeable price is an arbitrage-free price. Trivially, every claim $B \in \mathcal{R}^\infty$ is mutually-agreeable and the mutually-agreeable price is unique and equal to the unique arbitrage-free price.

Remark 3.1.5. Thanks to the choice of the exponential utility, the case where the agents have different subjective probability measures, say $\mathbb{P}_1 \neq \mathbb{P}_2$, is also covered. Indeed, if we assume that $\mathbb{P}_1 \approx \mathbb{P}_2$ and $\ln(\frac{d\mathbb{P}_1}{d\mathbb{P}_2}) \in \mathbb{L}^\infty$, we can reduce the analysis to the case of two agents with the same subjective measure, say $\mathbb{P}_2$, by adding $\gamma_1 \ln(\frac{d\mathbb{P}_1}{d\mathbb{P}_2})$ to the first agent's random endowment.

It will be important in the sequel to introduce the following notation for the set of all (strictly) mutually agreeable claims.

$$\mathcal{G} = \{B \in \mathbb{L}^\infty \,:\, B \text{ is mutually agreeable}\}\,, \text{ and,}$$

$$\mathcal{G}^\circ = \{B \in \mathbb{L}^\infty \,:\, B \text{ is strictly mutually agreeable}\}\,.$$

We remind the reader that $\mathcal{R}^\infty$ is the set of all replicable claims in $\mathbb{L}^\infty$.

Proposition 3.1.6. *The following statements hold*

1. $\mathcal{G}$ *is convex and* $\sigma(\mathbb{L}^\infty, \mathbb{L}^1)$*-closed.*

2. $\mathcal{G} \cap (-\mathcal{G}) = \mathcal{R}^\infty$, $\mathcal{G}^\circ \cap (-\mathcal{G}^\circ) = \emptyset$.

3. $\mathcal{G} = \mathbb{L}^\infty$ *if and only if* $\mathcal{R}^\infty = \mathbb{L}^\infty$.

Proof.

1. The convexity of $\mathcal{G}$ follows from the convexity of $\nu^{(w)}(\cdot;\gamma_1|\mathcal{E}_1)$ and the concavity of $\nu^{(b)}(\cdot;\gamma_2|\mathcal{E}_2)$ (see Proposition 2.2.6). As for its closedness, it is enough to note that $\nu^{(w)}(\cdot;\gamma_1|\mathcal{E}_1):\mathbb{L}^\infty\to\mathbb{R}$ is lower semi-continuous and $\nu^{(b)}(\cdot;\gamma_2|\mathcal{E}_2):\mathbb{L}^\infty\to\mathbb{R}$ is upper semi-continuous with respect to the weak-* topology $\sigma(\mathbb{L}^\infty,\mathbb{L}^1)$ (see Corollary 2.2.5).

2. Trivially, $\mathcal{R}^\infty\subseteq\mathcal{G}\cap(-\mathcal{G})$. For a claim $B\in\mathcal{G}\cap(-\mathcal{G})$, there exists $p,\hat{p}\in\mathbb{R}$ such that $p-B\in\mathcal{A}_{\gamma_1}(\mathcal{E}_1)$ and $B-p\in\mathcal{A}_{\gamma_2}(\mathcal{E}_2)$, as well as $\hat{p}+B\in\mathcal{A}_{\gamma_1}(\mathcal{E}_1)$ and $-B-\hat{p}\in\mathcal{A}_{\gamma_2}(\mathcal{E}_2)$. It follows, by the convexity of $\mathcal{A}_{\gamma_1}(\mathcal{E}_1)$ that
$$\frac{1}{2}(p+\hat{p})=\frac{1}{2}(p-B+\hat{p}+B)\in\mathcal{A}_{\gamma_1}(\mathcal{E}_1),$$
i.e., $u_{\gamma_1}(0|\mathcal{E}_1)\le u_{\gamma_1}(\frac{1}{2}(p+\hat{p})|\mathcal{E}_1)$. The strict monotonicity of the value function $u_{\gamma_1}(\cdot|\mathcal{E}_1)$, for deterministic arguments, implies that $\frac{1}{2}(p+\hat{p})\ge 0$. Applying the same line of reasoning to $\mathcal{A}_{\gamma_2}(\mathcal{E}_2)$ and the value function $u_{\gamma_2}(\cdot|\mathcal{E}_2)$, we get that $-\frac{1}{2}(p+\hat{p})\ge 0$, and, consequently, $p=-\hat{p}$. Using the definitions (2.1.6) and (2.1.7) of the conditional indifference prices, we easily get that
$$\nu^{(b)}(B;\gamma_1|\mathcal{E}_1)\ge p\ge\nu^{(w)}(B;\gamma_1|\mathcal{E}_1),$$
which, according to Corollary 3.2.2, implies that $B\in\mathcal{R}^\infty$.

   To prove the second claim, it suffices to observe that $\mathcal{G}^\circ\cap\mathcal{R}^\infty=\emptyset$. Indeed, $\nu^{(w)}(B;\gamma_1|\mathcal{E}_1)=\nu^{(b)}(B;\gamma_2|\mathcal{E}_2)=\mathbb{E}^{\mathbb{Q}}[B]$ for $B\in\mathcal{R}^\infty$ and all $\mathbb{Q}\in\mathcal{M}_a$.

3. If $\mathcal{G} = \mathbb{L}^\infty$ then $\mathbb{L}^\infty \subseteq \mathcal{G} \cap (-\mathcal{G})$ so $\mathbb{L}^\infty = \mathcal{R}^\infty$, by (2) above. Conversely, if $\mathbb{L}^\infty = \mathcal{R}^\infty$ then $\mathbb{L}^\infty = \mathcal{G} \cap (-\mathcal{G}) \subseteq \mathcal{G}$.

□

Remark 3.1.7. The weak-* topology $\sigma(\mathbb{L}^\infty, \mathbb{L}^1)$ in Proposition 3.1.6 can be replaced by an even weaker one, namely the coarsest topology τ on $\mathbb{L}^\infty$ which makes the expectation mappings $\mathbb{E}^{\mathbb{Q}}[\cdot] : \mathbb{L}^\infty \to \mathbb{R}$ continuous for each $\mathbb{Q} \in \mathcal{M}_{e,f}$.

3.2 No Agreement Without Random Endowments

In this section, we state an at first glance surprising result according to which mere difference in risk-aversion is not enough for two exponential agents to agree on a price for *any* contingent claim. Qualitatively, different random endowments are needed.

Proposition 3.2.1 (Non-agreement with replicable random endowments)**.** *Suppose that* $\mathcal{E}_1 \sim \mathcal{E}_2 \sim 0$*. Then* $\mathcal{G} = \mathcal{R}^\infty$ *and* $\mathcal{G}^\circ = \emptyset$*.*

Proof. The limiting relationships in (2.2.9) and the monotonicity properties of the indifference prices (see Proposition 2.2.13) imply that

$$\begin{aligned} \nu^{(w)}(B;\gamma_1|\mathcal{E}_1) &= \nu^{(w)}(B;\gamma_1) \geq \mathbb{E}^{\mathbb{Q}^{(0)}}[B] \\ &\geq \nu^{(b)}(B;\gamma_2) = \nu^{(b)}(B;\gamma_2|\mathcal{E}_2), \end{aligned} \tag{3.2.1}$$

for all $B \in \mathbb{L}^\infty$. Therefore, the strict inequality $\nu^{(w)}(B;\gamma_1|\mathcal{E}_1) < \nu^{(b)}(B;\gamma_2|\mathcal{E}_2)$ - needed for the strong agreement - cannot hold. Consequently, $\mathcal{G}^\circ = \emptyset$.

If $B \in \mathcal{G}$, (3.2.1) implies that $\nu^{(w)}(B;\gamma_1|\mathcal{E}_1) = \nu^{(w)}(B;\gamma_1) = \mathbb{E}^{\mathbb{Q}^{(0)}}[B] = \lim_{\gamma\to 0}\nu^{(w)}(B;\gamma)$. Therefore, the function $\gamma \mapsto \nu^{(w)}(B;\gamma)$ can not be strictly increasing on $(0,\infty)$, so, by Proposition 2.2.13, we must have $B \in \mathcal{R}^\infty$. Hence, $\mathcal{G} \subseteq \mathcal{R}^\infty$, which implies that $\mathcal{G} = \mathcal{R}^\infty$. □

The following result follows directly from Proposition 3.2.1 and the fact that the conditional indifference price becomes unconditional if the measure $\mathbb{P}$ is changed to $\mathbb{P}_{-\gamma\mathcal{E}}$ (see part 2 of Proposition 2.2.2).

Corollary 3.2.2. *Suppose that $\mathcal{E}_1 \sim \mathcal{E}_2$. Then for every $\gamma > 0$ and $B \in \mathbb{L}^\infty$, we have*

$$\begin{cases} \nu^{(w)}(B;\gamma|\mathcal{E}_1) > \nu^{(b)}(B;\gamma|\mathcal{E}_2) & \textit{for } B \notin \mathcal{R}^\infty, \\ \nu^{(w)}(B;\gamma|\mathcal{E}_1) = \nu^{(b)}(B;\gamma|\mathcal{E}_2) & \textit{otherwise.} \end{cases}$$

Remark 3.2.3. A non-agreement problem in the case where the agents' utility functions are defined on the positive real line is addressed in [23]. More precisely, the authors deal with the special case when the market incompleteness comes from the presence of "extraneous risk" and conclude that there is no agreement between the agents on non-replicable, not-necessarily-bounded claims. However, their arguments depend heavily on the special structure of the market and do not generalize directly to our setting.

3.3 Agreement With Random Endowments

Proposition 3.2.1 states that the absence of random endowments is a sufficient condition for the lack of (strict) agreement. Is it also necessary?

Given the result of Proposition 3.1.3, the question of the existence of non-replicable mutually agreeable claims leads to the following optimization problem with value function $\Sigma : (0,\infty)^2 \times (\mathbb{L}^\infty)^2 \to [0,+\infty]$, where

$$\Sigma(\gamma_1, \gamma_2, \mathcal{E}_1, \mathcal{E}_2) = \sup_{B\in\mathbb{L}^\infty} \big(\nu^{(b)}(B;\gamma_2|\mathcal{E}_2) - \nu^{(w)}(B;\gamma_1|\mathcal{E}_1)\big). \tag{3.3.1}$$

The following result holds directly from Definition 3.1.1 of the set $\mathcal{G}$.

Proposition 3.3.1. *For* $\mathcal{E}_1, \mathcal{E}_2 \in \mathbb{L}^\infty$, $\gamma_1, \gamma_2 \in (0,\infty)$ *and* $\Sigma = \Sigma(\gamma_1, \gamma_2, \mathcal{E}_1, \mathcal{E}_2)$, *the following two statements are equivalent*

1. $\mathcal{G}^\circ \neq \emptyset$

   and

2. $\Sigma > 0$.

Remark 3.3.2. The optimization problem above permits an interpretation in terms of the so-called *optimal risk-sharing problem*. In the case where the agents do not have access to a financial market, this problem has been addressed by many authors (see, e.g., [14] for its relation to the insurance-reinsurance problem, [48] and [9] for the exponential utility case, [52] for monetary utility functionals and [11] for concave preference functionals). When a financial market is present, the problem of optimal risk sharing when both agents have exponential utility has been studied in [10], where the authors focus on the form of the optimal structure.

Before we proceed, we introduce the following quantities.

Definition 3.3.3.

1. The sum $\mathcal{E} = \mathcal{E}_1 + \mathcal{E}_2$ of the random endowments of the agents is called the *aggregate endowment*.

2. A pair (B_1, B_2) in $(\mathbb{L}^\infty)^2$ is called an *allocation*, while an allocation (B_1, B_2) such that $B_1 + B_2 = \mathcal{E}$ is called a *feasible allocation*; the set of all feasible allocations will be denoted by $F(\mathcal{E})$

3. For an allocation (B_1, B_2), the sum $\nu^{(b)}(B_1; \gamma_1) + \nu^{(b)}(B_2; \gamma_2)$, denoted by $\sigma(B_1, B_2)$, is called the *score* of (B_1, B_2). The difference $\sigma(B_1, B_2) - \sigma(\mathcal{E}_1, \mathcal{E}_2)$ is called the *excess score* (where, for simplicity, the parameters γ_1 and γ_2 are omitted from the notation).

By Proposition 2.2.2, the expression $\nu^{(b)}(B; \gamma_2|\mathcal{E}_2) - \nu^{(w)}(B; \gamma_1|\mathcal{E}_1)$ appearing in (3.3.1) above can be rewritten as

$$\begin{aligned} \nu^{(b)}(B; \gamma_2|\mathcal{E}_2) - \nu^{(w)}(B; \gamma_1|\mathcal{E}_1) &= \nu^{(b)}(B; \gamma_2|\mathcal{E}_2) + \nu^{(b)}(-B; \gamma_1|\mathcal{E}_1) \\ &= \left(\nu^{(b)}(B + \mathcal{E}_2; \gamma_2) - \nu^{(b)}(\mathcal{E}_2; \gamma_2)\right) + \left(\nu^{(b)}(-B + \mathcal{E}_1; \gamma_1) - \nu^{(b)}(\mathcal{E}_1; \gamma_1)\right) \\ &= \sigma(\mathcal{E}_1 - B, \mathcal{E}_2 + B) - \sigma(\mathcal{E}_1, \mathcal{E}_2). \end{aligned} \tag{3.3.2}$$

Hence,

$$\Sigma(\gamma_1, \gamma_2, \mathcal{E}_1, \mathcal{E}_2) = \sup\{\sigma(B_1, B_2) : (B_1, B_2) \in F(\mathcal{E})\} - \sigma(\mathcal{E}_1, \mathcal{E}_2).$$

In other words, Σ is the maximized the excess score. If we think of the aggregate endowment $\mathcal{E}$ as the total wealth of our two-agent economy, the solution of

(3.3.1) (if it exists) will provide a redistribution of wealth so as to maximize the (improvement in) the score. Even though there is no direct economic reason why the sum of individual indifference prices should be maximized, Proposition 3.3.4 - which is a mere restatement of the discussion above - explains why the score is a useful concept.

Proposition 3.3.4. *For each $B \in \mathbb{L}^\infty$, the following two statements are equivalent*

1. $B \in \mathcal{G}^\circ$

   and

2. $\sigma(\mathcal{E}_1 - B, \mathcal{E}_2 + B) > \sigma(\mathcal{E}_1, \mathcal{E}_2)$.

The following proposition characterizes the score-optimal allocation, (compare to Theorem 2.3 in [10] and see also [14] and [48]).

Proposition 3.3.5. *For any $\mathcal{E}_1, \mathcal{E}_2 \in \mathbb{L}^\infty$ and $\gamma_1, \gamma_2 > 0$ there exists $B^* \in \mathbb{L}^\infty$ such that*

$$\sigma(\mathcal{E}_1 - B^*, \mathcal{E}_2 + B^*) \geq \sigma(B_1, B_2), \text{ for all } (B_1, B_2) \in F(\mathcal{E}).$$

Moreover, B^* is unique up to replicability and

$$B^* \sim \frac{\gamma_1 \mathcal{E}_1 - \gamma_2 \mathcal{E}_2}{\gamma_1 + \gamma_2}.$$

Proof. By (3.3.2), it suffices to show that

$$\begin{aligned} &\nu^{(w)}\left(-B^* - \mathcal{E}_2; \gamma_2\right) + \nu^{(w)}\left(B^* - \mathcal{E}_1; \gamma_1\right) \\ &\leq \nu^{(w)}\left(-B - \mathcal{E}_2; \gamma_2\right) + \nu^{(w)}\left(B - \mathcal{E}_1; \gamma_1\right), \end{aligned} \tag{3.3.3}$$

for all $B \in \mathbb{L}^\infty$. The left hand side of (3.3.3) equals to

$$\begin{aligned}
& \nu^{(w)}\left(-\frac{\gamma_1}{\gamma_1+\gamma_2}\mathcal{E};\gamma_2\right)+\nu^{(w)}\left(-\frac{\gamma_2}{\gamma_1+\gamma_2}\mathcal{E};\gamma_1\right) \\
= \; & \frac{1}{\gamma_2}v^{(w)}\left(-\frac{\gamma_1\gamma_2}{\gamma_1+\gamma_2}\mathcal{E};1\right)+\frac{1}{\gamma_1}v^{(w)}\left(-\frac{\gamma_1\gamma_2}{\gamma_1+\gamma_2}\mathcal{E};1\right) \\
= \; & v^{(w)}(-\mathcal{E};\tilde{\gamma}),
\end{aligned}$$

where we recall that $\tilde{\gamma} = \frac{1}{\gamma_1} + \frac{1}{\gamma_2}$. Hence (3.3.3) is equivalent to

$$\nu^{(w)}\left(B-\mathcal{E}_1;\gamma_1\right)+\nu^{(w)}\left(-B-\mathcal{E}_2;\gamma_2\right) \geq \nu^{(w)}\left(-\mathcal{E}_1-\mathcal{E}_2;\tilde{\gamma}\right),$$

which holds true by Lemma 2.2.9, where the equality is valid if and only if

$$\tfrac{\gamma_1}{\tilde{\gamma}}(B-\mathcal{E}_1) \sim \tfrac{\gamma_2}{\tilde{\gamma}}(-B-\mathcal{E}_2)$$

or equivalently if and only if

$$B \sim \frac{\gamma_1\mathcal{E}_1-\gamma_2\mathcal{E}_2}{\gamma_1+\gamma_2}.$$

□

Corollary 3.3.6. *The following statements are equivalent*

1. $\mathcal{G}^\circ = \emptyset$,
2. $B^* = \frac{\gamma_1\mathcal{E}_1-\gamma_2\mathcal{E}_2}{\gamma_1+\gamma_2}$ *is replicable*

   and

3. $\frac{\gamma_1}{\gamma_2}\mathcal{E}_1 \sim \mathcal{E}_2$.

Remark 3.3.7. We can relate the existence of mutually agreeable non-replicable claims with the well-known notion of *Pareto optimality*. More precisely, an allocation $(B_1, B_2) \in F(\mathcal{E})$ is called *Pareto optimal* if $\nexists (C_1, C_2) \in F(\mathcal{E})$ such that $B_i \preceq_{\gamma_i, \mathcal{E}_i} C_i$ for $i = 1, 2$ and $B_i \prec_{\gamma_i, \mathcal{E}_i} C_i$, for at least one $i = 1, 2$. It follows from Corollary 3.3.6, that the condition $\frac{\gamma_1}{\gamma_2}\mathcal{E}_1 \sim \mathcal{E}_2$ implies that the allocation $(\mathcal{E}_1, \mathcal{E}_2)$ is the unique (up to replicability) Pareto optimal one. If $\frac{\gamma_1}{\gamma_2}\mathcal{E}_1 \not\sim \mathcal{E}_2$, a transaction involving the optimal claim B^* will lead to a Pareto optimal allocation.

3.4 When is a Given Claim Mutually Agreeable?

In this section we attempt to answer the following question:

Provided that $\frac{\gamma_1}{\gamma_2}\mathcal{E}_1 \nsim \mathcal{E}_2$, *when does a given claim* B *belong to* $\mathcal{G}$*?*

For this, we propose two approaches: one through the notion of residual risk and the other based on the asymptotic approximation of the conditional indifference prices for small quantities.

3.4.1 Agreement and residual risk

As we have seen, the writer's residual risk of a claim B is the difference of the agent's liability and the optimal wealth at maturity. Intuitively, this means that a given claim is mutually agreeable if trading this claim results in improvement of the agents' residual risk exposure. Below, we make this statement precise.

First we recall from Remark 2.3.1 that

$$\nu^{(b)}(R^{(w)}(B;\gamma|\mathcal{E});\gamma|\mathcal{E}) = \nu^{(b)}(B;\gamma|\mathcal{E}) - \nu^{(w)}(B;\gamma|\mathcal{E}),$$

and hence the agreement condition (3.1.1) can also be written as

$$B \in \mathcal{G} \Leftrightarrow \nu^{(b)}(R^{(w)}(B;\gamma|\mathcal{E});\gamma|\mathcal{E}) \geq 0.$$

The following proposition gives a characterization of mutually agreeable contingent claims in terms of their residual risk.

Proposition 3.4.1. *For $B \in \mathbb{L}^\infty$, the following statements are equivalent*

1. *$B \in \mathcal{G}$,*

2. *the inequality*

$$\mathbb{E}^{\mathbb{Q}}\left[R^{(w)}(B;\gamma_1|\mathcal{E}_1)\right] + \mathbb{E}^{\mathbb{Q}}\left[R^{(b)}(B;\gamma_2|\mathcal{E}_2)\right] \geq 0 \tag{3.4.1}$$

   holds for some $\mathbb{Q} \in \mathcal{M}_{e,f}$,

3. *the inequality*

$$\begin{aligned} &\mathbb{E}^{\mathbb{Q}}\left[R^{(w)}(B-\mathcal{E}_1;\gamma_1) - R^{(w)}(-\mathcal{E}_1;\gamma_1)\right] + \\ &\mathbb{E}^{\mathbb{Q}}\left[R^{(w)}(-B-\mathcal{E}_2;\gamma_2) - R^{(w)}(-\mathcal{E}_2;\gamma_2)\right] \geq 0 \end{aligned} \tag{3.4.2}$$

   holds for some $\mathbb{Q} \in \mathcal{M}_{e,f}$,

4. *the inequality* (3.4.1) *holds for all $\mathbb{Q} \in \mathcal{M}_{e,f}$*

   and

5. *the inequality* (3.4.2) *holds for all $\mathbb{Q} \in \mathcal{M}_{e,f}$.*

Proof. It is enough to make the following two observations

(a) thanks to the definition (2.3.1) of residual risk, we have that the differences $\nu^{(w)}(B;\gamma_1|\mathcal{E}_1) - R^{(w)}(B;\gamma_1|\mathcal{E}_1)$ and $\nu^{(b)}(B;\gamma_2|\mathcal{E}_2) - R^{(b)}(B;\gamma_2|\mathcal{E}_2)$ are both of the form $\int_0^T \boldsymbol{\vartheta}_t \, d\mathbf{S}_t$ with $\boldsymbol{\vartheta} \in \boldsymbol{\Theta}$, and

(b) the following equality holds

$$\begin{aligned} R^{(w)}(B;\gamma_1|\mathcal{E}_1) - R^{(b)}(B;\gamma_2|\mathcal{E}_2) = R^{(w)}(B - \mathcal{E}_1;\gamma_1) - R^{(w)}(-\mathcal{E}_1;\gamma_1) \\ + R^{(w)}(-B - \mathcal{E}_2;\gamma_2) - R^{(w)}(-\mathcal{E}_2;\gamma_2). \end{aligned}$$

□

Remark 3.4.2. It should be pointed out that it is enough to check the above inequalities only for some probability measure in $\mathcal{M}_{e,f}$. Also, it follows from the definition of the residual risk that inequality (3.4.1) implies that the transaction of claim B at any price p will decrease the sum of expected residual risks. If in addition $p \in (\nu^{(w)}(B;\gamma_1|\mathcal{E}_1), \nu^{(b)}(B;\gamma_2|\mathcal{E}_2))$, each agent's expected residual risk will be decreased.

Under the additional mild assumption of continuity for the filtration $\mathbb{F}$, we can replace the criterion given in Proposition 3.4.1 by the following one, which sometimes is easier to check (see Section 2.3 for the additional notation).

Proposition 3.4.3. *Suppose that $\mathbb{F}$ is continuous. For $B \in \mathbb{L}^\infty$, the following two statements are equivalent*

1. $B \in \mathcal{G}$

and

2. $\gamma_1 \mathbb{E}^{\mathbb{Q}^{(0)}}\left[\left\langle R^{(w)}(B-\mathcal{E}_1;\gamma_1)\right\rangle_T - \left\langle R^{(w)}(-\mathcal{E}_1;\gamma_1)\right\rangle_T\right] +$
$$+\gamma_2 \mathbb{E}^{\mathbb{Q}^{(0)}}\left[\left\langle R^{(w)}(-B+\mathcal{E}_2;\gamma_2)\right\rangle_T - \left\langle R^{(w)}(\mathcal{E}_2;\gamma_2)\right\rangle_T\right] \geq 0.$$

Proof. It follows by part 2 of Theorem A.2.1 that $\left\langle R^{(w)}(B;\gamma)\right\rangle_t = \left\langle L^{(w)}(B;\gamma)\right\rangle_t$ for all $t \in [0,T]$. Hence, $R_t^{(w)}(B;\gamma) - \frac{\gamma}{2}\left\langle R^{(w)}(B;\gamma)\right\rangle_t$ is a $\mathbb{Q}^{(0)}-$martingale, for any $\gamma > 0$ and $B \in \mathbb{L}^\infty$. The desired equivalence then follows by Proposition 3.4.1. □

Remark 3.4.4. In the case where a claim B is mutually agreeable, the exact price $p \in \left(\nu^{(b)}(B;\gamma|\mathcal{E}), \nu^{(w)}(B;\gamma|\mathcal{E})\right)$ at which the transaction will take place in not determined by the arguments above. For the specification of this price, a negotiation model is necessary. A pricing scheme related to this problem is given in [3], where the agents' risk preferences are modelled by general utility functions.

Example 3.4.5. We adopt the setup of Example 2.2.3 and suppose that $\mathcal{E}_1 = g_1(Y_T)$ and $\mathcal{E}_2 = g_2(Y_T)$ for some Borel bounded functions, g_1 and g_2. Proposition 3.1.3 and representation (2.2.2) imply that $B = g(Y_T) \in \mathcal{G}$ if and only if

$$\left(\frac{\mathbb{E}^{\mathbb{Q}^{(0)}}\left(e^{\gamma_1 \tilde{B}} e^{-\gamma_1 \tilde{\mathcal{E}}_1}\right)}{\mathbb{E}^{\mathbb{Q}^{(0)}}\left(e^{-\gamma_1 \tilde{\mathcal{E}}_1}\right)}\right)^{\frac{\gamma_2}{\gamma_1}} \leq \frac{\mathbb{E}^{\mathbb{Q}^{(0)}}\left(e^{-\gamma_2 \tilde{\mathcal{E}}_2}\right)}{\mathbb{E}^{\mathbb{Q}^{(0)}}\left(e^{-\gamma_2 \tilde{\mathcal{E}}_2} e^{-\gamma_2 \tilde{B}}\right)},$$

where $\tilde{B} = (1-\rho^2)B$ and $\tilde{\mathcal{E}}_i = (1-\rho^2)\mathcal{E}_i$, $i = 1,2$.

As we have seen, $\nu^{(w)}(B;\gamma_1) > \nu^{(b)}(B;\gamma_2)$, $\forall B \nsim 0$. It is easy to verify that $\nu^{(w)}(B;\gamma_1|\mathcal{E}_1) \leq \nu^{(w)}(B;\gamma_1)$ if and only if $\mathrm{Cov}^{\mathbb{Q}^{(0)}}(\mathcal{E}_1, B) \geq 0$, where $\mathrm{Cov}^{\mathbb{Q}^{(0)}}(.,.)$ is the covariance under the measure $\mathbb{Q}^{(0)}$. This means that the presence of a random endowment which is positively correlated with the claim payoff, reduces the writer's indifference price.

Similarly, $\nu^{(b)}(B;\gamma_2|\mathcal{E}_2) \geq \nu^{(b)}(B;\gamma_2)$ if and only if $\mathrm{Cov}^{\mathbb{Q}^{(0)}}(\mathcal{E}_2, B) \leq 0$. Therefore, we infer that a necessary condition for a claim B to be mutually agreeable is that $\mathrm{Cov}^{\mathbb{Q}^{(0)}}(\mathcal{E}_1, B) > 0$, or $\mathrm{Cov}^{\mathbb{Q}^{(0)}}(\mathcal{E}_2, B) < 0$.

3.4.2 Agreement and price approximation

Although the asymptotic expansion (2.4.6) in Proposition 2.4.6 above is important in its own right, it can also be applied to provide the following criterion for mutual agreement for a small quantity of a given contingent claim. In other words, when the size of the claim whose price is negotiated is small compared to the sizes of agent's contingent claims (and this is typically the case in practice), one can use a Taylor-type expansion of the indifference price around 0, and obtain more precise quantitative answers to the agreement question. This is discussed next.

Proposition 3.4.6. *Suppose that $\mathbb{F}$ is continuous and that the random endowments $\mathcal{E}_1, \mathcal{E}_2 \in \mathbb{L}^\infty$ and risk-aversion coefficients $\gamma_1, \gamma_2 > 0$ are chosen. Let $B \in \mathbb{L}^\infty$ be a given contingent claim. The set $\mathcal{G}^\circ$ contains a segment of the form*

$$\{\alpha B: \ \alpha \in (0, \alpha_0)\} \text{ for some } \alpha_0 > 0 \text{ iff }, \mathbb{E}^{\mathbb{Q}^{(-\gamma_1 \mathcal{E}_1)}}[B] < \mathbb{E}^{\mathbb{Q}^{(-\gamma_2 \mathcal{E}_2)}}[B].$$

Similarly, the set $\mathcal{G}^\circ$ contains a segment of the form

$$\{\alpha B: \ \alpha \in (-\alpha_0, 0)\} \text{ for some } \alpha_0 > 0 \text{ iff }, \mathbb{E}^{\mathbb{Q}^{(-\gamma_1 \mathcal{E}_1)}}[B] > \mathbb{E}^{\mathbb{Q}^{(-\gamma_2 \mathcal{E}_2)}}[B].$$

Proof. We first note that the convexity of $\mathcal{G}^\circ$ implies that, if there exists $\alpha_0 > 0$ such that $\alpha_0 B \in \mathcal{G}^\circ$, then $\alpha B \in \mathcal{G}^\circ$, for all $\alpha \in (0, \alpha_0]$. By equation (2.4.6),

$$\nu^{(w)}(\alpha B; \gamma_1|\mathcal{E}_1) = \alpha \mathbb{E}^{\mathbb{Q}^{(-\gamma_1 \mathcal{E}_1)}}[B] + o(\alpha)$$

and

$$\nu^{(b)}(\alpha B; \gamma_2|\mathcal{E}_2) = \alpha \mathbb{E}^{\mathbb{Q}^{(-\gamma_2 \mathcal{E}_2)}}[B] + o(\alpha).$$

Hence, the inequality $\mathbb{E}^{\mathbb{Q}^{(-\gamma_1 \mathcal{E}_1)}}[B] < \mathbb{E}^{\mathbb{Q}^{(-\gamma_2 \mathcal{E}_2)}}[B]$ yields that there exists $\alpha_0 > 0$ small enough such that $\nu^{(w)}(\alpha_0 B; \gamma_1|\mathcal{E}_1) < \nu^{(b)}(\alpha_0 B; \gamma_2|\mathcal{E}_2)$, i.e., $\alpha_0 B \in \mathcal{G}^\circ$.

On the other hand, suppose that there exists $\alpha_0 > 0$ such that $\alpha_0 B \in \mathcal{G}^\circ$ and assume that $\mathbb{E}^{\mathbb{Q}^{(-\gamma_1 \mathcal{E}_1)}}[B] \geq \mathbb{E}^{\mathbb{Q}^{(-\gamma_2 \mathcal{E}_2)}}[B]$. It is easy to check that $\Delta^{\mathbb{Q}^{(-\gamma_i \mathcal{E}_i)}}(B) > 0$ for $i = 1, 2$ (since $B \notin \mathcal{R}^\infty$). Also, by (2.4.6) and its buyer's version, we get

$$\alpha \mathbb{E}^{\mathbb{Q}^{(-\gamma_1 \mathcal{E}_1)}}[B] + \frac{\alpha^2 \gamma_1}{2} \Delta^{\mathbb{Q}^{(-\gamma_1 \mathcal{E}_1)}}(B) < \alpha \mathbb{E}^{\mathbb{Q}^{(-\gamma_2 \mathcal{E}_2)}}[B] - \frac{\alpha^2 \gamma_2}{2} \Delta^{\mathbb{Q}^{(-\gamma_2 \mathcal{E}_2)}}(B) + o(\alpha^2)$$

for every α close to zero such that $0 < \alpha \leq \alpha_0$ (note that thanks to the linearity of $\mathbf{\Theta}^2_{\mathbb{Q}}$, we have $\Delta^{\mathbb{Q}}(B) = \Delta^{\mathbb{Q}}(-B)$, $\forall B \in \mathbb{L}^\infty$).

This implies that for any such α

$$\frac{\alpha^2}{2}(\gamma_1 \Delta^{\mathbb{Q}^{(-\gamma_1 \mathcal{E}_1)}}(B) + \gamma_2 \Delta^{\mathbb{Q}^{(-\gamma_2 \mathcal{E}_2)}}(B)) + o(\alpha^2) < 0,$$

Dividing through by α^2 and letting $\alpha \to 0$, we get that

$$\gamma_1 \Delta^{\mathbb{Q}^{(-\gamma_1 \mathcal{E}_1)}}(B) + \gamma_2 \Delta^{\mathbb{Q}^{(-\gamma_2 \mathcal{E}_2)}}(B) \leq 0,$$

which is a contradiction. The proof of the second argument is similar and hence omitted. □

Remark 3.4.7. It is clear from Proposition 3.4.6 that the order of the marginal prices, $\mathbb{E}^{\mathbb{Q}^{(-\gamma_1\mathcal{E}_1)}}[B]$ and $\mathbb{E}^{\mathbb{Q}^{(-\gamma_2\mathcal{E}_2)}}[B]$, specifies which of the agents is willing to be the writer and which the seller of some units of claim B. We can further provide an approximation of the size of the set of the agreement prices for small number of units. More precisely, if $\mathbb{E}^{\mathbb{Q}^{(-\gamma_1\mathcal{E}_1)}}[B] \neq \mathbb{E}^{\mathbb{Q}^{(-\gamma_2\mathcal{E}_2)}}[B]$, it holds that

$$\begin{aligned} &\nu^{(b)}(\alpha B;\gamma_2|\mathcal{E}_2) - \nu^{(w)}(\alpha B;\gamma_1|\mathcal{E}_1) \\ &= \alpha(\mathbb{E}^{\mathbb{Q}^{(-\gamma_2\mathcal{E}_2)}}[B]-\mathbb{E}^{\mathbb{Q}^{(-\gamma_1\mathcal{E}_1)}}[B])-\frac{\alpha^2}{2}(\gamma_1\Delta^{\mathbb{Q}^{(-\gamma_1\mathcal{E}_1)}}(B)+\gamma_2\Delta^{\mathbb{Q}^{(-\gamma_2\mathcal{E}_2)}}(B))+o(\alpha^2), \end{aligned} \tag{3.4.3}$$

for every $\alpha \in \mathbb{R}$ close to zero.

Chapter 4

The Partial Equilibrium Pricing

This chapter deals with the existence and uniqueness of a partial equilibrium price of a contingent claim in our two-agents economy. The discussion of mutual agreeability in the previous chapter assumed that the number of units is fixed and the claim is indivisible. If, however, the negotiation between agents involves both the quantity traded and the price, and if this quantity is not constrained by quantization, a great deal more can be said about the outcome of the negotiation. The main advantage is that the methodology of equilibrium theory can be applied and a unique price-quantity pair singled out on the basis of the fundamental economic principle of market clearing. We stress that the agents take the form of the claims as given and do not engage in any form of optimal design, as is the case in many real-world situations in insurance, illiquid markets, bulk sales of shares, etc.

We divide this chapter into two sections. In the first one, we introduce some helpful notation and we define and analyze the utility-based demand function on a given vector of claims. In the second section, we state the main theorem of our equilibrium pricing together with a number of related remarks.

Since agents' acceptance sets depend on every undertaken risky investment, it is more realistic for a pricing rule to apply to a vector of claims instead of a single one. In what follows, we fix a vector of contingent claims, $\boldsymbol{B} = (B_1, B_2, \ldots, B_n) \in (\mathbb{L}^\infty)^n$.

4.1 The Demand Function

In this section, we introduce the notion of the *demand function* and analyze some of its properties. First, we define the "restrictions" $\boldsymbol{U}_i : \overline{\mathbb{R}}^n \times \mathbb{R}^n \to \mathbb{R}$, $i \in \{1, 2\}$ of the value functions $u_{\gamma_1}(\cdot|\mathcal{E}_1)$ and $u_{\gamma_2}(\cdot|\mathcal{E}_2)$ (see (2.1.2)) by

$$\boldsymbol{U}_i(\boldsymbol{\alpha}; \boldsymbol{p}) = \begin{cases} u_{\gamma_i}(\boldsymbol{\alpha} \cdot (\boldsymbol{B} - \boldsymbol{p})|\mathcal{E}_i), & \boldsymbol{\alpha} \in \mathbb{R}^n, \\ \limsup_{\boldsymbol{\alpha}' \to \boldsymbol{\alpha},\, \boldsymbol{\alpha}' \in \mathbb{R}^n} \boldsymbol{U}_i(\boldsymbol{\alpha}'; \boldsymbol{p}), & \boldsymbol{\alpha} \in \overline{\mathbb{R}}^n \setminus \mathbb{R}^n, \end{cases} \qquad \boldsymbol{p} \in \mathbb{R}^n, \tag{4.1.1}$$

for $i \in \{1, 2\}$, where $\overline{\mathbb{R}}$ denotes the extended set $\mathbb{R} \cup \{\pm\infty\}$ of real numbers. In other words, $\boldsymbol{U}_i(\cdot; \boldsymbol{p})$ is the extension of the continuous function $\boldsymbol{U}_i(\cdot; \boldsymbol{p})\Big|_{\mathbb{R}^n}$ to $\overline{\mathbb{R}}^n$ by upper semi-continuity and gives the indirect utility of agent i when she holds $\boldsymbol{\alpha}$ units of $\boldsymbol{B}$, purchased at price $\boldsymbol{p}$.

Definition 4.1.1. The *demand correspondence* $Z_i : \mathbb{R}^n \to 2^{\overline{\mathbb{R}}^n}$, for the agent $i \in \{1, 2\}$, is defined by

$$Z_i(\boldsymbol{p}) = \operatorname{argmax} \left\{ \boldsymbol{U}_i(\boldsymbol{\alpha}, \boldsymbol{p}) \,:\, \boldsymbol{\alpha} \in \overline{\mathbb{R}}^n \right\}, \ \boldsymbol{p} \in \mathbb{R}^n. \tag{4.1.2}$$

Intuitively, the elements of $Z_i(\boldsymbol{p})$ give the numbers of units of $\boldsymbol{B}$ that agent i is willing to purchase at price $\boldsymbol{p}$ (the numbers of units that maximize her indirect utility).

For the remainder of this chapter we enforce the following assumption.

Assumption 4.1.2. There exists no $\boldsymbol{\alpha} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ such that $\boldsymbol{\alpha} \cdot \boldsymbol{B} \sim 0$.

In other words, through Assumption 4.1.2 we impose that there is no linear combination of the claims $B_1, B_2, ..., B_n$ which can be replicated by the traded assets.

Before we present some properties of the demand function, we need to introduce some further notation. We denote by $\mathcal{P}^{NA} \subseteq \mathbb{R}^n$ the set of all arbitrage-free price-vectors of the contingent claims $\boldsymbol{B}$, i.e.,

$$\mathcal{P}^{NA} = \left\{ \mathbb{E}^{\mathbb{Q}}[\boldsymbol{B}] \,:\, \mathbb{Q} \in \mathcal{M}_e \right\},$$

where, as usual, $\mathbb{E}^{\mathbb{Q}}[\boldsymbol{B}] = (\mathbb{E}^{\mathbb{Q}}[B_1], \dots, \mathbb{E}^{\mathbb{Q}}[B_n]) \in \mathbb{R}^n$. To simplify the notation, we introduce two n-dimensional families of measures in $\mathcal{M}_e$, parameterized by $\boldsymbol{\alpha} \in \mathbb{R}^n$,

$$\mathbb{Q}_i^{(\boldsymbol{\alpha})} = \mathbb{Q}^{(\gamma_i \boldsymbol{\alpha} \cdot \boldsymbol{B} - \gamma_i \mathcal{E}_i)}, \;\; \boldsymbol{\alpha} \in \mathbb{R}^n, \;\; i = 1, 2.$$

We, then, define the following sets

$$\mathcal{P}_i^U = \left\{ \mathbb{E}^{\mathbb{Q}_i^{(\boldsymbol{\alpha})}}[\boldsymbol{B}] \,:\, \boldsymbol{\alpha} \in \mathbb{R}^n \right\}, \text{ for } i = 1, 2.$$

In general, $\mathcal{P}_i^U \subseteq \mathcal{P}^{NA}$, for $i = 1, 2$. The equality holds when $\mathcal{E}_i \sim 0$ (see [51], Lemma 7.1). For future use, we define the function $\boldsymbol{u}_i : \mathbb{R}^n \to \overline{\mathbb{R}}$ by

$\boldsymbol{u}_i(\boldsymbol{p}) = \sup_{\boldsymbol{\alpha}\in\bar{\mathbb{R}}^n} \{\boldsymbol{U}_i(\boldsymbol{\alpha};\boldsymbol{p})\}$, for $i \in \{1,2\}$. Building on the notation of Section 2.4, we also introduce the following two shorthands

$$\left.\begin{aligned} w_i(\boldsymbol{\alpha}) &= \nu^{(w)}(\boldsymbol{\alpha}\cdot\boldsymbol{B};\gamma_i|\mathcal{E}_i) \\ b_i(\boldsymbol{\alpha}) &= \nu^{(b)}(\boldsymbol{\alpha}\cdot\boldsymbol{B};\gamma_i|\mathcal{E}_i) \end{aligned}\right\} \quad \boldsymbol{\alpha}\in\mathbb{R}^n,\ i\in\{1,2\}\,. \tag{4.1.3}$$

Lemma 4.1.3. *For $i = 1, 2$, w_i is strictly convex and b_i is strictly concave.*

Proof. A change of the probability measure $\mathbb{P}$ to $\mathbb{P}_{-\gamma_i\mathcal{E}_i}$ can be employed to justify no loss of generality if we assume that $\mathcal{E}_i = 0$ in this proof. The fact that $w_i(\cdot)$ is convex follows from the convexity of the indifference price. In order to establish that the convexity is, in fact, strict, we assume, to the contrary, that there exist $\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2 \in \mathbb{R}^n$ with $\boldsymbol{\alpha}_1 \neq \boldsymbol{\alpha}_2$ and $\lambda \in (0,1)$ such that

$$w_i(\lambda\boldsymbol{\alpha}_1 + (1-\lambda)\boldsymbol{\alpha}_2) = \lambda w_i(\boldsymbol{\alpha}_1) + (1-\lambda)w_i(\boldsymbol{\alpha}_2).$$

Equivalently, we then have

$$\nu^{(w)}\left((\lambda\boldsymbol{\alpha}_1 + (1-\lambda)\boldsymbol{\alpha}_2)\cdot\boldsymbol{B};\gamma_i\right) = \nu^{(w)}\left(\lambda\boldsymbol{\alpha}_1\cdot\boldsymbol{B};\tfrac{\gamma_i}{\lambda}\right) + \nu^{(w)}\left((1-\lambda)\boldsymbol{\alpha}_2\cdot\boldsymbol{B};\tfrac{\gamma_i}{1-\lambda}\right)$$

Since $(\frac{\gamma_i}{\lambda})^{-1} + (\frac{\gamma_i}{1-\lambda})^{-1} = (\gamma_i)^{-1}$, we can use Lemma 2.2.9 to conclude that

$$\boldsymbol{\alpha}_1\cdot\boldsymbol{B} \sim \boldsymbol{\alpha}_2\cdot\boldsymbol{B},\ \text{i.e.,}\ \boldsymbol{\alpha}\cdot\boldsymbol{B}\sim 0,\ \text{where}\ \boldsymbol{\alpha} = \boldsymbol{\alpha}_1 - \boldsymbol{\alpha}_2 \neq 0 \in \mathbb{R}^n,$$

a contradiction with Assumption 4.1.2. A similar argument can be employed to prove the strict concavity of b_i, $i \in \{1,2\}$. □

Proposition 4.1.4. *For $i \in \{1,2\}$, the functions $\boldsymbol{u}_i(\cdot)$ and $Z_i(\cdot)$ have the following properties*

1. *The maximum in* (4.1.2) *is always attained, i.e.,* $Z_i(\boldsymbol{p}) \neq \emptyset$, *for all* $\boldsymbol{p} \in \mathbb{R}^n$.

2. *For* $\boldsymbol{p} \in \mathbb{R}^n$, *we have*

$$Z_i(\boldsymbol{p}) = \underset{\boldsymbol{\alpha}\in\mathbb{R}^n}{\operatorname{argmax}}\{\nu^{(b)}(\boldsymbol{\alpha}\cdot\boldsymbol{B};\gamma_i|\mathcal{E}_i) - \boldsymbol{\alpha}\cdot\boldsymbol{p}\}. \tag{4.1.4}$$

3. *Either* $Z_i(\boldsymbol{p}) = \{\boldsymbol{\alpha}\}$ *for some* $\boldsymbol{\alpha} \in \mathbb{R}^n$ *or* $Z_i \subseteq \overline{\mathbb{R}}^n \setminus \mathbb{R}^n$.

4. $Z_i(\boldsymbol{p}) = \{\boldsymbol{\alpha}\}$ *if and only if* $\mathbb{E}^{\mathbb{Q}_i^{(\boldsymbol{\alpha})}}[\boldsymbol{B}] = \boldsymbol{p}$ *(in particular,* $\boldsymbol{p} \in \mathcal{P}_i^U$*).*

Proof.

1. It follows from the fact that the function Z_i is upper semi-continuous on the compact space $\overline{\mathbb{R}}^n$.

2. It suffices to observe that (2.1.7) implies that

$$\boldsymbol{u}_i(\boldsymbol{p}) = \exp\{-\gamma_i \sup_{\boldsymbol{\alpha}\in\bar{R}^n}(\nu^{(b)}(\boldsymbol{\alpha}\cdot\boldsymbol{B};\gamma_i|\mathcal{E}_i) - \boldsymbol{\alpha}\cdot\boldsymbol{p})\}\big(u_{\gamma_i}(0|\mathcal{E}_i)\big), \tag{4.1.5}$$

for all $\boldsymbol{p} \in \mathbb{R}^n$.

3. The set $Z_i(\boldsymbol{p})$ is convex, so if it contains a point in $\mathbb{R}^n$ and a point in $\overline{\mathbb{R}}^n \setminus \mathbb{R}^n$, it must contain infinitely many points in $\mathbb{R}^n$. This is in contradiction with the strict concavity of b_i on $\mathbb{R}^n$.

4. Proposition 2.4.1 states that b_i is continuously differentiable on $\mathbb{R}^n$ and that $\nabla b_i(\boldsymbol{\alpha}) = \mathbb{E}^{\mathbb{Q}_i^{(-\boldsymbol{\alpha})}}[\boldsymbol{B}]$. Therefore, $\nu^{(b)}(\boldsymbol{\alpha}\cdot\boldsymbol{B};\gamma_i|\mathcal{E}_i) - \boldsymbol{\alpha}\cdot\boldsymbol{p}$ is a concave and differentiable function of $\boldsymbol{\alpha} \in \mathbb{R}^n$ and its derivative is given by

$\mathbb{E}^{\mathbb{Q}_i^{(-\boldsymbol{\alpha})}}[\boldsymbol{B}]-\boldsymbol{p}$. Consequently, $\nu^{(b)}(\boldsymbol{\alpha}\cdot\boldsymbol{B};\gamma_i|\mathcal{E}_i)-\boldsymbol{\alpha}\cdot\boldsymbol{p}$ attains its maximum on $\mathbb{R}^n$ if and only if $\mathbb{E}^{\mathbb{Q}_i^{(-\boldsymbol{\alpha})}}[\boldsymbol{B}]=\boldsymbol{p}$ has a solution $\boldsymbol{\alpha}\in\mathbb{R}^n$. In that case, $Z_i(\boldsymbol{p})=\{\boldsymbol{\alpha}\}$.

□

4.2 Partial Equilibrium Price-Quantity

Using the definition of the demand function for a fixed vector of claims and the notation introduced above, we are ready to give the definition of the *partial equilibrium price-quantity.*

Definition 4.2.1. A pair $(\boldsymbol{p},\boldsymbol{\alpha})\in\mathbb{R}^n\times\mathbb{R}^n$ is called a *partial-equilibrium price-quantity (PEPQ)* if

$$\boldsymbol{\alpha}\in Z_1(\boldsymbol{p}) \text{ and } -\boldsymbol{\alpha}\in Z_2(\boldsymbol{p}). \tag{4.2.1}$$

A vector $\boldsymbol{p}\in\mathbb{R}^n$ for which there exists $\boldsymbol{\alpha}\in\mathbb{R}^n$ such that $(\boldsymbol{p},\boldsymbol{\alpha})$ is a PEPQ is called a *partial-equilibrium price (PEP).*

In other words, the PEP is the price-vector of the contingent claim $\boldsymbol{B}$ at which the quantity that one agent is willing to sell is equal to the quantity which the other agent is wants to buy.

Taking into account Proposition 4.1.4, we are able to provide the equation that the equilibrium quantity must satisfy.

Proposition 4.2.2. *A pair* $(\hat{\boldsymbol{p}}, \hat{\boldsymbol{\alpha}})$ *is a PEPQ if and only if* $\hat{\boldsymbol{p}} \in \mathcal{P}_1^U \cap \mathcal{P}_2^U$*,* $\boldsymbol{\alpha} \in \mathbb{R}^n$ *and*

$$\mathbb{E}^{\mathbb{Q}_1^{(\hat{\boldsymbol{\alpha}})}}[\boldsymbol{B}] = \mathbb{E}^{\mathbb{Q}_2^{(-\hat{\boldsymbol{\alpha}})}}[\boldsymbol{B}] = \hat{\boldsymbol{p}}. \tag{4.2.2}$$

Proof. If $(\hat{\boldsymbol{p}}, \hat{\boldsymbol{\alpha}})$ is a PEPQ, then $Z_i(\hat{\boldsymbol{p}}) \cap \mathbb{R}^n \neq \emptyset$ and, so, by Proposition 4.1.4, part 3., we must have $Z_i(\hat{\boldsymbol{p}}) = \{\boldsymbol{\alpha}_i\}$, for some $\boldsymbol{\alpha}_i \in \mathbb{R}^n$ and $\hat{\boldsymbol{p}} \in \mathcal{P}_i^U$, for $i = 1, 2$. By (4.2.1), we have $\boldsymbol{\alpha}_1 = -\boldsymbol{\alpha}_2$. The equalities in (5.4.8), with $\hat{\boldsymbol{\alpha}} = \boldsymbol{\alpha}_1$ follow directly from part 4. of Proposition 4.1.4.

Conversely, suppose that (5.4.8) holds. Then, by part 4. of Proposition 4.1.4, we have $Z_1(\boldsymbol{p}) = \{\hat{\boldsymbol{\alpha}}\}$ and $Z_2(\boldsymbol{p}) = \{-\hat{\boldsymbol{\alpha}}\}$, which, in turn, implies (4.2.1). □

We have also shown the following result which will be used shortly:

Corollary 4.2.3. *A pair* $(\hat{\boldsymbol{p}}, \hat{\boldsymbol{\alpha}}) \in (\mathcal{P}_1^U \cap \mathcal{P}_2^U) \times \mathbb{R}^n$ *is a PEPQ if and only if*

$$w_1(\hat{\boldsymbol{\alpha}}) - b_2(\hat{\boldsymbol{\alpha}}) \leq w_1(\boldsymbol{\alpha}) - b_2(\boldsymbol{\alpha}) \text{ for any } \boldsymbol{\alpha} \in \mathbb{R}^n, \text{ and } \hat{\boldsymbol{p}} = \nabla w_1(\hat{\boldsymbol{\alpha}}).$$

The main result of this section is presented in the following Theorem.

Theorem 4.2.4. *Let* $\mathcal{E}_1, \mathcal{E}_2 \in \mathbb{L}^\infty$*,* $\gamma_1, \gamma_2 > 0$ *and* $\boldsymbol{B} \in (\mathbb{L}^\infty)^n$ *be arbitrary, and suppose that the Assumption 4.1.2 is satisfied. Then, there exists a unique partial equilibrium price-quantity* $(\boldsymbol{\alpha}, \boldsymbol{p}) \in \mathbb{R}^n \times \mathbb{R}^n$*. Moreover,* $\boldsymbol{p} \in \mathcal{P}_1^U \cap \mathcal{P}_2^U$*.*

Proof. If the PEPQ $(\boldsymbol{\alpha}, \boldsymbol{p})$ exists, then $\boldsymbol{\alpha}$ globally minimizes the strictly concave function $w_1 - b_2$, so it must be unique. To establish existence, it will be enough to solve the equation $\nabla f = 0$, where $f = w_1 - b_2$. Assume, to the contrary, that $\nabla f(\boldsymbol{\alpha}) \neq 0$, for all $\boldsymbol{\alpha} \in \mathbb{R}^n$. The continuity of f implies that, for each $m \in \mathbb{N}$, there exists $\boldsymbol{\alpha}_m \in \overline{B}_m = \{\boldsymbol{\alpha} \in \mathbb{R}^n \,:\, \sum_{i=1}^n |\alpha_i| \leq m\}$ such that $f(\boldsymbol{\alpha}_m) \leq f(\boldsymbol{\alpha})$ for all $\boldsymbol{\alpha} \in \overline{B}_m$. Thanks to the strict convexity of f and the fact that $\nabla f \neq \mathbf{0}$ on $\overline{B}_m$, we must have $||\boldsymbol{\alpha}_m||_1 = m$, where $||\boldsymbol{\alpha}||_1 = \sum_{i=1}^n |\alpha_i|$. In order to obtain a contradiction, it will be enough to show that

$$\liminf_{m\to\infty} \tfrac{f(\boldsymbol{\alpha}_m)}{m} > 0. \tag{4.2.3}$$

Indeed, (4.2.3) would provide the following coercivity condition

$$\liminf_{m\to\infty} \inf \left\{ \frac{f(\boldsymbol{\alpha})}{||\boldsymbol{\alpha}||_1} \,:\, \boldsymbol{\alpha} \in \overline{B}_m \setminus \{\mathbf{0}\} \right\} > 0,$$

which, in turn, would guarantee existence of a global minimizer $\boldsymbol{\alpha}_0 \in \mathbb{R}^n$ for f (see Chapter 1 of [16]), at which $\nabla f(\boldsymbol{\alpha}_0) = 0$ holds.

The first step in the proof of (4.2.3) uses the representation given in part 2 of Proposition 2.2.2 and the risk-measure properties of $\nu^{(w)}(\cdot;\gamma|\mathcal{E})$ to obtain the following

$$\begin{aligned}
\liminf_{m\to\infty} \tfrac{f(\alpha_m)}{m} &= \liminf_{m\to\infty} \tfrac{1}{m}\Big(\nu^{(w)}\left(\boldsymbol{\alpha}_m \cdot \boldsymbol{B} - \mathcal{E}_1; \gamma_1\right) + \nu^{(w)}\left(-\boldsymbol{\alpha}_m \cdot \boldsymbol{B} - \mathcal{E}_2; \gamma_2\right) \Big) \\
&\geq \liminf_{m\to\infty} \frac{1}{m}\Big(\nu^{(w)}\left(\boldsymbol{\alpha}_m \cdot \boldsymbol{B}; \gamma_1\right) - ||\mathcal{E}_1||_{\mathbb{L}^\infty} + \nu^{(w)}\left(-\boldsymbol{\alpha}_m \cdot \boldsymbol{B}; \gamma_2\right) - ||\mathcal{E}_2||_{\mathbb{L}^\infty} \Big) \\
&= \liminf_{m\to\infty} \Big(\nu^{(w)}\left(\tfrac{1}{m}\boldsymbol{\alpha}_m \cdot \boldsymbol{B}; m\gamma_1\right) + \nu^{(w)}\left(-\tfrac{1}{m}\boldsymbol{\alpha}_m \cdot \boldsymbol{B}; m\gamma_2\right) \Big).
\end{aligned}$$

Any subsequence of $\mathbb{N}$, through which the $\liminf$ above is achieved admits a further subsequence $(m_k)_{k\in\mathbb{N}}$ such that the sequence $\frac{1}{m_k}\boldsymbol{\alpha}_{m_k}$ converges to some

$\boldsymbol{\alpha}_0 \in \mathbb{R}^n$ with $||\boldsymbol{\alpha}_0||_1 = 1$; indeed, the sequence $(\frac{1}{m}\boldsymbol{\alpha}_m)_{m\in\mathbb{N}}$ takes values in the compact set $\{\boldsymbol{\alpha} \in \mathbb{R}^n : ||\boldsymbol{\alpha}||_1 = 1\}$. Proposition 2.2.16 implies that

$$\begin{aligned} \nu^{(w)}\left(\tfrac{1}{m_k}\boldsymbol{\alpha}_{m_k} \cdot \boldsymbol{B}; m_k\gamma_1\right) &\to \sup_{\mathbb{Q}\in\mathcal{M}_e} \mathbb{E}^{\mathbb{Q}}[\boldsymbol{\alpha}_0 \cdot \boldsymbol{B}], \\ \text{and } \nu^{(w)}\left(-\tfrac{1}{m_k}\boldsymbol{\alpha}_{m_k} \cdot \boldsymbol{B}; m_k\gamma_2\right) &\to -\inf_{\mathbb{Q}\in\mathcal{M}_e} \mathbb{E}^{\mathbb{Q}}[\boldsymbol{\alpha}_0 \cdot \boldsymbol{B}], \end{aligned} \tag{4.2.4}$$

as $k \to \infty$. Therefore,

$$\liminf_{m\to\infty} \tfrac{1}{m} f(\boldsymbol{\alpha}_m) = \sup_{\mathbb{Q}\in\mathcal{M}_e} \mathbb{E}^{\mathbb{Q}}[\boldsymbol{\alpha}_0 \cdot \boldsymbol{B}] - \inf_{\mathbb{Q}\in\mathcal{M}_e} \mathbb{E}^{\mathbb{Q}}[\boldsymbol{\alpha}_0 \cdot \boldsymbol{B}].$$

It remains to observe that the equality $\sup_{\mathbb{Q}\in\mathcal{M}_e} \mathbb{E}^{\mathbb{Q}}[\boldsymbol{\alpha}_0 \cdot \boldsymbol{B}] = \inf_{\mathbb{Q}\in\mathcal{M}_e} \mathbb{E}^{\mathbb{Q}}[\boldsymbol{\alpha}_0 \cdot \boldsymbol{B}]$ cannot hold; if it did, Assumption 4.1.2 would be violated. □

Remark 4.2.5. The concept of the partial equilibrium pricing as stated in Definition 4.2.1 has its roots in the classic market clearing equilibrium arguments (see e.g., [15], Chapter 6 of [61], for the mathematical overview). The above proof of the existence of the PEPQ (and its uniqueness guaranteed by Assumption 4.1.2) differs from the classical scheme in the following sense: the special properties of the exponential utility and, in particular, the form of the induced demand function, allow us to derive an explicit expression for the PEPQ; namely, it is given as the minimizer of the difference $w_1(\boldsymbol{\alpha}) - b_2(\boldsymbol{\alpha})$. Additionally, and unlike the classical approach, such a constructive expression opens a possibility of efficient numerical computation in many cases of interest.

Remark 4.2.6.

1. When $n = 1$, the proof above can be simplified considerably; one can

show that

$$\lim_{\alpha\to\infty} w_1'(\alpha) > \lim_{\alpha\to\infty} b_2'(\alpha) \text{ and } \lim_{\alpha\to-\infty} w_1'(\alpha) < \lim_{\alpha\to-\infty} b_2'(\alpha),$$

and deduce the existence of the solution of the equation $w_1'(\alpha) = b_2'(\alpha)$ directly.

In addition, by Remark 3.4.7, we easily get that the quantity

$$\tilde{\alpha} = \frac{\mathbb{E}^{\mathbb{Q}^{(-\gamma_2\mathcal{E}_2)}}[B] - \mathbb{E}^{\mathbb{Q}^{(-\gamma_1\mathcal{E}_1)}}[B]}{\gamma_1\Delta^{\mathbb{Q}^{(-\gamma_1\mathcal{E}_1)}}(B) + \gamma_2\Delta^{\mathbb{Q}^{(-\gamma_2\mathcal{E}_2)}}(B)}$$

minimizes the second order approximation of the difference $w_1(\alpha)-b_2(\alpha)$. In view of Corollary 4.2.3, we can heuristically consider $\tilde{\alpha}$ as an approximation of the partial equilibrium quantity (PEQ), provided that $\tilde{\alpha}$ is close to zero.

2. Corollary 3.3.6 and the discussion preceding it show that when $\frac{\gamma_1}{\gamma_2}\mathcal{E}_1 \sim \mathcal{E}_2$, the unique PEPQ must be of the form $(\mathbf{0}, \boldsymbol{p})$, where $\boldsymbol{p} = \mathbb{E}^{\mathbb{Q}^{(-\gamma_1\mathcal{E}_1)}}[\boldsymbol{B}] = \mathbb{E}^{\mathbb{Q}^{(-\gamma_2\mathcal{E}_2)}}[\boldsymbol{B}]$ for every $\boldsymbol{B}$ which satisfies the Assumption 4.1.2. In such cases $\boldsymbol{p}$ should not be interpreted as a price of $\boldsymbol{B}$, since no transaction actually occurs. Furthermore, the strict agreement (in the sense of Definition 3.1.1) can then be reached for no contingent claim of the from $\boldsymbol{\alpha}\cdot\boldsymbol{B}$, $\boldsymbol{\alpha}\in\mathbb{R}^n$.

   Even when $\frac{\gamma_1}{\gamma_2}\mathcal{E}_1 \nsim \mathcal{E}_2$, there might exist claims for which the PEPQ is of the form $(\mathbf{0}, \boldsymbol{p})$. In fact, PEPQ is of the form $(\mathbf{0}, \boldsymbol{p})$ if and only if $\mathbb{E}^{\mathbb{Q}^{(-\gamma_1\mathcal{E}_1)}}[\boldsymbol{B}] = \mathbb{E}^{\mathbb{Q}^{(-\gamma_2\mathcal{E}_2)}}[\boldsymbol{B}]$ (see Proposition 3.4.6). As an example, consider a claim $\boldsymbol{B}$ which is independent of the stochastic process $\mathbf{S}$, as well as the two random endowments. The partial equilibrium price is then simply a certainty equivalent $\boldsymbol{p} = \mathbb{E}[\boldsymbol{B}] = \mathbb{E}^{\mathbb{Q}^{(-\gamma_1 \mathcal{E}_1)}}[\boldsymbol{B}] = \mathbb{E}^{\mathbb{Q}^{(-\gamma_2 \mathcal{E}_2)}}[\boldsymbol{B}]$.

If a vector of claims $\boldsymbol{B}$ satisfies the Assumption 4.1.2 and its PEPQ is of the form $(\mathbf{0}, \boldsymbol{p})$, then $\nu^{(w)}(\boldsymbol{\alpha} \cdot \boldsymbol{B}; \gamma_1 | \mathcal{E}_1) - \nu^{(b)}(\boldsymbol{\alpha} \cdot \boldsymbol{B}; \gamma_2 | \mathcal{E}_2) > 0$, for every $\boldsymbol{\alpha} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$, i.e., $\boldsymbol{\alpha} \cdot \boldsymbol{B} \notin \mathcal{G}$ for all $\boldsymbol{\alpha} \neq \mathbf{0}$. In other words, any trade in a nontrivial linear combination $\boldsymbol{\alpha} \cdot \boldsymbol{B}$ must make at least one of the agents strictly worse off.

Chapter 5

Agreement and Equilibrium Under Convex Capital Requirements

The aim of this chapter is to generalize the concept of agents' agreement, discussed in previous sections, by allowing a larger class of agents' acceptance sets and including more than two agents into the scheme. More precisely, we suppose that the agents' acceptance sets are not necessarily induced by utility maximization problems, but instead, we require that they satisfy some minimal financially rational axioms. These axioms are inspired by the relatively new and recently widely developed *Theory of Convex Risk Measures*. In Appendix B, a short introduction and a summary of the main results of this theory are provided. We consider the same locally bounded semimartingale financial market introduced in the previous chapters and assume that the agents' contingent claim valuation is based on their acceptance sets, in a similar fashion to the case of indifference pricing. We, then, generalize the notion of agreeable claims, and impose and discuss the necessary assumptions that lead to the existence and uniqueness of the partial equilibrium price for a vector of claims.

In Section 5.1, we state the axioms that an agent's acceptance set should satisfy. More precisely, we require that such a set is monotone, convex, contains zero and does not include non-positive payoffs. An axiom that connects the acceptance set with the market and the admissible strategies is also imposed (see axiom *Ax4*, at page 81). Following the arguments of the utility indifference pricing, we define the risk measure (induced from the acceptance set) of an arbitrary payoff to be the minimum amount of money such that, when added to the payoff, creates an acceptable position. A number of properties of this measure are also presented.

Section 5.2 is dedicated to the robust representation of the risk measure in the spirit of Theorem B.0.7. More precisely, we show that the effective domain of the corresponding penalty function is included in $\mathcal{M}_a$ (the absolutely continuous martingale probability measures). Similarly, as in the case of exponential utility, we define an equivalence relation (with respect to risk) among the essentially bounded payoffs. Using this relation, we introduce the property of *risk-strict convexity* of an acceptance set; roughly speaking, an acceptance set is risk-strictly convex if for every pair of acceptable claims, which do not belong to the same equivalence class, every convex combination of these claims results in strictly less risk. Imposing this assumption, we are able to restrict ourselves only to equivalent martingale measures.

In Section 5.3, we attempt to generalize the notion of agreement by allowing more than two agents and considering a vector of claims, instead of a single one. Given a vector of contingent claims $\boldsymbol{B} = (B_1, B_2, ..., B_n)$, we first define its set of *allocations*, i.e., the set of matrices that represent the feasible ways to share the units of claims in $\boldsymbol{B}$ among the agents. Then, we say that the pair of $\boldsymbol{B}$ and its allocation $\boldsymbol{a}$ is a *mutually-agreeable claim-allocation* if there is a price vector $\mathbf{p}$ at which the sharing of $\boldsymbol{B}$ according to allocation $\boldsymbol{a}$ is acceptable for every agents. In the last part of Section 5.3, we connect this generalized agreement notion with the so-called *inf-convolution* risk measure.

Finally, in Section 5.4, we deal with the market clearing equilibrium pricing arguments. We fix a vector of claims $\boldsymbol{B}$ and define the agent's demand function in a similar way as in Chapter 4. By imposing a strict convexity assumption on the agent's acceptance set, we show that the induced risk measure is differentiable as a function of units of the claims in $\boldsymbol{B}$. In turn, we provide a formula for its gradient. Using these results, we are able to exhibit some properties of the demand function such as monotonicity and continuity. In subsection 5.4.2, we include the definition of the *partial equilibrium price-allocation (PEPA)*; a price vector $\mathbf{p}$ is called a *partial equilibrium price* if the sum of agents' demand functions at $\mathbf{p}$ is zero. The corresponding allocation, determined by the demand functions, is called a *partial equilibrium allocation* (as mentioned in the Introduction, the term allocation is chosen instead of the term quantity to indicate the possibility of the participation of more than two agents). By imposing two necessary assumptions, we establish the existence and uniqueness of the PEPA.

5.1 Acceptance Sets and the Market

In this section, we adopt the market setting introduced in page 12, and consider a financial agent who invests in this market by creating self-financing portfolios. Since we do not restrict ourselves to exponential utility maximizers, we consider a set of admissible strategies different than the set $\boldsymbol{\Theta}$ defined in (2.1.1). First, we relax Assumption 2.1.1 by replacing it by the following weaker one.

Assumption 5.1.1. $\mathcal{M}_e \neq \emptyset$.

We call a strategy $\boldsymbol{h} \in L(\mathbf{S})$ *admissible* if the stochastic integral $(\boldsymbol{h} \cdot \mathbf{S})_t$ is uniformly bounded from below, i.e., the set of admissible strategies is given by

$$\boldsymbol{H} = \{\boldsymbol{h} \in L(\mathbf{S}) : \exists c \in \mathbb{R} \text{ such that } c \leq (\boldsymbol{h} \cdot \mathbf{S})_t,\ \forall t \in [0,T]\} \tag{5.1.1}$$

The gains process obtained by investing the initial wealth $x \in \mathbb{R}$ according to a strategy $\boldsymbol{h} \in \boldsymbol{H}$ is denoted by $X_t^{x,\boldsymbol{h}}$, i.e.,

$$X_t^{x,\boldsymbol{h}} = x + (\boldsymbol{h} \cdot \mathbf{S})_t = x + \int_0^t \boldsymbol{h}_u \, d\mathbf{S}_u,\ t \in [0,T]. \tag{5.1.2}$$

In addition, $\mathcal{X}(x) = \left\{ X_T^{x,\boldsymbol{h}} \,:\, \boldsymbol{h} \in \boldsymbol{H} \right\}$ is called the set of *admissible terminal gains* with initial wealth x, for $x \in \mathbb{R}$.

In what follows, we will use the notations $\mathcal{X} = \bigcup_{x \in \mathbb{R}} \mathcal{X}(x)$ and $\mathcal{X}^\infty = \mathcal{X} \cap \mathbb{L}^\infty$. We, also, define the set $\mathcal{R} = \{X \in \mathcal{X} \,:\, -X \in \mathcal{X}\}$, i.e., $X \in \mathcal{R}$ if there exists $x \in \mathbb{R}$ and $\boldsymbol{h} \in \boldsymbol{H}$ such that $X = x + (\boldsymbol{h} \cdot \mathbf{S})_T$ and $(\boldsymbol{h} \cdot \mathbf{S})_t$ is uniformly bounded. Note that $\mathcal{R} \subseteq \mathbb{L}^\infty$.

Remark 5.1.2. The lower bound on the losses of the admissible strategies is imposed in order to avoid the pathologies that the so-called *doubling strategies* create. Moreover, Assumption 5.1.1 excludes the existence of arbitrage opportunities in the market of traded assets $\mathbf{S}$ (see [30], Corollary 1.2). Note also that $X \in \mathcal{X}^{\infty}$ does not imply that $-X \in \mathcal{X}$, since there exist admissible strategies such that $(\boldsymbol{h} \cdot \mathbf{S})_T \in \mathbb{L}^{\infty}$ and $(-\boldsymbol{h} \cdot \mathbf{S})_t$ is not uniformly bounded from below.

Given the financial market $\mathbf{S}$ and the set of admissible strategies $\boldsymbol{H}$, we assume that each agent's risk preferences, investment goals and random endowment are incorporated in a set $\tilde{\mathcal{A}} \subseteq \mathbb{L}^0(\mathcal{F})$, called the *acceptance set*. In fact, $\tilde{\mathcal{A}}$ contains the discounted net wealth of all the investment positions the agent is willing to undertake at time $t = 0$. Following the literature on convex risk measures, we assume that $\tilde{\mathcal{A}}$ satisfies the following axioms.

Ax1. For $B, C \in \mathbb{L}^0(\mathcal{F})$, if $B \in \tilde{\mathcal{A}}$ and $B \leq C$, then $C \in \tilde{\mathcal{A}}$. If also $B, C \in \mathbb{L}^{\infty}$, then $B \leq C$, $\mathbb{P}-$a.s. and $B \in \tilde{\mathcal{A}}$ implies that $C \in \tilde{\mathcal{A}}$.

Ax2. $\tilde{\mathcal{A}}$ is convex.

Ax3. $\tilde{\mathcal{A}} \cap \mathbb{L}^0_-(\mathcal{F}) = \{0\}$.

Ax4. For every $B \in \tilde{\mathcal{A}}$, we have that $B - (\boldsymbol{h} \cdot \mathbf{S})_T \in \tilde{\mathcal{A}}$, for every $\boldsymbol{h} \in \boldsymbol{H}$.

A direct consequence of axiom *Ax4.* is the following property.

$$\text{If there exists } X \in \mathcal{X}(x) \text{ such that } B + X \in \tilde{\mathcal{A}}, \text{ then } x + B \in \tilde{\mathcal{A}}. \tag{5.1.3}$$

As discussed in Appendix B, axiom *Ax1* simply states that every investment with payoff higher than the payoff of an acceptable claim is also acceptable. Axiom *Ax2* reflects the fact that diversified portfolios of acceptable investments should also be acceptable, while Axiom *Ax3* means that the status quo (i.e., undertaking no investments) is an acceptable position and that the investments with non-positive, different than zero payoffs are not acceptable. Finally, axiom *Ax4* is the one that connects the market to the agent's acceptable positions. If $B \in \tilde{\mathcal{A}}$, the position $B - X + x$ belongs in $\tilde{\mathcal{A}}$, since the agent can use x to replicate X and add it to $B - X$. Similarly, if $B + X \in \tilde{\mathcal{A}}$ for some replicable position X, the payoff B plus the replication cost of X should also belong to $\tilde{\mathcal{A}}$.

An example of an acceptance set that satisfies these axioms is the set $\mathcal{A}_\gamma(\mathcal{E})$ defined in (2.1.4).

In what follows, we denote by $\mathcal{A}$ the set of essentially bounded acceptable claims, i.e., $\mathcal{A} = \tilde{\mathcal{A}} \cap \mathbb{L}^\infty$. Given the acceptance set $\tilde{\mathcal{A}}$, we call the mapping $\rho_\mathcal{A} : \mathbb{L}^\infty \to \bar{\mathbb{R}}$, defined by

$$\rho_\mathcal{A}(B) = \inf\{m \in \mathbb{R} : m + B \in \tilde{\mathcal{A}}\}, \text{ for every } B \in \mathbb{L}^\infty, \tag{5.1.4}$$

the agent's *convex capital requirement*. It follows (see B.0.5) that $\rho_\mathcal{A}(\cdot)$ is convex, decreasing and cash invariant (see also Definition B.0.3 for details).

By axioms *Ax1* and *Ax2*, we get that $\rho_\mathcal{A}(0) = 0$. Also, thanks to the inequality $-\|B\|_{\mathbb{L}^\infty} \leq B \leq \|B\|_{\mathbb{L}^\infty}$ and axiom *Ax1*, $\rho_\mathcal{A}(B) \in [-\|B\|_{\mathbb{L}^\infty}, \|B\|_{\mathbb{L}^\infty}] \subseteq \mathbb{R}$ for every $B \in \mathbb{L}^\infty$.

It is, also, straightforward to get that $\mathcal{A} \subseteq \{B \in \mathbb{L}^\infty : \rho_{\mathcal{A}}(B) \leq 0\}$. If, in addition, the acceptance set $\tilde{\mathcal{A}}$ satisfies the following closure property, the inverse inclusion also holds (see, also, Remark B.0.6).

$$\text{The set } \left\{\lambda \in [0,1] \,:\, \lambda m + (1-\lambda)B \in \tilde{\mathcal{A}}\right\} \text{ is closed in } [0,1], \tag{5.1.5}$$

for every $m \in \mathbb{R}_+$ and $B \in \mathbb{L}^\infty$.

Property (5.1.5) holds, in particular, if $\tilde{\mathcal{A}} \cap V$ is closed for any finite-dimensional subspace $V \subseteq \mathbb{L}^\infty$.

Summing up, we have shown the following proposition.

Proposition 5.1.3. *If an acceptance set $\tilde{\mathcal{A}}$ satisfies the axioms Ax1-Ax4 and property (5.1.5), the mapping $\rho_{\mathcal{A}} : \mathbb{L}^\infty \to \mathbb{R}$ defined in (5.1.4) is a convex risk measure, for which $\rho_{\mathcal{A}}(0) = 0$. Furthermore, the intersection $\mathcal{A} = \tilde{\mathcal{A}} \cap \mathbb{L}^\infty$ can be recovered from $\rho_{\mathcal{A}}(\cdot)$ through the equality $\mathcal{A} = \{B \in \mathbb{L}^\infty : \rho_{\mathcal{A}}(B) \leq 0\}$.*

In what follows, with a slight abuse of terminology, when we refer the term acceptance set we will refer to the set $\mathcal{A} = \tilde{\mathcal{A}} \cap \mathbb{L}^\infty$, for $\tilde{\mathcal{A}}$ that satisfies *Ax1-Ax4*.

Remark 5.1.4. Similar definitions of the convex capital requirement have been given in [37] (page 207) and [41]. In the former, a given acceptance set $\mathcal{A}$ is related to the market through a larger acceptance set $\hat{\mathcal{A}}$, defined as

$$\hat{\mathcal{A}} = \{B \in \mathbb{L}^\infty : \exists X \in \mathcal{X}(0), A \in \mathcal{A} \text{ such that } X + B \geq A, \mathbb{P}-\text{a.s.}\} \tag{5.1.6}$$

In our case, (5.1.3) yields that $\hat{\mathcal{A}} = \mathcal{A}$, which means that the definition of the acceptance set $\mathcal{A}$ has already taken the market into account.

In [41], the authors define the generalized capital requirement by

$$\hat{\rho}_{\mathcal{A}}(B) = \inf\left\{m \in \mathbb{R} : \exists X \in \mathcal{X}(m) \text{ such that } X + B \in \mathcal{A}\right\}.$$

If the acceptance set $\tilde{\mathcal{A}}$ satisfies the axioms *Ax1-Ax4*, it is straightforward to show that $\rho_{\mathcal{A}}(B) = \hat{\rho}_{\mathcal{A}}(B)$.

5.2 The Robust Representation

If we suppose that the set $\mathcal{A}$ is weak-*closed (i.e., closed in the topology $\sigma(\mathbb{L}^\infty, \mathbb{L}^1)$), the convex risk measure $\rho_{\mathcal{A}}(\cdot)$ admits a robust representation in the sense of [6], [28] and [36] (see Theorem B.0.7) and, also, satisfies the property (5.1.5). Axiom *Ax4* provides some further information about the penalty function and, in particular, about its effective domain, denoted by $\mathcal{M}_{\mathcal{A}}$.

Proposition 5.2.1. *If $\mathcal{A}$ is a weak-*closed acceptance set, then $\rho_{\mathcal{A}}$ admits a robust representation of the following form*

$$\rho_{\mathcal{A}}(B) = \sup_{\mathbb{Q} \in \mathcal{M}_a} \{\mathbb{E}^{\mathbb{Q}}[-B] - \alpha_{\mathcal{A}}(\mathbb{Q})\}, \tag{5.2.1}$$

for every $B \in \mathbb{L}^\infty$, where $\alpha_{\mathcal{A}}(\mathbb{Q}) = \sup_{B \in \mathcal{A}}\{\mathbb{E}^{\mathbb{Q}}[-B]\}$, i.e., $\mathcal{M}_{\mathcal{A}} \subseteq \mathcal{M}_a$. Also, the supremum is attained by a measure (possibly not unique), denoted by $\mathbb{Q}_{\mathcal{A}}^{(B)}$.

Proof. It is enough to show that for every $\mathbb{Q} \notin \mathcal{M}_a$, $\alpha_{\mathcal{A}}(\mathbb{Q}) = +\infty$.

For every such $\mathbb{Q}$, there exists an admissible terminal wealth $X \in \mathcal{X}(x)$, such that $\mathbb{E}^{\mathbb{Q}}[X] > x$, i.e., there exists a portfolio $\boldsymbol{h} \in \boldsymbol{H}$, such that $(\boldsymbol{h} \cdot \mathbf{S})_t$

is uniformly bounded from below and $\mathbb{E}^{\mathbb{Q}}[(\boldsymbol{h} \cdot \mathbf{S})_T] > 0$ (see Theorem 5.6 in [30]). Then, for every $k \in \mathbb{N}$, we define $B_k = -((\boldsymbol{h} \cdot \mathbf{S})_T \wedge k)$, which belongs to $\mathbb{L}^\infty$. Hence, $B_k + (\boldsymbol{h} \cdot \mathbf{S})_T = ((\boldsymbol{h} \cdot \mathbf{S})_T - k)\mathbf{1}_{\{(\boldsymbol{h} \cdot \mathbf{S})_T \geq k\}} \geq 0$, which means that $B_k + (\boldsymbol{h} \cdot \mathbf{S})_T \in \tilde{\mathcal{A}}$ for every $k \in \mathbb{N}$. Also, by (5.1.3) we have that $\lambda B_k \in \mathcal{A}$, for all $\lambda > 0$. Hence, $\alpha_{\mathcal{A}}(\mathbb{Q}) \geq \mathbb{E}^{\mathbb{Q}}[-\lambda B_k]$, for every $k \in \mathbb{N}$. Letting k go to infinity, we get from the monotone convergence theorem that

$$\alpha_{\mathcal{A}}(\mathbb{Q}) \geq \lambda \mathbb{E}^{\mathbb{Q}}[(\boldsymbol{h} \cdot \mathbf{S})_T].$$

If we let λ go to infinity, we conclude. □

Note that since $\rho_{\mathcal{A}}(0) = 0$, $\min_{\mathbb{Q} \in \mathcal{M}_{\mathcal{A}}} \{\alpha_{\mathcal{A}}(\mathbb{Q})\} = 0$.

Corollary 5.2.2. *If $\mathcal{A}$ is a weak-*closed acceptance set, $\rho_{\mathcal{A}}(\cdot)$ satisfies the following replication invariance property: For every $B \in \mathbb{L}^\infty$ and every $C \in \mathcal{R}$ such that $C = x + (\boldsymbol{h} \cdot \mathbf{S})_T$, $\mathbb{P}-a.s.$, for some $x \in \mathbb{R}$ and $\boldsymbol{h} \in \boldsymbol{H}$, it holds that*

$$\rho_{\mathcal{A}}(B + C) = \rho_{\mathcal{A}}(B) - x.$$

Below we give a definition (analogous to Definition 2.1.2) of equivalence classes with respect to risk.

Definition 5.2.3. We call two random variables $B, C \in \mathbb{L}^\infty$ equivalent with respect to their risk, and we write $B \sim C$, if $B - C \in \mathcal{R}$.

The condition $B \sim C$ means that the claims with payoffs B and C carry the same unhedgeable risk. $B \sim C$ also implies that for every $\lambda \in [0, 1]$,

$\rho_{\mathcal{A}}(\lambda B + (1-\lambda)C) = \lambda\rho_{\mathcal{A}}(B) + (1-\lambda)\rho_{\mathcal{A}}(C)$. This is because, $B \sim C$ is equivalent to the existence of some $x \in \mathbb{R}$ and $\boldsymbol{h} \in \boldsymbol{H}$ such that $B - C = x + (\boldsymbol{h}\cdot\mathbf{S})_T$, $\mathbb{P}-$a.s.. By Corollary 5.2.2, we get that

$$\rho_{\mathcal{A}}(\lambda B + (1-\lambda)C) = \rho_{\mathcal{A}}(C + \lambda x + (\lambda\boldsymbol{h}\cdot\mathbf{S})_T) = \rho_{\mathcal{A}}(C) - \lambda x$$

and also

$$\lambda\rho_{\mathcal{A}}(B) + (1-\lambda)\rho_{\mathcal{A}}(C) = \lambda\rho_{\mathcal{A}}(C) - \lambda x + (1-\lambda)\rho_{\mathcal{A}}(C) = \rho_{\mathcal{A}}(C) - \lambda x.$$

On the other hand, if $B \nsim C$, convex combinations of the payoffs B and C may lead to reduction of risk. If any such combination of claims (which do not belong in the same equivalence class) reduces the risk, the corresponding acceptance set $\mathcal{A}$ is called *risk-strictly convex*.

Definition 5.2.4. A weak-*closed acceptance set $\mathcal{A}$ is called *risk-strictly convex* if for every $B, C \in \mathcal{A}$ with $B \nsim C$, it holds that for every $\lambda \in (0,1)$ there exists a random variable $E \in \mathbb{L}^\infty_+$ such that, $\mathbb{Q}_{\mathcal{A}}^{(\lambda B+(1-\lambda)C)}(E > 0) > 0$, and

$$\lambda B + (1-\lambda)C - E \in \mathcal{A}.$$

Proposition 5.2.5. *Let $\mathcal{A}$ be a weak-*closed acceptance set. Then, $\mathcal{A}$ is risk-strictly convex if and only if its induced risk measure $\rho_{\mathcal{A}}$ is strictly convex up to the claims which do not belong to the same equivalence class.*

Proof. We first assume that $\mathcal{A}$ is risk-strictly convex. For arbitrarily chosen $B, C \in \mathbb{L}^\infty$ such that $B \nsim C$, we have that $B + \rho_{\mathcal{A}}(B), C + \rho_{\mathcal{A}}(C) \in \mathcal{A}$. Hence, for every $\lambda \in (0,1)$, there exists $E \in \mathbb{L}^\infty_+$ such that $\mathbb{Q}_\mathcal{A}^{(\lambda B+(1-\lambda)C)}(E>0)>0$ and

$$\lambda B + (1-\lambda)C + \lambda\rho_\mathcal{A}(B) + (1-\lambda)\rho_\mathcal{A}(C) - E \in \mathcal{A}.$$

This implies that $\rho_\mathcal{A}(\lambda B + (1-\lambda)C - E) \leq \lambda\rho_\mathcal{A}(B) + (1-\lambda)\rho_\mathcal{A}(C)$. Then,

$$\begin{aligned}\rho_\mathcal{A}(\lambda B + (1-\lambda)C) &= \mathbb{E}^{\mathbb{Q}_\mathcal{A}^{(\lambda B+(1-\lambda)C)}}[-\lambda B - (1-\lambda)C] - \alpha_\mathcal{A}(\mathbb{Q}_\mathcal{A}^{(\lambda B+(1-\lambda)C)}) \\ &< \mathbb{E}^{\mathbb{Q}_\mathcal{A}^{(\lambda B+(1-\lambda)C)}}[-\lambda B - (1-\lambda)C + E] - \alpha_\mathcal{A}(\mathbb{Q}_\mathcal{A}^{(\lambda B+(1-\lambda)C)}) \\ \leq \sup_{\mathbb{Q}\in\mathcal{M}_\mathcal{A}} \{\mathbb{E}^\mathbb{Q}[-\lambda B - (1-\lambda)C + E] - \alpha_\mathcal{A}(\mathbb{Q})\} &= \rho_\mathcal{A}(\lambda B + (1-\lambda)C - E) \\ &\leq \lambda\rho_\mathcal{A}(B) + (1-\lambda)\rho_\mathcal{A}(C).\end{aligned}$$

On the other hand, if for every $B, C \in \mathcal{A}$ such that $B \not\sim C$ and every $\lambda \in (0,1)$ we have that

$$\rho_\mathcal{A}(\lambda B + (1-\lambda)C) < \lambda\rho_\mathcal{A}(B) + (1-\lambda)\rho_\mathcal{A}(C),$$

it is enough to consider as the corresponding E the positive real number $-\rho_\mathcal{A}(\lambda B + (1-\lambda)C)$, for which $\lambda B + (1-\lambda)C - E \in \mathcal{A}$. □

Under the assumption that the acceptance set $\mathcal{A}$ is risk-strictly convex, we can say a bit more about the effective domain of the penalty function of the induced risk measure, $\mathcal{M}_\mathcal{A}$.

Proposition 5.2.6. *If an acceptance set $\mathcal{A}$ is weak-*closed and risk-strict convex, then $\mathcal{M}_\mathcal{A} \subseteq \mathcal{M}_e$.*

Proof. Assume that there exists $B \in \mathbb{L}^\infty$ such that $\mathbb{Q}_{\mathcal{A}}^{(B)} \notin \mathcal{M}_e$, i.e., there exists a measurable set $A \in \mathcal{F}$ such that $\mathbb{P}(A) > 0$ and $\mathbb{Q}_{\mathcal{A}}^{(B)}(A) = 0$. From the equation

$$\alpha_{\mathcal{A}}(\mathbb{Q}_{\mathcal{A}}^{(B)}) = \sup_{C \in \mathbb{L}^\infty} \{\mathbb{E}^{\mathbb{Q}_{\mathcal{A}}^{(B)}}[-C] - \rho_{\mathcal{A}}(C)\},$$

which follows from the definition of $\alpha_{\mathcal{A}}$, we get that

$$\rho_{\mathcal{A}}(B + k\mathbf{1}_A) \geq \rho_{\mathcal{A}}(B),$$

for all $k \in \mathbb{R}$. Furthermore, the monotonicity of $\rho_{\mathcal{A}}$ implies that $\rho_{\mathcal{A}}(B+k\mathbf{1}_A) = \rho_{\mathcal{A}}(B)$ for all $k \in \mathbb{R}_+$. By the risk-strictness assumption, the latter equality yields that $B + k\mathbf{1}_A \sim B$ for every $k \in \mathbb{R}_+$, i.e., $k\mathbf{1}_A \sim 0$, i.e., there exist $x \in \mathbb{R}$ and $\boldsymbol{h} \in \boldsymbol{H}$ such that $k\mathbf{1}_A = kx + (k\boldsymbol{h} \cdot \mathbf{S})_T \in \mathcal{R}$. Again by the equality $\rho_{\mathcal{A}}(B + k\mathbf{1}_A) = \rho_{\mathcal{A}}(B)$ and Corollary 5.2.2 we get that $x = 0$, and hence $\mathbf{1}_A = (\boldsymbol{h} \cdot \mathbf{S})_T$, which contradicts the non-arbitrage assumption. □

We recall the definition of the sensitivity of a risk measure, given in [37], page 173.

Definition 5.2.7. A risk measure ρ is called *sensitive* if $\rho(-B) > \rho(0)$ for every $B \in \mathbb{L}_+^\infty \backslash \{0\}$.

The fact that the minimizers of the penalty function $\alpha_{\mathcal{A}}(\cdot)$, belong in $\mathcal{P}_e(\mathbb{P})$ implies that the risk measure $\rho_{\mathcal{A}}(\cdot)$ is sensitive. Indeed, from the characterization (5.2.1),

$$\rho_{\mathcal{A}}(-B) \geq \mathbb{E}^{\mathbb{Q}_{\mathcal{A}}^{(0)}}[B] - \alpha_{\mathcal{A}}(\mathbb{Q}_{\mathcal{A}}^{(0)}) = \mathbb{E}^{\mathbb{Q}_{\mathcal{A}}^{(0)}}[B] > 0.$$

Similarly, we get that $\rho_{\mathcal{A}}(B) < 0$.

Another direct consequence for a risk-strictly convex acceptance set is the following proposition.

Proposition 5.2.8. *Let $B \in \mathbb{L}^\infty$ such that $B \notin \mathcal{R}$. Let $\rho_{\mathcal{A}}$ be the risk measure of a weak-*closed, risk-strictly convex acceptance set $\mathcal{A}$. Then,*

$$\rho_{\mathcal{A}}(B) + \rho_{\mathcal{A}}(-B) > 0.$$

In particular, $\mathcal{A} \cap (-\mathcal{A}) = \mathcal{X}(0) \cap \mathcal{R}$.

Proof. For any such B, $\rho_{\mathcal{A}}(B) + B \in \mathcal{A}$ and $\rho_{\mathcal{A}}(-B) - B \in \mathcal{A}$. By assumption (since $2B \notin \mathcal{R}$), there exists $E \in \mathbb{L}^\infty_+ \backslash \{0\}$ such that

$$\frac{1}{2}(\rho_{\mathcal{A}}(B) + \rho_{\mathcal{A}}(-B)) - E \in \mathcal{A}.$$

Hence, $\frac{1}{2}(\rho_{\mathcal{A}}(B) + \rho_{\mathcal{A}}(-B)) - \rho_{\mathcal{A}}(-E) \geq 0$ and by monotonicity of $\rho_{\mathcal{A}}$, we get that $\rho_{\mathcal{A}}(B) + \rho_{\mathcal{A}}(-B) > 0$.

The last statement then follows by Corollary 5.2.2. □

5.3 A Generalized Notion of Agreement

The aim of this section is to generalize the notion of mutually agreeable claims in the following sense:

- The number of agents is $I \geq 2$.
- A vector of claims is considered instead of a single one.

- The agents value claims using acceptance sets that satisfy axioms *Ax1*-*Ax4*.

- The agents may have access to different markets.

Before we state the corresponding definition, we introduce some further notation.

We suppose that each agent i has access to a sub-market $\mathbf{S}_i \subseteq \mathbf{S}$, i.e., she is allowed to trade (has access to) a vector of $d_i + 1 \leq d + 1$ traded assets $(S_t^{(0)}; S_t^{(i_1)}, \ldots, S_t^{(i_{d_i})})_{t\in[0,T]}$, where $i_k \in \{1, 2, ..., d\}$ and $i_k \neq i_l$ for $k \neq l$. Note that we assume that the numéraire $S^{(0)}$ is accessible to each agent. In order to take the whole market into account, we suppose that $\mathbf{S} = \bigcup_{i=1}^{I} \mathbf{S}_i$. Also note that the Assumption 5.1.1 implies that

$$\mathcal{M}_a^i = \{\mathbb{Q} \ll \mathbb{P} : \mathbf{S}_i \text{ is a local martingale under } \mathbb{Q}\} \neq \emptyset$$

$\forall i = 1, 2, ..., I$, since $\mathcal{M}_a \subseteq \mathcal{M}_a^i$. We, then, endow all agents with the corresponding to $\mathbf{S}_i$ set of admissible terminal gains $\mathcal{X}_i$ (terminal value of uniformly bounded from below gain processes) and an acceptance set $\tilde{\mathcal{A}}_i$ which satisfies the axioms *Ax1*-*Ax4*. The induced risk measure $\rho_{\mathcal{A}_i}$ is denoted by ρ_i and $\mathcal{M}_i$ stands for the effective domain of the corresponding penalty function α_i, i.e., the set $\mathcal{M}_{\mathcal{A}_i}$, for every $i = 1, 2, ..., I$.

If we further assume that the intersection $\tilde{\mathcal{A}}_i \cap \mathbb{L}^\infty$, denoted by $\mathcal{A}_i$, is a weak-*closed set, the induced risk measure $\rho_i = \rho_{\mathcal{A}_i}$ admits the following robust representation

$$\rho_i(B) = \sup_{\mathbb{Q}\in\mathcal{M}_i} \{\mathbb{E}^{\mathbb{Q}}[-B] - \alpha_i(\mathbb{Q})\}, \tag{5.3.1}$$

where $\mathcal{M}_i \subseteq \mathcal{M}_a^i$ for all i.

We use the notation $\mathbb{Q}_i^{(B)}$ for the attained supremum in (5.3.1) for $B \in \mathbb{L}^\infty$ and for every $i = 1, 2, ..., I$.

For vector of claims in $(\mathbb{L}^\infty)^n$, $n \in \mathbb{N}$, we call a matrix $(a_{i,k}) = \boldsymbol{a} \in \mathbb{R}^{I\times n}$ an *n-allocation* or simply an *allocation*, if $\sum_{i=1}^{I} a_{i,k} = 0$ for all $k = 1, 2, ..., n$. For convenience, we denote by $\boldsymbol{a}_i$ the vector $(a_{i,k})_{k=1}^n \in \mathbb{R}^n$ (the number of claims agent i holds according the allocation $\boldsymbol{a}$). The set of allocations is denoted by $\mathbf{F}$, i.e.,

$$\mathbf{F} = \{\boldsymbol{a} \in \mathbb{R}^{I\times n} \text{ such that } \sum_{i=1}^{I} \boldsymbol{a}_i = (0, 0, ..., 0)\}. \tag{5.3.2}$$

We are, now, ready to give a general definition of the notion of agreement.

Definition 5.3.1. The pair $(\boldsymbol{B}, \boldsymbol{a}) \in (\mathbb{L}^\infty)^n \times \mathbf{F}$ of a vector of claims and an allocation is called *mutually agreeable* if there exists a (price) vector $\mathbf{p} \in \mathbb{R}^n$ such that $\boldsymbol{a}_i \cdot \boldsymbol{B} - \boldsymbol{a}_i \cdot \mathbf{p} \in \mathcal{A}_i$, for all $i = 1, 2, ..., I$.

Intuitively, if a pair $(\boldsymbol{B}, \boldsymbol{a})$ is mutually agreeable, all agents are willing to allocate the claims in $\boldsymbol{B}$ according to the allocation $\boldsymbol{a}$ at price $\mathbf{p}$, where "willing" means that the investment position $\boldsymbol{a}_i \cdot \boldsymbol{B} - \boldsymbol{a}_i \cdot \mathbf{p}$ does not increase the $i-$agent's risk.

Remark 5.3.2. The Definition 3.1.1 of mutually agreeable claims can be interpreted as a special case of the above definition, where $I = 2$, $n = 1$, $\mathcal{A}_i = \mathcal{A}_{\gamma_i}(\mathcal{E}_i)$ and $\boldsymbol{a} = (-1, 1)$.

For every allocation $\boldsymbol{a}$, we define the set

$$\mathcal{G}^{\boldsymbol{a}} = \{\boldsymbol{B} \in (\mathbb{L}^\infty)^n : \ (\boldsymbol{B}, \boldsymbol{a}) \text{ is mutually agreeable}\}.$$

Note that $\mathcal{G}^{-\boldsymbol{a}} = -\mathcal{G}^{\boldsymbol{a}}$.

We also set $\hat{\mathcal{R}}_{\boldsymbol{a}} = \{\boldsymbol{B} \in (\mathbb{L}^\infty)^n : \boldsymbol{a}_i \cdot \boldsymbol{B} \in \mathcal{R}_i, \ \forall i = 1, 2, ..., I\}$ for $\boldsymbol{a} \in \mathbf{F}$. In the spirit of Proposition 3.1.6, we state the following properties of the set $\mathcal{G}^{\boldsymbol{a}}$.

Proposition 5.3.3. *For every allocation $\boldsymbol{a} \in \mathbf{F}$, $\mathcal{G}^{\boldsymbol{a}}$ is convex. If, also, $\mathcal{A}_i$ is weak*-closed and risk-strictly convex for every $i = 1, 2, ..., I$, then*

$$\mathcal{G}^{\boldsymbol{a}} \cap (-\mathcal{G}^{\boldsymbol{a}}) \subseteq \hat{\mathcal{R}}_{\boldsymbol{a}},$$

for all $\boldsymbol{a} \in \mathbf{F}$.

Proof. The convexity follows directly from the convexity of $\mathcal{A}_i$'s. For the second statement, suppose that $\boldsymbol{B} \in \mathcal{G}^{\boldsymbol{a}} \cap (-\mathcal{G}^{\boldsymbol{a}})$, i.e., there exist $\mathbf{p}, \hat{\mathbf{p}} \in \mathbb{R}^n$ such that $\boldsymbol{a}_i \cdot \boldsymbol{B} - \boldsymbol{a}_i \cdot \mathbf{p} \in \mathcal{A}_i$ and $-\boldsymbol{a}_i \cdot \boldsymbol{B} + \boldsymbol{a}_i \cdot \hat{\mathbf{p}} \in \mathcal{A}_i$, for all i. The convexity of $\mathcal{A}_i$ then implies that $\frac{1}{2}\boldsymbol{a}_i \cdot (\hat{\mathbf{p}} - \mathbf{p}) \in \mathcal{A}_i$, which yields that $\boldsymbol{a}_i \cdot (\hat{\mathbf{p}} - \mathbf{p}) \leq 0$, for all $i = 1, 2, ..., I$. Since $\sum_{i=1}^{I} \boldsymbol{a}_i = 0$, we conclude that for every agent $\boldsymbol{a}_i \cdot \mathbf{p} = \boldsymbol{a}_i \cdot \hat{\mathbf{p}}$. From this equality and the acceptance conditions above, we get that $\rho_i(\boldsymbol{a}_i \cdot \boldsymbol{B}) \leq -\boldsymbol{a}_i \cdot \mathbf{p}$ and also $\rho_i(-\boldsymbol{a}_i \cdot \boldsymbol{B}) \leq \boldsymbol{a}_i \cdot \mathbf{p}$ for all i. But in this case $\rho_i(\boldsymbol{a}_i \cdot \boldsymbol{B}) + \rho_i(-\boldsymbol{a}_i \cdot \boldsymbol{B}) \leq 0$, which means (thanks to the risk-strict convexity of ρ_i) that $\boldsymbol{a}_i \cdot \boldsymbol{B} \in \mathcal{R}_i$ for all i, i.e., $\boldsymbol{B} \in \hat{\mathcal{R}}_{\boldsymbol{a}}$. □

Another useful notion in this concept is the *inf-convolution* of risk measures, first introduced in [9].

Definition 5.3.4. The *inf-convolution* of the risk measures $\rho_1, \rho_2, ..., \rho_I$ is the mapping $\Diamond\rho : \mathbb{L}^\infty \to \mathbb{R} \cup \{-\infty\}$, defined by

$$\Diamond\rho(C) = \inf_{B_1, B_2, ..., B_{I-1} \in \mathbb{L}^\infty} \left\{ \sum_{i=1}^{I-1} \rho_i(B_i) + \rho_I(C - (B_1 + B_2 + ... + B_{I-1})) \right\}, \tag{5.3.3}$$

for $C \in \mathbb{L}^\infty$.

In what follows, we denote by $\mathcal{M}$ the intersection $\bigcap_{i=1}^{I} \mathcal{M}_i$ and we impose the following assumption.

Assumption 5.3.5. $\mathcal{M} \neq \emptyset$.

Since $\mathbf{S} = \bigcup_{i=1}^{I} \mathbf{S}_i$ we have that $\bigcap_{i=1}^{I} \mathcal{M}_e^i = \mathcal{M}_e$. Hence, if $\mathcal{A}_i$'s are risk-strictly convex, it holds that $\mathcal{M} \subseteq \mathcal{M}_e$ and, hence, Assumption 5.3.5 is a stronger version of Assumption 5.1.1.

The following proposition is a slight generalization of Theorem 3.6 in [10], where the case of $I = 2$ is addressed. The proof when $I \geq 2$ is similar and, hence, omitted.

Proposition 5.3.6. *Under Assumption 5.3.5 and if $\mathcal{A}_i$ is weak-*closed for every $i = 1, 2, ..., I$, the map $\Diamond\rho : \mathbb{L}^\infty \to \mathbb{R}$ is a convex risk measure with penalty function $h(\mathbb{Q}) = \sum_{i=1}^{I} \alpha_i(\mathbb{Q})$ whose effective domain is $\mathcal{M}$.*

Definition 5.3.7. We say that the agents are in *Pareto-optimal condition* or in *Pareto equilibrium* if $\diamondsuit\rho(0) = 0$.

In other words, the Pareto-optimal condition implies that there is no transaction that can strictly decrease the sum of the agent's risk measures, i.e., there is no transaction that strictly improves their risk exposure.

Below, we state a characterization of the Pareto-optimal condition in terms of the minimizers of the penalty functions α_i.

Proposition 5.3.8. *The agents are in a Pareto-optimal condition if and only if $\mathbb{Q}_i^{(0)} = \mathbb{Q}_j^{(0)}$ for every $i, j = 1, 2, ..., I$.*

Proof. We observe that

$$\begin{aligned} \diamondsuit\rho(0) &= \sup_{\mathbb{Q}\in\mathcal{M}}\{-h(\mathbb{Q})\} \\ &= \sup_{\mathbb{Q}\in\mathcal{M}}\{-\sum_{i=1}^{I}\alpha_i(\mathbb{Q})\} = -\inf_{\mathbb{Q}\in\mathcal{M}}\{\sum_{i=1}^{I}\alpha_i(\mathbb{Q})\} \leq 0. \end{aligned}$$

and the equality holds if and only if α_i's have a common minimizer. □

The following proposition states that if the agents are in Pareto-optimal condition, i.e., when $\diamondsuit\rho(0) = 0$, the risk-strict convexity assumption implies that the transaction of any non-replicable claim results in increased risk for at least one of the agents involved in this transaction.

Proposition 5.3.9. *Assume that $\mathcal{A}_i$ are weak-*closed and risk-strictly convex for all $i = 1, 2, .., I$ and suppose that $\diamondsuit\rho(0) = 0$. For any claims $B_1, B_2, ..., B_{I-1}$*

in $\mathbb{L}^\infty$*, it holds that*

$$\sum_{i=1}^{I-1} \rho_i(B_i) + \rho_I(-\sum_{i=1}^{I-1} B_i) = 0$$

if and only if $B_i \in \mathcal{R}_i$ *for all* $i = 1, 2, ..., I$*, where* $B_I = -\sum_{i=1}^{I-1} B_i$*.*

Proof. Assume that there exists $k \in \{1, 2, ..., I\}$ such that $B_k \notin \mathcal{R}_k$. Then, by the strict convexity assumption, $\forall \lambda \in (0, 1)$ there exists $E \in \mathbb{L}^\infty_+ \backslash \{0\}$ such that $\rho_k(\lambda B_k - E) \leq \lambda \rho_k(B_k)$. This implies that $\rho_k(\lambda B_k) < \lambda \rho_k(B_k)$. Note that $\rho_i(\lambda B_i) - \lambda \rho_i(B_i) \leq 0$, $\forall i = 1, 2, ..., I$. Hence,

$$\sum_{i=1}^{I} \rho_i(\lambda B_i) < \lambda \sum_{i=1}^{I} \rho_i(B_i) = 0,$$

which contradicts the assumption $\Diamond\rho(0) = 0$.

On the other hand, if $B_i \in \mathcal{R}_i$ for all $i = 1, 2, ..., I$, then

$$\rho_i(B_i) = -\rho_i(-B_i).$$

Hence, if $\sum_{i=1}^{I} \rho_i(B_i) > 0$, we have that $\sum_{i=1}^{I} \rho_i(-B_i) < 0$, which also contradicts the assumption $\Diamond\rho(0) = 0$. □

Corollary 5.3.10. *Suppose that* $\mathbb{Q}_i^{(0)} = \mathbb{Q}_j^{(0)}$ *for every* $i, j = 1, 2, ..., I$*. Then, for* $\boldsymbol{B} \in (\mathbb{L}^\infty)^n$ *and* $\boldsymbol{a} \in \mathbf{F}$*, if* $\boldsymbol{B} \in \mathcal{G}^{\boldsymbol{a}}$*, it holds that* $\boldsymbol{a}_i \cdot \boldsymbol{B} \in \mathcal{R}_i$*, for every* $i = 1, 2, ..., I$*.*

Proof. The fact that $\boldsymbol{B} \in \mathcal{G}^{\boldsymbol{a}}$ yields that there exists a price vector $\mathbf{p}$ such that $\boldsymbol{a}_i \cdot \boldsymbol{B} - \boldsymbol{a}_i \cdot \mathbf{p} \in \mathcal{A}_i$, for all i. This implies that $\sum_{i=1}^{I} \rho_i(\boldsymbol{a}_i \cdot \boldsymbol{B}) \leq 0$, which by the hypothesis and Proposition 5.3.9 yields that $\boldsymbol{a}_i \cdot \boldsymbol{B} \in \mathcal{R}_i$ for all $i = 1, 2, ..., I$. □

Example 5.3.11. Suppose that we adopt the exponential utility setup of Chapter 3 for every agent i, where by γ_i and $\mathcal{E}_i$ we denote the agents' risk aversion coefficients and random endowments, respectively. Then, we can conclude by following the same lines as in the proof of Proposition 3.3.5 that exponential utility maximizers are in a Pareto-optimal condition if and only if $\frac{\gamma_i}{\gamma_j}\mathcal{E}_i \sim \mathcal{E}_j$ for all $i, j = 1, 2, ..., I$.

5.4 The Partial Equilibrium Price Allocation

Having introduced the setup of I agents, their acceptance sets and sets of admissible terminal gains, we can establish a partial equilibrium pricing for a fixed vector of claims $\boldsymbol{B} \in (\mathbb{L}^\infty)^n$. We start with the demand correspondence for $\boldsymbol{B}$ of the agent i.

5.4.1 The demand function

For every agent i, and for the fixed vector $\boldsymbol{B} \in (\mathbb{L}^\infty)^n$, we give the following generalization of Definition 4.1.1.

Definition 5.4.1. For the agent i, $i = 1, 2, ..., I$, the demand correspondence of the vector of claims $\boldsymbol{B}$, $Z_i : \mathbb{R}^n \to 2^{\bar{\mathbb{R}}^n}$ is defined by

$$Z_i(\mathbf{p}) = \underset{\boldsymbol{a}\in\bar{\mathbb{R}}^n}{\operatorname{argmin}}\{\rho_i(\boldsymbol{a}\cdot\boldsymbol{B} - \boldsymbol{a}\cdot\mathbf{p})\} = \underset{\boldsymbol{a}\in\bar{\mathbb{R}}^n}{\operatorname{argmin}}\{\rho_i(\boldsymbol{a}\cdot\boldsymbol{B}) + \boldsymbol{a}\cdot\mathbf{p}\} \qquad (5.4.1)$$

In other words, $Z_i(\mathbf{p})$ gives the vectors of units of $\boldsymbol{B}$ that the agent i is willing (in terms of risk minimization) to buy/sell at price $\mathbf{p}$. From Definition 5.4.1, it follows that the differentiability of the mapping $\boldsymbol{a} \mapsto \rho_i(\boldsymbol{a}\cdot\boldsymbol{B})$, $\boldsymbol{a} \in \mathbb{R}^n$

is an important issue. For this, we need the following definition, which is a weaker version of the risk-strict convexity.

Definition 5.4.2. Let $\mathcal{A}$ be a weak-*closed acceptance set and $\boldsymbol{B} \in (\mathbb{L}^\infty)^n$ a vector of claims. We call $\mathcal{A}$ *strictly convex with respect to* $\boldsymbol{B}$ if for every $\lambda \in (0,1)$ and every $\boldsymbol{a}, \boldsymbol{\delta} \in \mathbb{R}^n$ and $m, k \in \mathbb{R}$ such that $\boldsymbol{a} \neq \boldsymbol{\delta}$ and $\boldsymbol{a} \cdot \boldsymbol{B} + m, \boldsymbol{\delta} \cdot \boldsymbol{B} + k \in \mathcal{A}$, there exists a random variable $E \in \mathbb{L}^\infty_+$, such that $\mathbb{Q}_{\mathcal{A}}^{((\lambda \boldsymbol{a} + (1-\lambda)\boldsymbol{\delta}) \cdot \boldsymbol{B})}(E > 0) > 0$ and

$$\lambda(\boldsymbol{a} \cdot \boldsymbol{B} + m) + (1-\lambda)(\boldsymbol{\delta} \cdot \boldsymbol{B} + k) - E \in \mathcal{A}.$$

Following the proof of Proposition 5.2.5, we can show that an acceptance set $\mathcal{A}_i$ is strictly convex with respect to $\boldsymbol{B}$ if and only if the function $\mathbb{R}^n \ni \boldsymbol{a} \mapsto \rho_i(\boldsymbol{a} \cdot \boldsymbol{B}) \in \mathbb{R}$ is strictly convex. In fact, it is also differentiable in $\mathbb{R}^n$. For the proof of this argument, we need the following lemma.

Lemma 5.4.3. *Let $i \in \{1, 2, ..., I\}$ and $\boldsymbol{B} = (B_1, B_2, ..., B_n) \in (\mathbb{L}^\infty)^n$ be a vector of claims for which there is no $\boldsymbol{a} \in \mathbb{R}^n$ such that $\boldsymbol{a} \cdot \boldsymbol{B} \in \mathcal{R}_i$. If $\mathcal{A}_i$ is weak-*closed and strictly convex with respect to $\boldsymbol{B}$, it holds that*

$$\lim_{a_k \to +\infty} \mathbb{E}^{\mathbb{Q}_i^{(\boldsymbol{a} \cdot \boldsymbol{B})}}[-B_k] = \sup_{\mathbb{Q} \in \mathcal{M}_i} \{\mathbb{E}^{\mathbb{Q}}[-B_k]\} = \sup_{\boldsymbol{\delta} \in \mathbb{R}^n} \{\mathbb{E}^{\mathbb{Q}_i^{(\boldsymbol{\delta} \cdot \boldsymbol{B})}}[-B_k]\} \tag{5.4.2}$$

and

$$\lim_{a_k \to -\infty} \mathbb{E}^{\mathbb{Q}_i^{(\boldsymbol{a} \cdot \boldsymbol{B})}}[-B_k] = \inf_{\mathbb{Q} \in \mathcal{M}_i} \{\mathbb{E}^{\mathbb{Q}}[-B_k]\} = \inf_{\boldsymbol{\delta} \in \mathbb{R}^n} \{\mathbb{E}^{\mathbb{Q}_i^{(\boldsymbol{\delta} \cdot \boldsymbol{B})}}[-B_k]\}, \tag{5.4.3}$$

for all $\boldsymbol{a} = (a_1, a_2, ..., a_n) \in \mathbb{R}^n$ and $k = 1, 2, ..., n$.

Proof. Without loss of generality, we assume that $k = 1$. Note that by convexity of ρ_i, the ratio $\frac{\rho_i(\boldsymbol{a}\cdot\boldsymbol{B})}{a_1}$ is an increasing function of $a_1 > 0$ and, hence, its limit as $a_1 \to +\infty$ exists (in $\bar{\mathbb{R}}$). We claim that

$$\lim_{a_1\to+\infty} \frac{\rho_i(\boldsymbol{a}\cdot\boldsymbol{B})}{a_1} \geq \sup_{\mathbb{Q}\in\mathcal{M}_i} \{\mathbb{E}^{\mathbb{Q}}[-B_1]\}. \tag{5.4.4}$$

Indeed,

$$\begin{aligned}
\lim_{a_1\to+\infty} \frac{\rho_i(\boldsymbol{a}\cdot\boldsymbol{B})}{a_1} &= \sup_{a_1\in\mathbb{R}_+} \frac{\rho_i(\boldsymbol{a}\cdot\boldsymbol{B})}{a_1} \\
&= \sup_{a_1\in\mathbb{R}_+}\sup_{\mathbb{Q}\in\mathcal{M}_i} \left\{ \frac{\mathbb{E}^{\mathbb{Q}}[-(\boldsymbol{a}\cdot\boldsymbol{B})]}{a_1} - \frac{\alpha_i(\mathbb{Q})}{a_1} \right\} \\
&= \sup_{\mathbb{Q}\in\mathcal{M}_i}\sup_{a_1\in\mathbb{R}_+} \left\{ \frac{\mathbb{E}^{\mathbb{Q}}[-(\boldsymbol{a}\cdot\boldsymbol{B})]}{a_1} - \frac{\alpha_i(\mathbb{Q})}{a_1} \right\} \\
&= \sup_{\mathbb{Q}\in\mathcal{M}_i}\sup_{a_1\in\mathbb{R}_+} \left\{ \mathbb{E}^{\mathbb{Q}}[-B_1] + \frac{\mathbb{E}^{\mathbb{Q}}[-\sum_{j=2}^n a_i B_i] - \alpha_i(\mathbb{Q})}{a_1} \right\} \\
&\geq \sup_{\mathbb{Q}\in\mathcal{M}_i}\sup_{a_1\in\mathbb{R}_+} \left\{ \mathbb{E}^{\mathbb{Q}}[-B_1] + \frac{-\sum_{j=2}^n |a_i|||B_i||_\infty - \alpha_i(\mathbb{Q})}{a_1} \right\} \\
&= \sup_{\mathbb{Q}\in\mathcal{M}_i} \{\mathbb{E}^{\mathbb{Q}}[-B_1]\}.
\end{aligned}$$

Also, from the inequality $\frac{\rho_i(\boldsymbol{a}\cdot\boldsymbol{B})}{a_1} + \frac{1}{a_1}\mathbb{E}^{\mathbb{Q}_i^{(\boldsymbol{a}\cdot\boldsymbol{B})}}[(\boldsymbol{a}\cdot\boldsymbol{B})] \leq 0$, which holds for all $(a_2, a_3, ..., a_n) \in \mathbb{R}^{n-1}$ and $a_1 > 0$, we get that

$$\begin{aligned}
\frac{\rho_i(\boldsymbol{a}\cdot\boldsymbol{B})}{a_1} &\leq \mathbb{E}^{\mathbb{Q}_i^{(\boldsymbol{a}\cdot\boldsymbol{B})}}[-B_1] + \frac{1}{a_1}\mathbb{E}^{\mathbb{Q}_i^{(\boldsymbol{a}\cdot\boldsymbol{B})}}[-\sum_{j=2}^n a_i B_i] \\
&\leq \mathbb{E}^{\mathbb{Q}_i^{(\boldsymbol{a}\cdot\boldsymbol{B})}}[-B_1] + \frac{\sum_{j=2}^n |a_i|||B_i||_\infty}{a_1}.
\end{aligned}$$

Hence, $\lim_{a_1\to+\infty} \frac{\rho_i(\boldsymbol{a}\cdot\boldsymbol{B})}{a_1} \leq \lim_{a_1\to+\infty} \mathbb{E}^{\mathbb{Q}_i^{(\boldsymbol{a}\cdot\boldsymbol{B})}}[-B_1]$. Hence, by (5.4.4) we get that

$$\lim_{a_1\to+\infty} \mathbb{E}^{\mathbb{Q}_i^{(\boldsymbol{a}\cdot\boldsymbol{B})}}[-B_1] = \sup_{\mathbb{Q}\in\mathcal{M}_i} \{\mathbb{E}^{\mathbb{Q}}[-B_1]\}.$$

It is left to recall that $\mathbb{Q}_i^{(\boldsymbol{\delta}\cdot\boldsymbol{B})} \in \mathcal{M}_i$ for all $\boldsymbol{\delta} \in \mathbb{R}^n$.

The proof of the equality (5.4.3) follows the same lines. $\square$

For every vector of claims $\boldsymbol{B} \in (\mathbb{L}^\infty)^n$ and for every agent i, we define the set

$$\mathcal{P}_i(\boldsymbol{B}) = \{\mathbb{E}^{\mathbb{Q}_i^{(\boldsymbol{a}\cdot\boldsymbol{B})}}[\boldsymbol{B}] : \boldsymbol{a} \in \mathbb{R}^n\} \subseteq \mathbb{R}^n. \tag{5.4.5}$$

Note that Lemma 5.4.3 implies, in particular, that $\sup\{\mathcal{P}_i(B)\} = \sup_{\mathbb{Q}\in\mathcal{M}_i}\{\mathbb{E}_{\mathbb{Q}}[-B]\}$, for every $B \in \mathbb{L}^\infty$.

Proposition 5.4.4. *Let $i \in \{1, 2, ..., I\}$ and $\boldsymbol{B} = (B_1, B_2, ..., B_n) \in (\mathbb{L}^\infty)^n$ be a vector of claims for which there is no $\boldsymbol{a} \in \mathbb{R}^n$ such that $\boldsymbol{a}\cdot\boldsymbol{B} \in \mathcal{R}_i$. If $\mathcal{A}_i$ is weak-*closed and strictly convex with respect to $\boldsymbol{B}$, then the function $\boldsymbol{a} \mapsto \rho_i(\boldsymbol{a}\cdot\boldsymbol{B})$ is differentiable for every $\boldsymbol{a} \in \mathbb{R}^n$ and $\nabla\rho_i(\boldsymbol{a}\cdot\boldsymbol{B}) = \mathbb{E}^{\mathbb{Q}_i^{(\boldsymbol{a}\cdot\boldsymbol{B})}}[-\boldsymbol{B}]$, i.e.,*

$$\frac{\partial}{\partial a_k}\rho_i(\boldsymbol{a}\cdot\boldsymbol{B}) = \mathbb{E}^{\mathbb{Q}_i^{(\boldsymbol{a}\cdot\boldsymbol{B})}}[-B_k], \tag{5.4.6}$$

for every $k = 1, 2, ..., n$ and $\boldsymbol{a} \in \mathbb{R}^n$.

Proof. According to Proposition I.5.3 in [33], we need to show that $\mathbb{E}^{\mathbb{Q}_i^{(\boldsymbol{a}\cdot\boldsymbol{B})}}[-\boldsymbol{B}]$ is the unique subgradient of the function $\boldsymbol{\delta} \ni \mathbb{R}^n \mapsto \rho_i(\boldsymbol{\delta}\cdot\boldsymbol{B}) \in \mathbb{R}$ at $\boldsymbol{a} = (a_1, a_2, ..., a_n) \in \mathbb{R}^n$, i.e., if for some $\boldsymbol{a}^* = (a_1^*, a_2^*, ..., a_n^*) \in \mathbb{R}^n$ it holds that

$$\rho_i(\boldsymbol{\delta}\cdot\boldsymbol{B}) \geq \rho_i(\boldsymbol{a}\cdot\boldsymbol{B}) - \boldsymbol{a}^*\cdot(\boldsymbol{a} - \boldsymbol{\delta}),\ \forall\boldsymbol{\delta} \in \mathbb{R}^n, \tag{5.4.7}$$

then $\boldsymbol{a}^* = \mathbb{E}^{\mathbb{Q}_i^{(\boldsymbol{a}\cdot\boldsymbol{B})}}[-\boldsymbol{B}]$.

Assume that $\boldsymbol{a}^*$ satisfies (5.4.7) and $\boldsymbol{a}^* \neq \mathbb{E}^{\mathbb{Q}_i^{(\boldsymbol{a}\cdot\boldsymbol{B})}}[-\boldsymbol{B}]$. Suppose, first, that $-\boldsymbol{a}^* \in \mathcal{P}_i(\boldsymbol{B})$, i.e., that there exists $\hat{\boldsymbol{\delta}} \in \mathbb{R}^n$ such that $\hat{\boldsymbol{\delta}} \neq \boldsymbol{a}$ and $\boldsymbol{a}^* = \mathbb{E}^{\mathbb{Q}_i^{(\hat{\boldsymbol{\delta}}\cdot\boldsymbol{B})}}[-\boldsymbol{B}]$. For $\boldsymbol{\delta} = \hat{\boldsymbol{\delta}}$ in (5.4.7) we get

$$\rho_i(\hat{\boldsymbol{\delta}} \cdot \boldsymbol{B}) + \mathbb{E}^{\mathbb{Q}_i^{(\hat{\boldsymbol{\delta}}\cdot\boldsymbol{B})}}[\hat{\boldsymbol{\delta}} \cdot \boldsymbol{B}] \geq \rho_i(\boldsymbol{a} \cdot \boldsymbol{B}) + \mathbb{E}^{\mathbb{Q}_i^{(\hat{\boldsymbol{\delta}}\cdot\boldsymbol{B})}}[\boldsymbol{a} \cdot \boldsymbol{B}].$$

Note, however, that

$$\begin{aligned} \rho_i(\boldsymbol{a} \cdot \boldsymbol{B}) &\geq \mathbb{E}^{\mathbb{Q}_i^{(\hat{\boldsymbol{\delta}}\cdot\boldsymbol{B})}}[-\boldsymbol{a} \cdot \boldsymbol{B}] - \alpha_i(\mathbb{Q}_i^{(\hat{\boldsymbol{\delta}}\cdot\boldsymbol{B})}) \\ &= \mathbb{E}^{\mathbb{Q}_i^{(\hat{\boldsymbol{\delta}}\cdot\boldsymbol{B})}}[-\boldsymbol{a} \cdot \boldsymbol{B}] + \rho_i(\hat{\boldsymbol{\delta}} \cdot \boldsymbol{B}) + \mathbb{E}^{\mathbb{Q}_i^{(\hat{\boldsymbol{\delta}}\cdot\boldsymbol{B})}}[\hat{\boldsymbol{\delta}} \cdot \boldsymbol{B}], \end{aligned}$$

and the equality holds if and only if the measure $\mathbb{Q}_i^{(\hat{\boldsymbol{\delta}}\cdot\boldsymbol{B})}$ is a maximizer of the difference $\mathbb{E}^{\mathbb{Q}}[-\boldsymbol{a} \cdot \boldsymbol{B}] - \alpha_i(\mathbb{Q})$. But this implies that

$$\frac{1}{2}(\rho_i(\boldsymbol{a} \cdot \boldsymbol{B}) + \rho_i(\hat{\boldsymbol{\delta}} \cdot \boldsymbol{B})) = \rho_i\left(\frac{(\hat{\boldsymbol{\delta}} + \boldsymbol{a}) \cdot \boldsymbol{B}}{2}\right),$$

which contradicts the assumption of strict convexity.

It is left to show that if $\boldsymbol{a}^* \in \mathbb{R}^n$ satisfies (5.4.7), then $-\boldsymbol{a}^* \in \mathcal{P}_i(\boldsymbol{B})$. We argue by contradiction and assume that this is not the case. By Lemma 5.4.3 this means that there exists $l = 1, 2, ..., n$ such that $a_l^* \geq \sup_{\mathbb{Q}\in\mathcal{M}_i} \{\mathbb{E}^{\mathbb{Q}}[-B_l]\}$ or $a_l^* \leq \inf_{\mathbb{Q}\in\mathcal{M}_i} \{\mathbb{E}^{\mathbb{Q}}[-B_l]\}$. Without loss of generality, suppose that the former holds and that $l = 1$. Then, applying (5.4.7) for $\boldsymbol{\delta} = \tilde{\boldsymbol{a}} = (a_1 + \epsilon, a_2, a_3, ..., a_n)$, where $\epsilon > 0$, yields that

$$\rho_i(\tilde{\boldsymbol{a}} \cdot \boldsymbol{B}) + \epsilon \inf_{\mathbb{Q}\in\mathcal{M}_i} \{\mathbb{E}^{\mathbb{Q}}[B_1]\} \geq \rho_i(\boldsymbol{a} \cdot \boldsymbol{B}).$$

But this contradicts that

$$\begin{aligned} \rho_i(\boldsymbol{a} \cdot \boldsymbol{B}) &\geq \mathbb{E}^{\mathbb{Q}_i^{(\tilde{\boldsymbol{a}}\cdot\boldsymbol{B})}}[-\boldsymbol{a} \cdot \boldsymbol{B}] - \alpha_i(\mathbb{Q}_i^{(\tilde{\boldsymbol{a}}\cdot\boldsymbol{B})}) \\ &= \rho_i(\tilde{\boldsymbol{a}} \cdot \boldsymbol{B}) + \epsilon \mathbb{E}^{\mathbb{Q}_i^{(\tilde{\boldsymbol{a}}\cdot\boldsymbol{B})}}[B_1] > \rho_i(\tilde{\boldsymbol{a}} \cdot \boldsymbol{B}) + \epsilon \inf_{\mathbb{Q}\in\mathcal{M}_i} \{\mathbb{E}^{\mathbb{Q}}[B_1]\}. \end{aligned}$$

This completes the proof. □

Remark 5.4.5. If for $\boldsymbol{B} \in (\mathbb{L}^\infty)^n$ there is no $\boldsymbol{a} \in \mathbb{R}^n$ such that $\boldsymbol{a} \cdot \boldsymbol{B} \in \mathcal{R}_i$ and if $\mathcal{A}_i$ is strictly convex with respect to $\boldsymbol{B}$, we can restrict ourselves only to prices which belong to $\mathcal{P}_i(\boldsymbol{B})$, for the specification of the corresponding demand $Z_i(\mathbf{p})$. This is because by its definition, the demand correspondence, $Z_i(\mathbf{p})$, consists of the solutions, $\boldsymbol{a}$, of the equation $-\nabla\rho_i(\boldsymbol{a}\cdot\boldsymbol{B}) = \mathbb{E}^{\mathbb{Q}_i^{(\boldsymbol{a}\cdot\boldsymbol{B})}}[\boldsymbol{B}] = \mathbf{p}$. For $\mathbf{p} \in \mathcal{P}_i(\boldsymbol{B})$, the mapping $Z_i(\mathbf{p})$. In fact a function, i.e., $Z_i(\mathbf{p})$ is a singleton. Indeed, if we assume that for some price vector $\hat{\mathbf{p}}$, there exist $\hat{\boldsymbol{\zeta}}$, $\tilde{\boldsymbol{\zeta}} \in \mathbb{R}^n$ such that $\hat{\boldsymbol{\zeta}} \neq \tilde{\boldsymbol{\zeta}}$ and $\mathbb{E}^{\mathbb{Q}^{(\hat{\boldsymbol{\zeta}}\cdot\boldsymbol{B})}}[\boldsymbol{B}] = \mathbb{E}^{\mathbb{Q}^{(\tilde{\boldsymbol{\zeta}}\cdot\boldsymbol{B})}}[\boldsymbol{B}] = \hat{\mathbf{p}}$, then

$$\rho_i(Z_i(\hat{\mathbf{p}}) \cdot \boldsymbol{B}) + Z_i(\hat{\mathbf{p}}) \cdot \hat{\mathbf{p}} = \rho_i(\hat{\boldsymbol{\zeta}} \cdot \boldsymbol{B}) + \hat{\boldsymbol{\zeta}} \cdot \hat{\mathbf{p}} = \rho_i(\tilde{\boldsymbol{\zeta}} \cdot \boldsymbol{B}) + \tilde{\boldsymbol{\zeta}} \cdot \hat{\mathbf{p}}.$$

Thanks to the strict convexity of ρ_i, for every $\lambda \in (0, 1)$, we have that

$$\rho_i((\lambda\hat{\boldsymbol{\zeta}} + (1-\lambda)\tilde{\boldsymbol{\zeta}}) \cdot \boldsymbol{B}) + (\lambda\hat{\boldsymbol{\zeta}} + (1-\lambda)\tilde{\boldsymbol{\zeta}}) \cdot \hat{\mathbf{p}} < \rho_i(Z_i(\hat{\mathbf{p}}) \cdot \boldsymbol{B}) + Z_i(\hat{\mathbf{p}}) \cdot \hat{\mathbf{p}}.$$

The last inequality contradicts the fact that

$$\rho_i(Z_i(\hat{\mathbf{p}}) \cdot \boldsymbol{B}) + Z_i(\hat{\mathbf{p}}) \cdot \hat{\mathbf{p}} \leq \rho_i(\boldsymbol{a} \cdot \boldsymbol{B}) + \boldsymbol{a} \cdot \boldsymbol{B},$$

which holds for every $\boldsymbol{a} \in \mathbb{R}^n$.

Proposition 5.4.6. *Let $i \in \{1, 2, ..., I\}$ and $\boldsymbol{B} \in (\mathbb{L}^\infty)^n$ be a vector of claims for which there is no $\boldsymbol{a} \in \mathbb{R}^n$ such that $\boldsymbol{a} \cdot \boldsymbol{B} \in \mathcal{R}_i$. If $\mathcal{A}_i$ is weak-*closed and strictly convex with respect to $\boldsymbol{B}$, then the demand function Z_i is continuous in $\mathcal{P}_i(\boldsymbol{B})$ and satisfies the monotonicity property*

$$(Z_i(\mathbf{p}_1) - Z_i(\mathbf{p}_2)) \cdot (\mathbf{p}_1 - \mathbf{p}_2) < 0,$$

for every $\mathbf{p}_1, \mathbf{p}_2 \in \mathcal{P}_i(\boldsymbol{B})$ with $\mathbf{p}_1 \neq \mathbf{p}_2$.

Proof. The continuity is a direct application of the Berge's Maximum Theorem (see, for instance, Theorem 17.31 in [1]). From the definition of the demand function, we have that

$$\rho_i(Z_i(\mathbf{p}_j) \cdot \boldsymbol{B}) + Z_i(\mathbf{p}_j) \cdot \mathbf{p}_j < \rho_i(\boldsymbol{a} \cdot \boldsymbol{B}) + \boldsymbol{a} \cdot \mathbf{p}_j$$

for every $\boldsymbol{a} \in \mathbb{R}^n$ with $\boldsymbol{a} \neq Z_i(\mathbf{p}_j)$, for $j = 1, 2$. Note also that $Z_i(\mathbf{p}_1) \neq Z_i(\mathbf{p}_2)$ since $Z_i(\mathbf{p}_j)$ is the solution $\hat{\boldsymbol{a}}$ of the equation $\mathbf{p}_j = \mathbb{E}^{\mathbb{Q}_i^{(\hat{\boldsymbol{a}} \cdot \boldsymbol{B})}}[\boldsymbol{B}]$. Thus,

$$\rho_i(Z_i(\mathbf{p}_1) \cdot \boldsymbol{B}) - \rho_i(Z_i(\mathbf{p}_2) \cdot \boldsymbol{B}) + (Z_i(\mathbf{p}_1) - Z_i(\mathbf{p}_2)) \cdot \mathbf{p}_1 < 0$$

and

$$\rho_i(Z_i(\mathbf{p}_2) \cdot \boldsymbol{B}) - \rho_i(Z_i(\mathbf{p}_1) \cdot \boldsymbol{B}) - (Z_i(\mathbf{p}_1) - Z_i(\mathbf{p}_2)) \cdot \mathbf{p}_2 < 0$$

By adding the above inequalities we conclude. □

5.4.2 The equilibrium pricing

Given our fixed vector of claims $\boldsymbol{B}$, if its price is given by vector $\mathbf{p}$, each agent sells/buys the units of $\boldsymbol{B}$ that minimizes her risk. As in the classical market clearing, we follow the ideas stated in Section 4.2 to define a partial equilibrium price for $\boldsymbol{B}$.

Definition 5.4.7. We say that the pair $(\mathbf{p}, \boldsymbol{a}) \in \bigcap_{i=1}^{I} \mathcal{P}_i(\boldsymbol{B}) \times \mathbf{F}$ is a *partial equilibrium price allocation (PEPA)*, if $\boldsymbol{a}_i = Z_i(\mathbf{p})$ for every i, i.e.,

$$\sum_{i=1}^{I} Z_i(\mathbf{p}) = (0, 0, ..., 0). \tag{5.4.8}$$

In remainder of this chapter, we impose the following assumptions.

Assumption 5.4.8. The acceptance set $\mathcal{A}_i$ is weak-*closed and strictly convex with respect to $\boldsymbol{B}$, for all $i = 1, 2, ..., I$.

Assumption 5.4.9. There is no $\boldsymbol{\delta} \in \mathbb{R}^I \setminus \{\mathbf{0}\}$ for which $\mathbb{E}^{\mathbb{Q}}[\boldsymbol{\delta} \cdot \boldsymbol{B}]$ is constant function of $\mathbb{Q} \in \mathcal{M}$.

Remark 5.4.10. Assumption 5.4.9 implies, in particular, that for every $i = 1, 2, ..., I$, there is no $\boldsymbol{\delta} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ such that $\boldsymbol{\delta} \cdot \boldsymbol{B} \in \mathcal{X}_i^\infty$. This means that there is no $\boldsymbol{\delta} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ such $\boldsymbol{\delta} \cdot \boldsymbol{B} \in \mathcal{R}_i$.

Remark 5.4.11. If Assumptions 5.4.8 and 5.4.9 hold true and the agents are in Pareto optimal condition (i.e., $\mathbb{Q}_i^{(0)} = \mathbb{Q}_j^{(0)}$, $\forall i, j$), then the only PEPA is of the form $(\mathbf{p}, \mathbf{0})$. To show this, we assume that there exists an $\boldsymbol{a} \in \mathbf{F} \setminus \{\mathbf{0}\}$ such that $(\mathbf{p}, \boldsymbol{a})$ is a PEPA. Then, $\rho_i(\boldsymbol{a}_i \cdot \boldsymbol{B}) + \boldsymbol{a}_i \cdot \mathbf{p} \leq 0$ for all i. Therefore, by the strict convexity of the risk measures for at least two agents the inequality is strict. This implies that $\sum_{i=1}^{I} \rho_i(\boldsymbol{a}_i \cdot \boldsymbol{B}) < 0$, which contradicts the assumption of Pareto optimality. In other words, the notion of the PEPA is meaningful only when the probability measures that minimize the penalty functions are not the same for all the agents.

Theorem 5.4.12. *For a vector of claims $\boldsymbol{B} \in (\mathbb{L}^\infty)^n$ satisfying Assumption 5.4.9 and under Assumptions 5.3.5 and 5.4.8, there exists a unique PEPA $(\hat{\mathbf{p}}, \hat{\boldsymbol{a}}) \in \bigcap_{i=1}^{I} \mathcal{P}_i(\boldsymbol{B}) \times \mathbf{F}$.*

Proof. We first define the strictly convex function $f : \mathbb{R}^{(I-1)\times n} \to \mathbb{R}$ by

$$f(\boldsymbol{a}) = \rho_1(\boldsymbol{a}_1 \cdot \boldsymbol{B}) + \rho_2(\boldsymbol{a}_2 \cdot \boldsymbol{B}) + ... + \rho_{I-1}(\boldsymbol{a}_{I-1} \cdot \boldsymbol{B}) + \rho_I((-\sum_{i=1}^{I-1} \boldsymbol{a}_i) \cdot \boldsymbol{B}).$$

If for some $\tilde{\boldsymbol{a}} \in \mathbb{R}^{(I-1)\times n}$, we have that $\nabla f(\tilde{\boldsymbol{a}}) = \mathbf{0}$, then $\tilde{\boldsymbol{a}}$ is the unique minimizer of f, for which $\nabla \rho_i(\tilde{\boldsymbol{a}}_i \cdot \boldsymbol{B}) = \nabla \rho_I(-\left(\sum_{i=1}^{I-1} \tilde{\boldsymbol{a}}_i\right) \cdot \boldsymbol{B})$, for every $i = 1, 2, ..., I-1$, where $\tilde{\boldsymbol{a}}_i$ denotes the vector $(\tilde{a}_{i,k})_{k=1,2,...,n}$. The latter means that

$$\mathbb{E}^{\mathbb{Q}_i^{(\tilde{\boldsymbol{a}}_i \cdot \boldsymbol{B})}}[\boldsymbol{B}] = \mathbb{E}^{\mathbb{Q}_I^{(-(\sum_{i=1}^{I-1} \tilde{\boldsymbol{a}}_i)\cdot \boldsymbol{B})}}[\boldsymbol{B}],$$

for every $i = 1, 2, ..., I-1$. Then, for the price $\hat{\mathbf{p}} = \mathbb{E}^{\mathbb{Q}_i^{(\tilde{\boldsymbol{a}}_i \cdot \boldsymbol{B})}}[\boldsymbol{B}]$, we have that $Z_i(\hat{\mathbf{p}}) = \tilde{\boldsymbol{a}}_i$ for every $i = 1, 2, ..., I-1$ and $Z_I(\hat{\mathbf{p}}) = -\sum_{i=1}^{I-1} \tilde{\boldsymbol{a}}_i$.

In other words, if we denote by $\hat{\boldsymbol{a}}$ the allocation whose rows are given by $\hat{\boldsymbol{a}}_i = \tilde{\boldsymbol{a}}_i$, for $i = 1, 2, ..., I-1$ and $\hat{\boldsymbol{a}}_I = -\sum_{i=1}^{I-1} \tilde{\boldsymbol{a}}_i$, the pair $(\hat{\mathbf{p}}, \hat{\boldsymbol{a}})$ is a partial equilibrium price allocation. In fact, it is the unique one, since, if we assume the existence of another PEPA $(\check{\mathbf{p}}, \check{\boldsymbol{a}}) \neq (\hat{\mathbf{p}}, \hat{\boldsymbol{a}})$, we get that $\check{\mathbf{p}} = \mathbb{E}^{\mathbb{Q}_i^{(\check{\boldsymbol{a}}_i \cdot \boldsymbol{B})}}[\boldsymbol{B}]$, which in turn implies that $\nabla f(\check{\boldsymbol{a}}) = \mathbf{0}$. The latter equation contradicts the uniqueness of the minimizer of the function f.

Hence, it suffices to show that $\nabla f(\boldsymbol{a})$ has a root. Assume that this is not the case. Then, by the continuity of f, we deduce that for, each $m \in \mathbb{N}$, there exists $\boldsymbol{a}^{(m)} \in \mathbf{D}_m = \{\boldsymbol{a} \in \mathbb{R}^{(I-1)\times n} : \sum_{k=1}^{(I-1)n} |a_k| \leq m\}$ such that $f(\boldsymbol{a}^{(m)}) \leq f(\boldsymbol{a})$ for all $\boldsymbol{a} \in \mathbf{D}_m$. Furthermore, by the strict convexity of f it follows that $||\boldsymbol{a}^{(m)}||_1 = m$. This means that, in order to reach a contradiction, it is enough to establish

$$\liminf_{m\to\infty} \frac{f(\boldsymbol{a}^{(m)})}{m} > 0. \tag{5.4.9}$$

(see e.g. Chapter 1 in [16]). Any sequence of $\mathbb{N}$ through which the lim inf in (5.4.9) is achieved admits a further subsequence $(m_k)_{k\in\mathbb{N}}$, for which $\boldsymbol{a}^{(m_k)}\frac{1}{m_k}$

converges to some $\boldsymbol{a}^{(0)} \in \mathbb{R}^{(I-1)\times n}$ with $||\boldsymbol{a}^{(0)}||_1 = 1$ (and in particular $\boldsymbol{a}_i^{(m_k)}$ converges to $\boldsymbol{a}_i^{(0)}$ for every $i = 1, 2, ..., I-1$). We consider this subsequence and, for each i, we have that

$$\liminf_{k\to\infty} \frac{\rho_i(\boldsymbol{a}_i^{(k)} \cdot \boldsymbol{B})}{k} = \lim_{k\to\infty} \rho_i'(k(\boldsymbol{a}_i^{(0)} \cdot \boldsymbol{B})),$$

where $\rho_i'(\beta(\boldsymbol{a}_i^{(0)} \cdot \boldsymbol{B}))$ denotes the derivative of the function $\beta \mapsto \rho_i((\beta(\boldsymbol{a}_i^{(0)} \cdot \boldsymbol{B})))$ for $\beta \in \mathbb{R}$ (which exists by Proposition 5.4.4). Indeed, by the convexity and the Lipschitz continuity of ρ_i (see discussion after Remark B.0.4),

$$\begin{aligned}\liminf_{k\to\infty} \left| \frac{\rho_i(\boldsymbol{a}_i^{(k)} \cdot \boldsymbol{B})}{k} - \rho_i'(k(\boldsymbol{a}_i^{(0)} \cdot \boldsymbol{B}) \right| &= \liminf_{k\to\infty} \left| \frac{\rho_i(\boldsymbol{a}_i^{(k)} \cdot \boldsymbol{B})}{k} - \frac{\rho_i(k(\boldsymbol{a}_i^{(0)} \cdot \boldsymbol{B})}{k} \right| \\ &\leq ||\boldsymbol{a}_i^{(k)} - \boldsymbol{a}_i^{(0)}||_1 ||\boldsymbol{B}||_{(\mathbb{L}^\infty)^d} \to 0.\end{aligned}$$

Thus, it is left to show that

$$\lim_{k\to\infty} \left(\sum_{i=1}^{I-1} \rho_i'(k(\boldsymbol{a}_i^{(0)} \cdot \boldsymbol{B})) \right) + \rho_I' \left(k(-\sum_{i=1}^{I-1} \boldsymbol{a}_i^{(0)} \cdot \boldsymbol{B}) \right) > 0.$$

Taking also Lemma 5.4.3 into account, we have that

$$\lim_{k\to\infty} \rho_i'(k(\boldsymbol{a}_i^{(0)} \cdot \boldsymbol{B})) = \sup\{\mathcal{P}_i((-\boldsymbol{a}_i^{(0)} \cdot \boldsymbol{B}))\}.$$

Hence,

$$\begin{aligned}\liminf_{k\to\infty} \frac{f(\boldsymbol{a}^{(k)})}{k} &= \sum_{i=1}^{I-1} \sup_{\mathbb{Q}\in\mathcal{M}_i} \{\mathbb{E}^{\mathbb{Q}}[-\boldsymbol{a}_i^{(0)} \cdot \boldsymbol{B}]\} + \sup_{\mathbb{Q}\in\mathcal{M}_I} \{\mathbb{E}^{\mathbb{Q}}[(\sum_{i=1}^{I-1} \boldsymbol{a}_i^{(0)}) \cdot \boldsymbol{B}]\} \\ &\geq \sum_{i=1}^{I-1} \sup_{\mathbb{Q}\in\mathcal{M}} \{\mathbb{E}^{\mathbb{Q}}[-\boldsymbol{a}_i^{(0)} \cdot \boldsymbol{B}]\} + \sup_{\mathbb{Q}\in\mathcal{M}} \{\mathbb{E}^{\mathbb{Q}}[(\sum_{i=1}^{I-1} \boldsymbol{a}_i^{(0)}) \cdot \boldsymbol{B}]\} \\ &\geq \sup_{\mathbb{Q}\in\mathcal{M}} \{\mathbb{E}^{\mathbb{Q}}[(-\sum_{i=1}^{I-1} \boldsymbol{a}_i^{(0)}) \cdot \boldsymbol{B}]\} - \inf_{\mathbb{Q}\in\mathcal{M}} \mathbb{E}^{\mathbb{Q}}[(-\sum_{i=1}^{I-1} \boldsymbol{a}_i^{(0)}) \cdot \boldsymbol{B}] > 0,\end{aligned}$$

where the last strict inequality follows directly from Assumption 5.4.9. □

Remark 5.4.13. By the uniqueness of the minimizer $\hat{\boldsymbol{a}} \in \mathbf{F}$ of the function f, we get that any agent i who participates in the equilibrium (that is $\hat{\boldsymbol{a}}_i \neq 0$) enjoys a risk reduction (improvement), i.e, $\rho_i(\hat{\boldsymbol{a}}_i \cdot \boldsymbol{B} - \hat{\boldsymbol{a}}_i \cdot \hat{\mathbf{p}}) < 0$, where $\hat{\mathbf{p}}$ is the partial equilibrium price (PEP).

Remark 5.4.14. It follows from Theorem 5.4.12 that the PEPA on a vector of claims $\boldsymbol{B}$ is of the form $(\hat{\mathbf{p}}, \mathbf{0})$, if and only if $\mathbb{E}^{\mathbb{Q}_i^{(0)}}[\boldsymbol{B}] = \mathbb{E}^{\mathbb{Q}_j^{(0)}}[\boldsymbol{B}]$ for every $i, j \in \{1, 2, ..., n\}$. In other words, for every vector of claims $\boldsymbol{B} \in (\mathbb{L}^\infty)^n$ for which the hypotheses of Theorem 5.4.12 hold and $\mathbb{E}^{\mathbb{Q}_i^{(0)}}[\boldsymbol{B}] \neq \mathbb{E}^{\mathbb{Q}_j^{(0)}}[\boldsymbol{B}]$, for some $i \neq j$, there exists an allocation $\hat{\boldsymbol{a}} \in \mathbf{F}$ such that $\hat{\boldsymbol{a}} \neq \mathbf{0}$ and $\boldsymbol{B} \in \mathcal{G}^{\hat{\boldsymbol{a}}}$.

Appendices

Appendix A

The Dynamic Version of the Indifference Price

In addition to the study of the indifference prices $\nu^{(w)}(B;\gamma|\mathcal{E})$ and $\nu^{(b)}(B;\gamma|\mathcal{E})$ defined at time $t=0$, one can restrict attention to any subinterval $[t,T]$ of $[0,T]$, and consider the filtered probability space $(\Omega,\mathcal{F},(\mathcal{F}_u)_{u\in[t,T]},\mathbb{P})$ and the traded-assets process $(\mathbf{S}_u)_{u\in[t,T]}$. To keep this text self-contained, we state the definition of the dynamic version of the (conditional) indifference price, together with some results used in the previous chapters. In Section A.2, we define the residual risk process of the conditional indifference price valuation and state a characterization of it in the case of continuous filtrations. In Section A.3, we give a brief discussion on the class of *BMO*-martingales.

A.1 The Conditional Indifference Price Process

The value function defined in (2.1.3) is the indirect utility of the agent at time $t=0$. In a similar way, one can define the agent's value function at any time $t\in[0,T]$. More precisely, for every claim payoff $B\in\mathbb{L}^\infty$, any random endowment $\mathcal{E}\in\mathbb{L}^\infty$, any risk aversion coefficient $\gamma>0$ and any time

$t \in [0,T]$ we define the *indirect utility process* as

$$
\begin{aligned}
u_\gamma(B,t|\mathcal{E}) &= \operatorname*{esssup}_{\boldsymbol{\vartheta}\in\boldsymbol{\Theta}} \mathbb{E}\Big[U\left(\int_t^T \boldsymbol{\vartheta}_u d\mathbf{S}_u + \mathcal{E} + B\right)\Big|\mathcal{F}_t\Big] \\
&= \operatorname*{esssup}_{\boldsymbol{\vartheta}\in\boldsymbol{\Theta}} \mathbb{E}\Big[-\exp\left(-\gamma\left(\int_t^T \boldsymbol{\vartheta}_u d\mathbf{S}_u + \mathcal{E} + B\right)\right)\Big|\mathcal{F}_t\Big].
\end{aligned}
\tag{A.1.1}
$$

Similarly as in the static case, the indirect utility process induces an acceptance set at any time $t \in [0,T]$,

$$
\mathcal{A}_\gamma(\mathcal{E},t) = \{B \in \mathbb{L}^\infty \,:\, u_\gamma(B,t|\mathcal{E}) \geq u_\gamma(0,t|\mathcal{E})\} \tag{A.1.2}
$$

and its strict version

$$
\mathcal{A}^\circ_\gamma(\mathcal{E},t) = \{B \in \mathbb{L}^\infty \,:\, u_\gamma(B,t|\mathcal{E}) > u_\gamma(0,t|\mathcal{E})\}\,. \tag{A.1.3}
$$

Therefore, the (conditional) indifference price of a contingent claim at time t, $\nu_t^{(w)}(B;\gamma|\mathcal{E})$ is given by

$$
\nu_t^{(w)}(B;\gamma|\mathcal{E}) = \inf\left\{p \in \mathbb{L}^0(\mathcal{F}_t) \,:\, p - B \in \mathcal{A}_\gamma(\mathcal{E},t)\right\}, \tag{A.1.4}
$$

for any $t \in [0,T]$. In other words, $\nu_t^{(w)}(B;\gamma|\mathcal{E})$ is the minimum amount at which the agent with risk aversion coefficient γ and random endowment $\mathcal{E}$ is willing to sell at time t a claim with payoff B. Equivalently, we can define $\nu_t^{(w)}(B;\gamma|\mathcal{E})$ as the $\mathbb{P}-$a.s. unique solution of the equation

$$
\begin{aligned}
&\operatorname*{esssup}_{\boldsymbol{\vartheta}\in\boldsymbol{\Theta}} \mathbb{E}\Big[-\exp\Big(-\gamma\big(\mathcal{E} + \int_t^T \boldsymbol{\vartheta}_u\, d\mathbf{S}_u + \nu_t^{(w)}(B;\gamma|\mathcal{E}) - B\big)\Big)\Big|\mathcal{F}_t\Big] \\
&\qquad = \operatorname*{esssup}_{\boldsymbol{\vartheta}\in\boldsymbol{\Theta}} \mathbb{E}\Big[-\exp\Big(-\gamma\big(\mathcal{E} + \int_t^T \boldsymbol{\vartheta}_u\, d\mathbf{S}_u\big)\Big)\Big|\mathcal{F}_t\Big].
\end{aligned}
\tag{A.1.5}
$$

One can show, using the standard dynamic-programming method (see e.g., [62]) that $(\nu_t^{(w)}(B;\gamma|\mathcal{E}))_{t\in[0,T]}$ admits a cádlág modification. The cádlág process $(\nu_t^{(w)}(B;\gamma|\mathcal{E}))_{t\in[0,T]}$ is called the *writer's conditional indifference price process* for the claim B.

A natural analogue corresponding to the buyer's side can be introduced in a similar fashion. Namely, the *conditional buyer's indifference price process* of claim B, $(\nu_t^{(b)}(B;\gamma|\mathcal{E}))_{t\in[0,T]}$, is defined as the cádlág modification of the $\mathbb{P}-$a.s. unique solution of the equation

$$\begin{aligned}\operatorname*{esssup}_{\boldsymbol{\vartheta}\in\boldsymbol{\Theta}}\mathbb{E}\Big[-\exp\Big(-\gamma\big(\mathcal{E}+\int_t^T\boldsymbol{\vartheta}_u\,d\mathbf{S}_u-\nu_t^{(b)}(B;\gamma|\mathcal{E})+B\big)\Big)\Big|\mathcal{F}_t\Big]\\ =\operatorname*{esssup}_{\boldsymbol{\vartheta}\in\boldsymbol{\Theta}}\mathbb{E}\Big[-\exp\Big(-\gamma\big(\mathcal{E}+\int_t^T\boldsymbol{\vartheta}_u\,d\mathbf{S}_u\big)\Big)\Big|\mathcal{F}_t\Big]. \quad \text{(A.1.6)}\end{aligned}$$

In other words, $\nu_t^{(b)}(B;\gamma|\mathcal{E})$ is the maximum amount at which the agent with risk aversion coefficient γ and random endowment $\mathcal{E}$ is willing to buy at time t a contingent claim with payoff B.

Below, we state some properties of the process $(\nu_t^{(w)}(B;\gamma|\mathcal{E}))_{t\in[0,T]}$, for the proof of which we refer the reader to [62] (see also Proposition 2.2.2).

Proposition A.1.1. *For fixed $\mathcal{E}\in\mathbb{L}^\infty$, $\gamma>0$ and $t\in[0,T]$, the mapping $B\mapsto\nu_t^{(w)}(B;\gamma|\mathcal{E})$ satisfies the following properties*

1. *It is increasing, i.e., for every $B,C\in\mathbb{L}^\infty$ such that $B\le C$, $\mathbb{P}-$a.s., then $\nu_t^{(w)}(B;\gamma|\mathcal{E})\le\nu_t^{(w)}(C;\gamma|\mathcal{E})$, $\mathbb{P}-$a.s..*

2. *It is* $\mathcal{F}_t-$*convex, i.e., for every* $B, C \in \mathbb{L}^\infty$ *and* $\lambda \in \mathbb{L}^\infty(\mathcal{F}_t)$*, such that* $\lambda \in [0,1]$, $\nu_t^{(w)}(\lambda B+(1-\lambda)C;\gamma|\mathcal{E}) \leq \lambda\nu_t^{(w)}(B;\gamma|\mathcal{E})+(1-\lambda)\nu_t^{(w)}(C;\gamma|\mathcal{E})$, $\mathbb{P}-$*a.s.*.

3. *It is replication-invariant, i.e., for every* $\boldsymbol{\vartheta} \in \boldsymbol{\Theta}$*,* $D \in \mathbb{L}^\infty(\mathcal{F}_t)$ *and* $B \in \mathbb{L}^\infty$, $\nu_t^{(w)}(B + D + \int_t^T \boldsymbol{\vartheta}_u d\mathbf{S}_u;\gamma|\mathcal{E}) = \nu_t^{(w)}(B;\gamma|\mathcal{E}) + D$, $\mathbb{P}-$*a.s.*.

4. *It is time-consistent, i.e., for all stopping times* $\sigma, \tau \in [0,T]$ *such that* $\sigma \leq \tau$, $v_\sigma^{(w)}(B|\mathcal{E};\gamma) = v_\sigma^{(w)}\left(v_\tau^{(w)}(B|\mathcal{E};\gamma)|\mathcal{E};\gamma\right)$, $\mathbb{P}-$*a.s.*.

A.2 The Residual Risk Process

Having defined the dynamic versions of the indifference price processes, $(\nu_t^{(b)}(B;\gamma|\mathcal{E}))_{t\in[0,T]}$ and $(\nu_t^{(w)}(B;\gamma|\mathcal{E}))_{t\in[0,T]}$, one can extent the notion of the residual risk introduced in Section 2.3, to the dynamic setting. More precisely, the *writer's residual risk process* of a claim $B \in \mathbb{L}^\infty$, $(R_t^{(w)}(B;\gamma|\mathcal{E}))_{t\in[0,T]}$ is defined by

$$R_t^{(w)}(B;\gamma|\mathcal{E}) = \nu_t^{(w)}(B;\gamma|\mathcal{E}) - \nu^{(w)}(B;\gamma|\mathcal{E}) - \int_0^t \boldsymbol{\vartheta}_u^{(B|\mathcal{E})}\, d\mathbf{S}_u,$$

(note that $R_T^{(w)}(B;\gamma|\mathcal{E}) = R^{(w)}(B;\gamma|\mathcal{E})$).

We can define the corresponding *buyer's residual risk process* by

$$R_t^{(b)}(B;\gamma|\mathcal{E}) = R_t^{(w)}(-B;\gamma|\mathcal{E}).$$

It is straightforward that

$$R_t^{(w)}(B;\gamma|\mathcal{E}) = R_t^{(w)}(B-\mathcal{E};\gamma) - R_t^{(w)}(-\mathcal{E};\gamma),\ t \in [0,T], \tag{A.2.1}$$

where $R_t^{(w)}(B;\gamma)$ stands for $R_t^{(w)}(B;\gamma|0)$. It is, also, clear from the discussion in Section A.1 that the processes $(R_t^{(w)}(B;\gamma|\mathcal{E}))_{t\in[0,T]}$ and $(R_t^{(b)}(B;\gamma|\mathcal{E}))_{t\in[0,T]}$ admit cádlág modifications.

It has been shown in [62] (see Theorem 13) that when $\mathbb{F}$ is continuous, the residual risk process admits a representation in terms of a martingale orthogonal to $\mathbf{S}$. Below, we state the straightforward extension of this result to the conditional case.

Theorem A.2.1. *([62])*
Suppose that the filtration $\mathbb{F}$ is continuous, and let the process $(R_t^{(w)}(B;\gamma|\mathcal{E}))_{t\in[0,T]}$, defined in (A.2.1). Then, there exists a process $(L_t^{(w)}(B;\gamma|\mathcal{E}))_{t\in[0,T]}$ such that

1. *$(L_t^{(w)}(B;\gamma|\mathcal{E}))_{t\in[0,T]}$ is a $\mathbb{Q}^{(-\gamma\mathcal{E})}-$martingale in the space $BMO(\mathbb{Q}^{(-\gamma\mathcal{E})})$*
2. *$R_t^{(w)}(B;\gamma|\mathcal{E}) = L_t^{(w)}(B;\gamma|\mathcal{E}) - \frac{\gamma}{2}\langle L^{(w)}(B;\gamma|\mathcal{E})\rangle_t$.*

When $\mathcal{E} \sim 0$, the family $\{L_t^{(w)}(B;\gamma)\}_{\gamma>0}$ (where $L_t^{(w)}(B;\gamma)$ denotes the process $L_t^{(w)}(B;\gamma|0)$) admits a limit $L_t^{(w)}(B;0)$, as $\gamma \searrow 0$, in $BMO(\mathbb{Q}^{(0)})$ sense. The process $L_t^{(w)}(B;0)$ can be identified as a term in the Kunita-Watanabe decomposition

$$B_t = \mathbb{E}^{\mathbb{Q}^{(0)}}[B] + \int_0^t \hat{\boldsymbol{\vartheta}}_u^{(B)}\, d\mathbf{S}_u + L_t^{(w)}(B;0),\ t \in [0,T], \tag{A.2.2}$$

of the $\mathbb{Q}^{(0)}-$martingale $B_t = \mathbb{E}^{\mathbb{Q}^{(0)}}[B|\mathcal{F}_t]$, where $\hat{\boldsymbol{\vartheta}}^{(B)}$ is an $\mathbf{S}-$integrable predictable process for which $(\hat{\boldsymbol{\vartheta}}^{(B)}\cdot\mathbf{S})$ is a $\mathbb{Q}^{(0)}-$square integrable martingale. In particular, $L^{(w)}(B;0)$ is strongly orthogonal to any $\mathbb{Q}^{(0)}-$local martingale of the form $(\boldsymbol{\vartheta}\cdot\mathbf{S})$, $\boldsymbol{\vartheta} \in L(\mathbf{S})$.

A.3 The *BMO* Martingales

To facilitate the exposition, we state in this section the definition of the class of Bounded Mean Oscillation-martingales, usually refereed as *BMO*-martingales, which are mentioned in Sections 2.4, 3.4 and A.2. For a detailed review on this class, we refer the reader to [56].

Consider the probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{Q})$, where $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ is a filtration that satisfies the usual conditions. Let $(M_t)_{t\in[0,T]}$ be a uniformly integrable $\mathbb{Q}-$martingale with respect to $\mathcal{F}_t$, with $M_0 = 0$. Then, we define the norm

$$\|M\|_{BMO(\mathbb{Q})} = \sup_{\tau} \left\| \mathbb{E}^{\mathbb{Q}} \left[|M_T - M_{\tau-}| \middle| \mathcal{F}_\tau \right] \right\|_\infty \tag{A.3.1}$$

where the supremum is taken over all stopping times τ and $||\cdot||_\infty$ stands for the $\mathbb{L}^\infty(\Omega, \mathcal{F}, \mathbb{Q})$ norm.

Definition A.3.1. A uniformly integrable $\mathbb{Q}-$martingale $(M_t)_{t\in[0,T]}$, with $M_0 = 0$, belongs in the $BMO(\mathbb{Q})$ class if $\|M\|_{BMO(\mathbb{Q})} < \infty$.

For the proof of following theorem, which is used in the proof of Lemma 2.4.4, we refer the reader to [56], page 28.

Theorem A.3.2. *For $p \in \mathbb{R}$ such that $1 < p < \infty$, there exists a positive constant C_p such that for any uniformly integrable $\mathbb{Q}-$martingale $(M_t)_{t\in[0,T]}$,*

$$\|M\|_{BMO(\mathbb{Q})} \leq \sup_{\tau} \left\| \mathbb{E}^{\mathbb{Q}} \left[|M_T - M_{\tau-}|^p \middle| \mathcal{F}_\tau \right]^{1/p} \right\|_\infty \leq C_p \|M\|_{BMO(\mathbb{Q})}$$

where the supremum is taken under all stopping times.

Appendix B

A Short Survey of Convex Risk Measures

A *risk measure* is a quantitative way to assess the risk involved in a financial position. If any investment is described by its discounted net payoff at the end of a certain time period, a risk measure maps any such position to the real line with the goal to give an assessment of its riskyness. Clearly, such a map should satisfy a number of axioms in order to be a "rational" measure of risk.

The axiomatic definition of such a measure of risk was introduced in [5] and [6], where the notion of the *coherent risk measure* (see Remark B.0.4) was defined for a finite Ω and its characterization, usually called *robust representation*, was established. Their results have been generalized for a larger family of probability models in [28] (see also [27] for a related overview). In [36] (see also Chapter 4 of [37]), the more general notion of convex risk measures was introduced and the corresponding robust representation was provided. More recently, the dynamic versions of risk measures have been developed by many authors (see among others, [7], [21], [32], [40], [68]) together with discussions on related issues such as time consistency (see [21] and [57]) and equilibrium arguments (see [8], [9], [11], [18], [34], [35], [43] and [52]). The concept of a risk measure in a financial market setting, discussed in Chapter 5, was first analyzed in [19] and [36] (see also Section 4.8 in [37]) and, then, extensively developed for a variety of financial models (see among others [41], [57], [67], [76]).

In this chapter, we state the axiomatic definition of risk measures together with some of their main properties used in Chapter 5. In order to be consistent with the main body of this text, we fix a time horizon and consider $\mathbb{L}^\infty(\Omega, \mathcal{F}, \mathbb{P})$ (as usually denoted by $\mathbb{L}^\infty$) as the set of the investment payoffs. Similar results as the ones stated below can also be obtained for larger payoff spaces (see, e.g., [20], [22]).

Definition B.0.3. A mapping $\rho : \mathbb{L}^\infty \to \mathbb{R}$ is called a *convex risk measure* or *convex capital requirement*, if it satisfies the following conditions, for every $B, C \in \mathbb{L}^\infty$:

1. *Monotonicity*, i.e., if $B \leq C$, $\mathbb{P}-$a.s., then $\rho(B) \geq \rho(C)$.

2. *Cash invariance* or *translation invariance*, i.e., for every $m \in \mathbb{R}$,

$$\rho(B + m) = \rho(B) - m.$$

3. *Convexity*, i.e., for every $\lambda \in [0, 1]$,

$$\rho(\lambda B + (1 - \lambda)C) \leq \lambda\rho(B) + (1 - \lambda)\rho(C).$$

Remark B.0.4. In the case where a convex risk measure is also positively homogenous, that is for all $\lambda > 0$, and $B \in \mathbb{L}^\infty$, $\rho(\lambda B) = \lambda\rho(B)$, then ρ is called a *coherent risk measure*.

The axiomatic properties in Definition B.0.3 are the minimal ones that any map whose aim is to qualify the risk involved in investments should satisfy. The monotonicity simply reflects the fact that investments with higher payoff should not increase the risk. Similarly, the convexity states that the diversification of investment positions results in less or equal risk. The property of cash invariance, and in particular the fact that $\rho(B + \rho(B)) = 0$, implies that $\rho(B)$ is in fact the amount of money which, if added to the payoff B, makes the risk of the position equal to zero. This property, also, means that $\rho(B)$ is measured in currency units.

A straightforward consequence of monotonicity and cash invariance property is that ρ is a Lipschitz continuous function with respect to the essential norm $||\cdot||_{\mathbb{L}^\infty}$. Indeed, for every $B, C \in \mathbb{L}^\infty$, $B \leq C + ||B - C||_{\mathbb{L}^\infty}$ and hence $\rho(C) - \rho(B) \leq ||B - C||_{\mathbb{L}^\infty}$.

The class of financial positions whose risk is non-positive are called *acceptable*. In other words, any convex risk measure ρ induces the class $\mathcal{A}_\rho$, defined as

$$\mathcal{A}_\rho = \left\{ B \in \mathbb{L}^\infty \,:\, \rho(B) \leq 0 \right\}. \tag{B.0.1}$$

Some properties of the acceptance set $\mathcal{A}_\rho$, which follows directly from its definition, are stated below.

Proposition B.0.5. *([36])*

Let ρ be a convex risk measure and $\mathcal{A}_\rho$ be its induced acceptance set. Then the following statements hold

1. $\mathcal{A}_\rho$ is monotone and convex.

2. For any $B \in \mathbb{L}^\infty$, $\rho(B) = \inf\{m \in \mathbb{R} \,:\, m + B \in \mathcal{A}_\rho\}$.

If in addition ρ is coherent, then $\mathcal{A}_\rho$ is a cone in $\mathbb{L}^\infty$.

Remark B.0.6. Proposition B.0.5 (especially its second part) signifies that one can consider a given non-empty, monotone and convex class $\mathcal{A} \subseteq \mathbb{L}^\infty$ as the set of agent's acceptable financial positions and, then, induce the corresponding risk measure $\rho_\mathcal{A}$ by the formula

$$\rho_\mathcal{A}(B) = \inf\{m \in \mathbb{R} \,:\, m + B \in \mathcal{A}\} \tag{B.0.2}$$

(see also (5.1.4)). As proven in Proposition 4.7 of [37], (B.0.2) holds if we assume in addition that there exists a constant c, such that $c \in \mathcal{A}$ and that for every $m \in \mathbb{R}_+$ and $B \in \mathbb{L}^\infty$, the set

$$\{\lambda \in [0,1] \,:\, \lambda m + (1-\lambda)B \in \mathcal{A}\} \text{ is closed in } [0,1].$$

The dual representation of the convex map ρ is usually called robust representation and is a way to characterize the class of convex risk measures in terms of expectations and a penalty function. Below, we state this characterization, which is established in Theorems 4.15 and 4.31 of [37].

Theorem B.0.7. *([36])*
*Let ρ be a lower semi-continuous (for the weak-*topology) convex risk measure on $\mathbb{L}^\infty$. Then ρ admits the following characterization*

$$\rho(B) = \sup_{\mathbb{Q} \in \mathcal{P}_a} \left\{\mathbb{E}^\mathbb{Q}[-B] - \alpha(\mathbb{Q})\right\} \tag{B.0.3}$$

for any $B \in \mathbb{L}^\infty$*, where* $\mathcal{P}_a$ *denotes the set of all absolutely continuous with respect to* $\mathbb{P}$ *probability measures in* $(\Omega, \mathcal{F})$ *and* $\alpha : \mathcal{P}_a \to \bar{\mathbb{R}}$ *is called penalty function, given by* $\alpha(\mathbb{Q}) = \underset{B \in \mathcal{A}_\rho}{sup} \mathbb{E}^{\mathbb{Q}}[-B]$*, for any* $\mathbb{Q} \in \mathcal{P}_a$*.*

Remark B.0.8.

1. The penalty function α defined above is the minimum one, that is, if there exists another map $\hat{\alpha} : \mathcal{P}_a \to \bar{\mathbb{R}}$ for which (B.0.3) holds, then $\alpha(\mathbb{Q}) \leq \hat{\alpha}(\mathbb{Q})$, for all $\mathbb{Q} \in \mathcal{P}_a$.

2. It can, also, been shown that the supremum in (B.0.3) is attained.

3. It follows by formula (B.0.1) that ρ is lower semi-continuous for the weak-*topology if and only if $\mathcal{A}$ is weak-*closed.

Remark B.0.9. Representation (B.0.3) can also be understood as a structural way to obtain a convex risk measure. Namely, (B.0.3) implies that $B \in \mathcal{A}_\rho$ if and only if $\mathbb{E}^{\mathbb{Q}}[B] \geq -\alpha(\mathbb{Q})$, for all $\mathbb{Q} \in \mathcal{P}_a$. In other words, for a given penalty function α, a position is acceptable if the expectation of its payoff under any probability measure absolutely continuous with respect to $\mathbb{P}$ is higher that the level $-\alpha(\mathbb{Q})$.

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Vita

Michail Anthropelos was born in Athens, Greece, on 21 October 1980, the son of Thomas Anthropelos and Irini Spinari. He received with honors the Bachelor of Science degree in Statistics and Insurance Science from the University of Piraeus in July 2002. He got his Master's in Mathematics with specialization in Mathematics of Finance from Columbia University in May 2003. Before he entered the Graduate School of University of Texas at Austin in August 2004, he enrolled to Master's program in Applied Mathematics, in the department of Applied Mathematics and Physical Science of the National and Technical University of Athens.

Permanent address: Iras str. 4, Agios Ioannis Rentis
Piraeus, 182 33, Greece

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