The efficiency of risk sharing between UK and US: Robust estimation and calibration under market incompleteness

Marcelo Fernandes
Economics Department, Queen Mary, University of London
E-mail: m.fernandes@qmul.ac.uk

José Gil Ferreira Vieira-Filho
Department of Economics, University of Chicago
E-mail: josegil@uchicago.edu

Abstract: We propose a novel estimator for the amount of international risk sharing that depends exclusively on asset returns data. In particular, our estimator has a nonparametric flavor in that it makes no parametric assumption on preferences and on the stochastic process that governs the dynamics of asset returns. This is in contrast with the existing estimators in the literature that either assume a specific utility function or that asset returns follow a geometric Brownian motion (GBM). Our estimates reveal there is less risk sharing between UK and US than one would find under the GBM assumption, though much more than what consumption data might suggest. Moreover, a simple calibration analysis shows that market incompleteness alone is enough to explain the difference between the consumption-based estimate of the risk-sharing index and ours.

JEL classification numbers: G12, C23, E44
Keywords: asset pricing, fixed-effects panel regression, incomplete markets, mimicking portfolio, stochastic discount factor.

Acknowledgments: We are indebted to Marco Bonomo, Carlos Eugênio da Costa, Giulio Fella, and Daniel Ferreira as well as seminar participants at Queen Mary, RiskControl/Coppead, University of Manchester, Brazilian Econometric Society Meeting (Salvador, Brazil), Brazilian Finance Society Meeting (Vitória, Brazil), Financial Econometrics Conference (University of York, UK), and Nonlinear Methods in Time Series (Udine, Italy) for valuable comments. The authors thank the financial support from the ESRC under the grant RES-062-23-0311 and from CAPES, respectively. The usual disclaimer applies.

1 Introduction

The amount of international risk sharing quantifies how efficient is the allocation of consumption among economies in the different states of nature. The usual framework then consists of measuring international risk sharing using consumption data (see Cochrane, 2001). In a frictionless economy, the law of one price ensures that the stochastic discount factor (SDF) exists and coincides with the marginal rate of intertemporal substitution of consumption (Harrison and Kreps, 1979; Hansen and Richard, 1987; Hansen and Jagannathan, 1991). It thus follows that one may measure the degree of international risk sharing by estimating the association between the domestic and foreign stochastic discount factors. To avoid the excess smoothness of consumption data, it is convenient to estimate the latter using only asset returns data (Campbell, 1993).

Brandt, Cochrane and Santa-Clara (2006) propose an index of international risk sharing that gauges the association between the domestic and foreign stochastic discount factors. They estimate the stochastic discount factor under the parametric assumption that asset prices follow a geometric Brownian motion (GBM) and hence are lognormal. This means that the resulting estimator of the international risk-sharing index depends heavily on the GBM and lognormality assumptions. This is very unfortunate in view that the empirical literature strongly rejects such hypotheses for asset prices (Mandelbrot, 1963; Aït-Sahalia, 1996; Schwartz, 1997).

This paper proposes an estimation strategy for the index of international risk sharing that relies exclusively on asset returns data, without making any parametric assumption on preferences and on the stochastic process that governs the dynamics of asset prices. In particular, we focus on a panel-regression framework to derive an estimator for the variance of the minimum-variance stochastic discount factor (MVSDF). The precision of our risk-sharing index estimator ameliorates as both the number of time series observations and the number of assets in the economy increase. This is in contrast, for instance, with Brandt et al.'s (2006) parametric estimator, whose performance does not necessarily improve with the number of assets.

To justify the new estimation strategy, we perform a small scale Monte Carlo experiment focusing on our ability to precisely estimate the stochastic discount factor. Although our estimation strategy applies for a general class of stochastic processes, the Monte Carlo study focuses on two simple models for asset prices, namely, Black and Scholes's (1973) geometric Brownian motion and Vasicek's (1977) Ornstein-Uhlenbeck process. The motivation of the first setting is to assess the price we pay for using a more robust estimator that does not assume any particular stochastic process for asset prices. As for the second setup, the idea is to consider a marginal departure from the GBM assumption so as to deal with a slightly unfavorable setting for Brandt et al.'s (2006) estimator. As expected, the results show that our nonparametric estimator easily beats the GBM-based estimator in the Ornstein-Uhlenbeck world. What is more surprising, however, is that it also entails a better out-of-sample relative performance in the GBM setting. We thus conclude that the nonparametric character that our estimator brings about comes without any loss of efficiency.

In our empirical application, we investigate the amount of risk sharing between UK and US. We employ monthly data on 404 assets to estimate the variances of the UK and US minimum-variance stochastic discount factors. The resulting estimate of the risk-sharing index amounts to 87%, evincing that risks are very well shared through the existing asset markets. We then carefully examine the amount of overall risk sharing by running a calibration exercise that accounts for the additional uninsurable risks that are orthogonal to asset markets under market incompleteness. Under plausible assumptions on these nonspanned risks, we find that one could indeed observe poor risk sharing in consumption despite of the large degree of risk sharing in asset markets. We thus conclude that market incompleteness suffices to reconcile the difference between the asset market view of risk sharing and the view based on consumption data.

The remainder of this paper ensues as follows. Section 2 describes the theoretical foundation for the risk-sharing index based on the SDF approach to asset pricing. Section 3 discusses our two-step estimation procedure for the analysis of international risk sharing. The first step involves estimating the domestic and foreign stochastic discount factors, whereas the second step consists of computing their long-run variances so as to gauge the amount of international risk sharing. Section 4 reports some Monte Carlo results that indicate that our MVSDF estimator performs extremely well in finite samples. Section 5 then investigates the efficiency of risk sharing between UK and US. Section 6 offers some concluding remarks, whereas the appendix collects some technical derivations.

2 Risk sharing and the stochastic discount factor

The uncertainty in the economy intuitively relates to the variation of consumption among the different states of nature. To measure the degree of risk sharing between two economies, one must then employ a quantity that depends on how the growth rates of the marginal utility of consumption in the two economies relate. We therefore adopt Brandt et al.'s (2006) risk-sharing index that hinges on the fact that the SDF corresponds to the intertemporal marginal rate of substitution and to the growth rate of the marginal utility of consumption.

Harrison and Kreps (1979), Hansen and Richard (1987) and Hansen and Jagannathan (1991) describe a general framework to asset pricing that relies on the pricing equation

$$\mathbb{E}_t(M_{t+1}R_{i,t+1}) = 1, \qquad i \in \{1, \ldots, N\} \tag{1}$$

where $\mathbb{E}_t(\cdot)$ denotes the conditional expectation given the available information at time t, M_t is the stochastic discount factor, $R_{i,t}$ denotes the gross return on the ith risky asset, and N is the number of risky assets in a frictionless economy. In the CCAPM context, for instance, the SDF corresponds to the intertemporal marginal rate of substitution, hence the pricing equation (1) mainly illustrates the fact that consumers equate marginal rates of substitution to prices.

The existence of a stochastic discount factor M_t that prices assets according to (1) only requires that the law of one price holds (Cochrane, 2001). It is not necessary to assume, for instance, that there is a complete set of security markets. It is only to ensure the uniqueness of the SDF that one must assume away incomplete markets. Under market incompleteness, there will exist an infinite number of stochastic discount factors M_t that correctly price all traded securities. Notwithstanding, there still exists a unique discount factor M_t^* in the payoff space. Moreover, projecting any SDF onto the payoff space recovers M_t^*, so that one may decompose any SDF as $M_t = M_t^* + \xi_t$, where the noise ξ_t is orthogonal to any linear combination of the asset payoffs. The pricing implications of any stochastic discount factor M_t thus are the same as those of M_t^*, also known as the minimum-variance stochastic discount factor (MVSDF) or the mimicking portfolio for marginal utility growth (see Cochrane, 2001).

We will implicitly assume that the mimicking portfolio M_t^* is always positive given that our estimation strategy basically identifies its logarithm. This assumption implies that there are no arbitrage opportunities in the payoff space. It requires local non-satiability given that the latter implies that one may express the stochastic discount factor as $M_{t+1} = \beta \frac{U'(C_{t+1})}{U'(C_t)} > 0$, where β and C_t denote the subjective discount rate and the aggregate consumption at time t. Apart from nonsatiation, we impose no other restriction on the preferences representation.

We now turn to the definitions of the domestic and foreign stochastic discount factors so as to build a coherent framework for the analysis of international risk sharing.

2.1 The domestic and foreign stochastic discount factors

Consider an asset A that trades on the domestic market, but generates payoff at the foreign market. The law of one price across countries ensures that the price in foreign currency at the domestic market must coincide with the price at the foreign market. Applying (1) to the foreign market gives way to

$$\mathbb{E}_t\left(M_{f,t+1}\, R^f_{A,t+1}\right) = \mathbb{E}_t\left(M_{d,t+1}\, \frac{E_{t+1}}{E_t}\, R^f_{A,t+1}\right), \tag{2}$$

where E_t is the real exchange rate gauging how many units of domestic asset trade for one unit of foreign asset, $R^f_{A,t+1}$ is the gross return on asset A at the foreign market, and $M_{d,t+1}$ and $M_{f,t+1}$ are the domestic and foreign stochastic discount factors, respectively.

This means that, to ensure the consistency of the stochastic discount factor approach, one must enable investors to trade both domestic and foreign assets regardless of the country they come from. The domestic market thus include assets traded on the domestic country as well as on the foreign country. As the latter generates payoffs in a foreign currency, one must consider the exchange rate when computing their gross returns. The gross returns on the assets available to domestic investors thus are

$$\left(R^d_{1,t}, \ldots, R^d_{N^d,t}, \frac{E_t}{E_{t-1}}\, R^f_{1,t}, \ldots, \frac{E_t}{E_{t-1}}\, R^f_{N^f,t}\right), \tag{3}$$

where $R^d_{i,t}$ is the gross return on ith domestic asset, N^d is the number of domestic assets, $R^f_{j,t}$ is the gross return on jth foreign asset, and N^f is the number of foreign assets. Similarly, the vector of gross returns on assets available to foreign investors is

$$\left(\frac{E_{t-1}}{E_t}\, R^d_{1,t}, \ldots, \frac{E_{t-1}}{E_t}\, R^d_{N^d,t}, R^f_{1,t}, \ldots, R^f_{N^f,t}\right). \tag{4}$$

We next discuss how the domestic and foreign stochastic discount factors relate to the degree of international risk sharing so as to motivate the index we employ to gauge the latter.

2.2 International risk-sharing index

Backus, Foresi and Telmer (2001) show that, in the absence of arbitrage opportunities, (2) implies that there exists at least one pair of domestic and foreign stochastic discount factors such that

$$M_{f,t} = \frac{E_t}{E_{t-1}}\, M_{d,t}, \tag{5}$$

and hence

$$\Delta e_t = m_{f,t} - m_{d,t}, \tag{6}$$

where $m_{d,t} = \ln M_{d,t}$, $m_{f,t} = \ln M_{f,t}$, $e_t = \ln E_t$, and Δ denotes the first-difference operator. Brandt et al. (2006) thus propose to measure the degree of international risk sharing by

$$\lambda = 1 - \frac{\sigma^2(\Delta e_t)}{\sigma^2\left(m_{d,t}\right) + \sigma^2(m_{f,t})}, \tag{7}$$

where $\sigma^2(Z_t)$ denotes the unconditional variance of stochastic process Z.

The risk-sharing index λ in (7) is convenient for it relates to the covariance between the domestic and foreign stochastic discount factors in view that

$$\sigma^2(\Delta e_t) = \sigma^2\left(m_{f,t} - m_{d,t}\right) = \sigma^2\left(m_{d,t}\right) + \sigma^2\left(m_{f,t}\right) - 2\operatorname{cov}\left(m_{f,t}, m_{d,t}\right).$$

In particular, if there is no linear association between the domestic and foreign stochastic discount factors, λ will equal to zero, indicating the absence of international risk sharing. If the variances of the domestic and foreign stochastic discount factors coincide, then $|\lambda| = 1$ only if there is a perfect linear association between the domestic and foreign stochastic discount factors.

To estimate the variances of the domestic and foreign stochastic discount factors, Brandt et al. (2006) assume that both domestic and foreign asset prices vary according to a multivariate geometric Brownian motion. This is convenient because it allows estimating the stochastic discount factors using only asset returns data, avoiding the excessive smoothness of consumption data (Campbell, 1993). The price to pay is that the GBM assumption is not at all consistent with the dynamics that asset prices exhibit (Mandelbrot, 1963; Aït-Sahalia, 1996; Schwartz, 1997). In the next section, we show how to relax such unrealistic assumption by building on Araujo, Issler and Fernandes's (2006) identification strategy. In particular, we derive consistent panel-data estimators for the domestic and foreign stochastic discount factors that employ only asset returns data.

3 Estimation strategy

To conduct statistical inference, we must impose some restrictions on the stochastic nature of asset returns and of the stochastic discount factor. For the sake of exposition, we start with the stringent assumption that, for every asset $i \in \{1, \ldots, N\}$ in the economy, $M_{t+1}R_{i,t+1}$ is conditionally lognormal with constant variance given the available information at time t.[1] This setting is slightly more general than the GBM setup put forth by Brandt et al. (2006). In fact, conditional lognormality and homoskedasticity has a long pedigree in macroeconomics (see, e.g., Hansen and Singleton, 1983; Campbell, 1993; Lettau and Ludvigson, 2001). As well, conditional lognormality is also standard in financial econometrics, especially in the context of conditional heteroskedasticity (e.g., Meddahi and Renault, 2004). Although the appendix shows that our approach is valid in a much more general context, the conditional homoskedasticity assumption is quite fair if one restricts attention to low frequency data. It is well known, for instance, that GARCH-type effects fade away

[1] For notational simplicity, in this section, we denote the vector of gross returns at time t by $(R_{1,t}, \ldots, R_{N,t})$ rather than using the vectors in (3) and (4). Accordingly, we denote the stochastic discount factor by M_t without any reference to whether it relates to the domestic or foreign economy.

as the sampling frequency decreases (see Diebold, 1988; Drost and Nijman, 1993; Meddahi and Renault, 2004).

Taking logs of both sides of the pricing equation (1) gives way to

$$\ln \mathbb{E}_t(M_{t+1} R_{i,t+1}) = 0, \qquad i \in \{1, \ldots, N\}. \tag{8}$$

Conditional lognormality then implies that, for every $i \in \{1, \ldots, N\}$,

$$\mathbb{E}_t(m_{t+1} + r_{i,t+1}) + \tfrac{1}{2}\,\mathbb{V}_t(m_{t+1} + r_{i,t+1}) = 0, \tag{9}$$

where $m_t = \ln M_t$, $r_{i,t} = \ln R_{i,t}$, and the conditional variance $\mathbb{V}_t(\cdot)$ given the available information at time t is constant, say σ_i^2, under the conditional homoskedasticity assumption. Decomposing $r_{i,t+1} + m_{t+1}$ into the projection on the information set at time t and an orthogonal error results in

$$r_{i,t+1} + m_{t+1} = \mathbb{E}_t(r_{i,t+1} + m_{t+1}) + \epsilon_{i,t+1}, \tag{10}$$

where $\epsilon_{i,t+1}$ is Gaussian with mean zero and variance σ_i^2 given the conditional lognormality and homoskedasticity assumptions.

From (9) and (10), it follows that

$$r_{i,t+1} = -\, m_{t+1} - \tfrac{1}{2}\,\sigma_i^2 + \epsilon_{i,t+1}. \tag{11}$$

In the context of panel-data regression, (11) corresponds to a standard unobserved fixed-effects model with no explanatory variables other than time dummies, also known as the two-way fixed-effects model (see Baltagi, 2001). Stacking the estimates of the coefficients of the time dummies then presumably provide a consistent estimate for the series of the log-SDF, whereas the fixed-effects capture the individual heterogeneity stemming from the second moment of the log-returns. It turns out however that (11) identifies the coefficients of the time dummies only up to a normalization constant. In particular, Wallace and Hussain's (1969) within transformation yields the following consistent estimator

$$\widehat{m_t - \bar{m}} = \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} r_{i,t} - \frac{1}{N} \sum_{i=1}^{N} r_{i,t}, \tag{12}$$

where $\bar{m}$ denotes the sample average of log-SDF over time. It is also interesting to note that (12) depends exclusively on appropriate geometric averages of the asset gross returns, viz. $\ln\big(\bar{R}/\bar{R}_t\big)$ with $\bar{R} = \prod_{t=1}^{T} \prod_{i=1}^{N} R_{i,t}^{\frac{1}{NT}}$ and $\bar{R}_t = \prod_{i=1}^{N} R_{i,t}^{1/N}$ for $t = 1, \ldots, T$. It is therefore not surprising that, under market incompleteness, (12) relates to the minimum-variance stochastic discount factor (MVSDF), i.e., the unique orthogonal projection of any SDF onto the payoffs space.

We deem that our estimator has some interesting properties. First, the estimation procedure is straightforward in that it relies on a standard two-way fixed-effects regression. In fact, the resulting estimator is also very simple given that it depends only on averages of the asset log-returns. Second, it postulates neither a particular preference representation nor any pricing theory. Third, our estimator is consistent with the risk-premium for it avoids the excessive smoothness of consumption data.

Araujo et al. (2006) actually go one step further in that they actually estimate the level of the MVSDF at times $t = 2, \ldots, T$. Under some additional conditions, they show that

$$\widehat{M}_t = \left(\frac{\bar{R}_t^G}{T-1} \sum_{s=2}^{T} \frac{\bar{R}_s}{\bar{R}_s^G} \right)^{-1}, \tag{13}$$

where $\bar{R}_t$ and $\bar{R}_t^G$ are respectively the cross-sectional arithmetic and geometric average of assets' gross returns, converges in probability to the realization of the MVSDF as both the time-series and cross-section sample sizes grow without bound.

3.1 Estimating the degree of international risk sharing

The estimation of the risk-sharing index given by (7) involves two steps. The first estimates the domestic and foreign minimum-variance stochastic discount factors by (12) using the vectors of domestic and foreign gross returns given by (3) and (4), viz.

$$\widehat{m_{d,t} - \bar{m}_d} = \frac{1}{(N^d + N^f)T} \sum_{t=1}^{T} \left[\sum_{i=1}^{N^d} r_{i,t}^d + \sum_{i=1}^{N^f} \left(r_{i,t}^f + \Delta e_t \right) \right] - \frac{1}{N^d + N^f} \left[\sum_{i=1}^{N^d} r_{i,t}^d + \sum_{i=1}^{N^f} \left(r_{i,t}^f + \Delta e_t \right) \right]$$

$$\widehat{m_{f,t} - \bar{m}_f} = \frac{1}{(N^d + N^f)T} \sum_{t=1}^{T} \left[\sum_{i=1}^{N^f} r_{i,t}^f + \sum_{i=1}^{N^d} \left(r_{i,t}^d - \Delta e_t \right) \right] - \frac{1}{N^d + N^f} \left[\sum_{i=1}^{N^f} r_{i,t}^f + \sum_{i=1}^{N^d} \left(r_{i,t}^d - \Delta e_t \right) \right].$$

The second step consists of estimating the long-run variances of the domestic and foreign stochastic discount factors as well as of the exchange rate, giving way to

$$\hat{\lambda} = 1 - \frac{\hat{\sigma}^2(\Delta e_t)}{\hat{\sigma}^2 \left(\widehat{m_{d,t} - \bar{m}_d} \right) + \hat{\sigma}^2 \left(\widehat{m_{f,t} - \bar{m}_f} \right)}, \tag{14}$$

where $\hat{\sigma}$ denote Newey and West's (1987) estimator for the long-run variance.

The statistic in (14) consistently estimates the international risk-sharing index in (7) because the normalization of the stochastic discount factor does not affect the estimation of the long-run variance. Consistency actually depends on a double asymptotic argument in that both the cross-sectional and time-series dimensions of the panel must go to infinity. A large number of assets is necessary to ensure the consistency of the MVSDF estimator, whereas the large number of time-series observations ensures the estimation of the long-run variances in a consistent manner. In particular, if T is of order $o(N)$, it readily follows that $\hat{\lambda}$ converges in probability to λ at the usual $\sqrt{T}$ rate.

The estimator in (14) inherits the properties of the MVSDF estimator. This means that (14) entails a nonparametric character to the analysis of international risk sharing in that it makes no assumption on preference representations and on the stochastic process that governs the dynamics of asset returns. This is in stark contrast with Brandt et al.'s (2006) estimator that heavily depends on the parametric assumption that asset prices follow a multivariate geometric Brownian motion.

4 Monte Carlo simulations

In theory, the main advantage of our approach rests on the robustness of our MVSDF estimator in that it is consistent under very mild conditions. In particular, it does not rely on the very unrealistic GBM assumption that permeates analysis. It is therefore natural to ask whether this feature actually holds in practice. In what follows, we provide indirect numerical evidence that our risk-sharing index estimator is indeed more robust. Our evidence is only indirect because our Monte Carlo study actually compares the performance of Araujo et al.'s (2006) MVSDF estimator relative to the MVSDF estimator based on the GBM assumption.

We consider two different scenarios. In the first setting, asset prices follow a geometric Brownian motion as postulated by Brandt et al. (2006). The motivation is to assess the price we pay for using a more robust estimator that does not assume any particular stochastic process for asset prices. In the second, we generate asset prices that vary according to a stationary Ornstein-Uhlenbeck process as in Vasicek (1977). The idea is to consider a marginal departure from the GBM assumption so as to deal with a slightly unfavorable setting for Brandt et al.'s (2006) estimator. In particular, the Ornstein-Uhlenbeck assumption yields prices that are conditionally normal with constant variance, so that the exact transition density for returns with continuous compounding (i.e., log-returns) is conditionally non-normal and heteroskedastic, exhibiting nonlinear serial dependence.

The main advantage of confining our attention to the above specifications is that they entail closed-form analytical solutions for the transition densities, facilitating substantially the simulation task. Unfortunately, the same does not hold for the transition density of the stochastic discount factor in that it has closed-form solution only under the GBM assumption. In particular, the transition density of the stochastic discount factor process is lognormal and such that

$$m_{t+\triangle} \sim \mathcal{N}\left(\mu_m \triangle, \sigma_m^2 \triangle\right), \tag{15}$$

where $\triangle$ denotes the length of the discrete time interval, $\mu_m = -\left(r + \frac{1}{2}\,\boldsymbol{\mu}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}\right)$, $\sigma_m^2 = \boldsymbol{\mu}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}$, r is the risk-free rate of return, $\boldsymbol{\mu}$ is the $N \times 1$ vector of instantaneous risk premia, and $\boldsymbol{\Sigma}$ is a $N \times N$ instantaneous covariance matrix of asset returns.

In fact, Brandt et al. (2006) take advantage of (15) to estimate the SDF variance by plugging in the sample counterparts to the instantaneous risk premia and covariance matrix of asset returns:

$$\hat{\boldsymbol{\mu}} = \frac{1}{\triangle}\,\frac{1}{T}\sum_{t=1}^{T} \boldsymbol{R}_t^e \tag{16}$$

$$\hat{\boldsymbol{\Sigma}} = \frac{1}{\triangle}\,\frac{1}{T}\sum_{t=1}^{T} \left(\boldsymbol{R}_t^e - \hat{\boldsymbol{\mu}}\triangle\right)\left(\boldsymbol{R}_t^e - \hat{\boldsymbol{\mu}}\triangle\right)', \tag{17}$$

where $\boldsymbol{R}_t^e = (R_{1,t} - r_t, \ldots, R_{N,t} - r_t)'$ denotes the vector of excess returns over the risk-free rate r_t at time t. Moreover, It is straightforward that Brandt et al. (2006) implicitly estimate the SDF by

$$\widetilde{M}_t = \exp\left[-\left(\bar{r} + \frac{1}{2}\,\hat{\boldsymbol{\mu}}'\hat{\boldsymbol{\Sigma}}^{-1}\hat{\boldsymbol{\mu}}\right)\triangle - \hat{\boldsymbol{\mu}}'\hat{\boldsymbol{\Sigma}}^{-1}(\boldsymbol{R}_t - \bar{\boldsymbol{R}})\right], \tag{18}$$

where $\boldsymbol{R}_t = (R_{1,t}, \ldots, R_{N,t})'$ and $\bar{\boldsymbol{R}} = \frac{1}{T}\sum_{t=1}^{T} \boldsymbol{R}_t$.

In contrast, Araujo et al.'s (2006) estimator given by (13) does not depend on any particular specification for the stochastic process that governs the dynamics of asset prices. To compare the relative performance of (18) relative to Araujo et al.'s (2006) MVSDF estimator in (13), we employ three goodness-of-fit statistics to assess whether the estimators correctly price the values of the assets, which we set to one without any loss of generality. The first measure verifies whether the MVSDF estimates satisfy the unconditional version of the pricing equation (1), though we also compute the same goodness-of-fit statistic for the true SDF in the GBM context for the sake of comparison. The other goodness-of-fit measures are feasible only in the GBM context given that they also depend on the actual realization of the SDF process. The second relates to the standardized mean squared error of the estimators, whereas the third gauges the sample correlation between the actual and estimated stochastic discount factors.

The Monte Carlo design is very similar in both settings, hinging on $1,000$ replications. We first generate $T + 500$ time-series observations of N^* asset returns from the transition density. To alleviate initial condition effects, we discard the first 500 observations of each time series, forming a panel with T time-series observations of N^* asset returns. We then split the panel data for in-sample and out-of-sample purposes. We employ the first $N < N^*$ asset returns to estimate the

MVSDF according to (13) and (18), whereas we save the remaining $N^* - N$ returns to compute our main goodness-of-fit statistic:

$$\hat{p} = \frac{1}{T(N^* - N)} \sum_{i=N+1}^{N^*} \sum_{t=1}^{T} \left(\widehat{M}_t R_{i,t} - 1 \right)^2 \tag{19}$$

$$\tilde{p} = \frac{1}{T(N^* - N)} \sum_{i=N+1}^{N^*} \sum_{t=1}^{T} \left(\widetilde{M}_t R_{i,t} - 1 \right)^2 . \tag{20}$$

In the GBM context, we also compute

$$\bar{p} = \frac{1}{T(N^* - N)} \sum_{i=N+1}^{N^*} \sum_{t=1}^{T} \left(M_t R_{i,t} - 1 \right)^2 \tag{21}$$

to act as a sanity check, as well as the two other goodness-of-fit statistics. More specifically, we standardize the mean squared error of the estimators by the sample variance of the actual SDF realizations:

$$\widehat{\mathrm{MSE}} = \frac{\sum_{t=1}^{T} \left(\widehat{M}_t - M_t \right)^2}{\sum_{t=1}^{T} M_t^2} \qquad \text{and} \qquad \widetilde{\mathrm{MSE}} = \frac{\sum_{t=1}^{T} (\widetilde{M}_t - M_t)^2}{\sum_{t=1}^{T} M_t^2}, \tag{22}$$

and compute the sample correlation between the actual and estimated stochastic discount factors, i.e., corr $\left(\widehat{M}_t, M_t \right)$ and corr $\left(\widetilde{M}_t, M_t \right)$. Finally, we aggregate the Monte Carlo results by averaging the goodness-of-fit statistics in (19)–(22) across the 1000 replications.

In line with our empirical exercise in Section 5, we consider 10 years of monthly data by fixing the time interval at $\triangle = 1/12$ and the number of time-series observations at $T = 120$ for each asset return. We then estimate the stochastic discount factor using only $N = 15$ out of the $N^* = 25$ asset returns that we simulate according either to the GBM or to the Ornstein-Uhlenbeck process. We restrict attention to a small set of assets given that Araujo et al.'s (2006) MVSDF estimator improves with the number of assets in the economy as opposed to the estimator based on the GBM assumption. As for the parameter values, we consider in both settings a full covariance matrix (i.e., all entries are different from zero) with diagonal elements varying from 0.00025 to 0.04840, with mean of 0.22. In contrast, the risk premia vector $\boldsymbol{\mu}$ takes values ranging from 0.02 and 0.09, with mean of 0.07, whereas the risk-free rate of return is $r = \ln(1.10)$ in the GBM world. As for the second design, we sample asset prices from an Ornstein-Uhlenbeck process characterized by a long-run mean ranging from 2 to 85, with cross-sectional mean of 39, and a mean-reversion parameter varying from 0.02 and 0.09, with cross-sectional mean of 0.07.[2]

[2] Although the Ornstein-Uhlenbeck process does not ensure the positivity of prices, it is very unlikely to simulate a path with negative prices given the parameter values we consider in our design. There are indeed no negative values for any asset price in our simulations.

4.1 Numerical results

Tables 1 and 2 report the Monte Carlo results for the GBM and Ornstein-Uhlenbeck settings, respectively. In the first instance, there is no significative difference between the two estimators given that their mean squared errors are $\widehat{\text{MSE}} = 0.28$ and $\widetilde{\text{MSE}} = 0.30$ with standard errors of 0.048 and 0.13, respectively. As for the correlation-based measures, our results once more indicate that the estimators have very similar performances. More precisely, $\text{corr}(\hat{M}_t, M_t) = 0.67$ and $\text{corr}(\tilde{M}_t, M_t) = 0.53$ with standard errors of 0.0474 and 0.10, respectively. Finally, our main goodness-of-fit measures tell a similar story, though there is some evidence that Araujo et al.'s (2006) estimator entails a better out-of-sample performance: $\hat{p} = 0.00266$, $\tilde{p} = 0.46$ and $p = 0.42$, with standard errors of 0.000117, 0.22 and 0.13, respectively.

The simulations rooted in the Ornstein-Uhlenbeck data generating mechanism confirm the lack of robustness of the MVSDF estimator based on the GBM assumption, and hence cast doubts on Brandt et al.'s (2006) risk-sharing analysis. In contrast, the performance of Araujo et al.'s (2006) MVSDF estimator is really excellent, suggesting that our approach for the estimation of the risk-sharing index is very promising. In particular, we find that $\hat{p} = 3.56 \times 10^{-6}$ and $\tilde{p} = 7.69$ with standard errors of 9.33×10^{-7} and 43.61, respectively.

The numerical results are striking in that the nonparametric MVSDF estimator dominates the estimator based on the GBM assumption. The price we pay for a more robust estimator indeed is negligible, if any, under the GBM assumption. As the risk-sharing index estimator presumably inherits the properties of the MVSDF estimator, we conjecture that Brandt et al.'s (2006) empirical analysis may not entail a definitive answer for the efficiency of risk sharing among countries. In the next section, we thus revisit their study of the amount of risk sharing between UK and US.

5 Risk sharing between UK and US

The consistency of our estimator in (14) requires a large panel, especially in the cross-sectional dimension. As the number of assets increases, the payoff space also spans more consumption contingent plans and so the mimicking portfolio gets closer to the marginal utility growth rate. Our sample consists of 404 monthly asset returns in the period ranging from January 1990 to June 1999. In particular, we consider interest rates, stock market indexes, and the 200 most liquid stocks in UK and in US.

As in Brandt et al. (2006), the stock market indexes are total market returns from Datastream, the interest rates stand for one-month Eurocurrency deposits from Datastream, and the consumer price indexes are from the International Monetary Fund's IFS database. The real returns on the GBP/USD exchange rate are excess returns for borrowing in US dollars, converting to sterling pounds, lending at the UK interest rate, and converting the proceeds back to US dollars. In addition, we collect equity price data from CRSP and Datastream relating to the 200 stocks with largest traded volume in US and to the 200 stocks with highest average turnover in the London Stock Exchange, respectively.

Table 3 reports the main descriptive statistics concerning the series of real returns with continuous compounding.[3] The preliminary results are consistent with the stylized facts in the literature. First, the sample mean and standard deviations of real equity returns are well above those of the returns on interest rates, reflecting a large equity premium. Second, the evidence against the GBM assumption is very strong in that all equity returns and interest rates exhibit leptokurtosis. This not only motivates our nonparametric approach, but also casts doubts on Brandt et al.'s (2006) empirical results given that they employ exactly the same data.[4]

To form a preliminary view about the efficiency of risk sharing between UK and US, it is natural to observe more closely the interaction between their stock market indexes given that they reflect to some extent wealth in the economy. The evidence seems to suggest there exists a large degree of risk sharing between the two economies in view that their stock market indexes clearly co-move over time as illustrated by Figure 1 and by the sample correlation of about 70%. Figure 2 confirms our intuition. It plots the demeaned realizations of the minimum-variance stochastic discount factors that we estimate for UK and US using (12). Their sample correlation is about 96%, suggesting that the risk sharing between UK and US is highly efficient.

To complete the picture, we now estimate the risk-sharing index using (14) so as to consider the information that exchange rate movements convey about the amount of risk sharing. Table 4 reports that the index estimate is around 87%, indicating that exchange rate movements do not blunt much the risk-sharing possibilities of the existing asset markets. This corroborates the main conclusion put forth by Brandt et al. (2006) that risk sharing is better than most people think. It remains to examine how our result compares with the degree of risk sharing implied by consumption data. For instance, Brandt et al. (2006) estimate risk-sharing indexes of 0.99 using monthly asset price data from 1975 to 1999 and of 0.25 (0.36) using quarterly (annually, respectively) consumption data from 1975 to 1998. In what follows, we show that accounting for market incompleteness suffices to

[3] In what follows, unless we specifically say otherwise, all sample variances and covariances refer to Newey and West's (1987) HAC estimates with 8 lags so as to cope with the time-series dependence.

[4] Although we employ a shorter sample period than Brandt et al. (2006), the evidence against lognormality is extremely robust to variations in the sample period.

reconcile the difference between our index estimates based on asset market and consumption data.

As in Brandt et al. (2006), we employ a consumption-based estimate of the risk-sharing index that assumes the domestic and foreign representative investors have the same constant relative risk aversion (CRRA) utility function. This implies that

$$\lambda_C = 1 - \frac{\sigma^2(\Delta c_{f,t} - \Delta c_{d,t})}{\sigma^2(\Delta c_{d,t}) + \sigma^2(\Delta c_{f,t})}, \tag{23}$$

where $\Delta c_{d,t}$ and $\Delta c_{f,t}$ denote the aggregated consumption growth in the domestic and foreign economies. To compute the right-hand side of (23), we take up the same quarterly consumption data as Brandt et al. (2006), corresponding to the real per-capita consumption of nondurables and services from the IFS database. The only difference is that we consider a shorter sample period ranging from the first quarter of 1990 to the third quarter of 1996. As expected, we obtain a much smaller value for the consumption-based risk-sharing index, namely 0.38, indicating a much poorer risk sharing in consumption between UK and US.

The difference between the risk-sharing index estimates using consumption and asset data is consistent with market incompleteness. Because there may exist additional risks that are orthogonal to asset markets, the index in (7) exclusively gauges the component of risk sharing achieved through asset markets, whereas consumption-based calculations measure overall risk sharing. We thus perform a simple calibration study, as in Brandt et al. (2006), so as to verify whether one could indeed observe a large degree of risk sharing in asset markets, but poor risk sharing in consumption under plausible assumptions on these nonspanned risks.

Let $\xi_{d,t}$ and $\xi_{f,t}$ correspond to the nonspanned risks in the domestic and foreign economies, respectively. By definition, it follows that $\mathrm{cov}(M^*_{d,t}, \xi_{d,t}) = \mathrm{cov}(M^*_{f,t}, \xi_{f,t}) = 0$, where $M^*_{d,t}$ and $M^*_{f,t}$ are the minimum-variance stochastic discount factors (or mimicking portfolios) of the domestic and foreign economies. Our estimates show that $M^*_{f,t}$ and $M^*_{d,t}$ are close to one, whereas $\xi_{f,t}$ and $\xi_{d,t}$ are on average zero (so as to correctly price interest rates). This means that we may approximate $\ln(M^*_{d,t} + \xi_{d,t})$ by $m^*_{d,t} + \xi_{d,t}$ and $\ln(M^*_{f,t} + \xi_{f,t})$ by $m^*_{f,t} + \xi_{f,t}$, giving way to

$$\begin{aligned} \lambda_\xi &= 1 - \frac{\sigma^2\left[\ln(M^*_{f,t} + \xi_{d,t}) - \ln(M^*_{d,t} + \xi_{f,t})\right]}{\sigma^2\left[\ln(M^*_{d,t} + \xi_{d,t})\right] + \sigma^2\left[\ln(M^*_{f,t} + \xi_{f,t})\right]} \\ &\cong 1 - \frac{\sigma^2(m^*_{f,t} - m^*_{d,t}) + \sigma^2(\xi_{d,t}) + \sigma^2(\xi_{f,t}) - 2\,\mathrm{cov}(m^*_{d,t}, \xi_{f,t}) - 2\,\mathrm{cov}(m^*_{f,t}, \xi_{d,t}) - 2\,\mathrm{cov}(\xi_{d,t}, \xi_{f,t})}{\sigma^2(m^*_{d,t}) + \sigma^2(m^*_{f,t}) + \sigma^2(\xi_{d,t}) + \sigma^2(\xi_{f,t})}. \end{aligned} \tag{24}$$

Given that the sample correlation between our domestic and foreign MVSDF estimates are very high, it is reasonable to assume that $\mathrm{cov}(\xi_{d,t}, M^*_{f,t}) = \mathrm{cov}(\xi_{f,t}, M^*_{d,t}) = 0$. After some straightforward manipulation, (24) reduces to

$$\lambda_\xi \;\cong\; 2\,\frac{\operatorname{corr}(m_{d,t}, m_{f,t})\,\sigma(m^*_{d,t})\,\sigma(m^*_{f,t}) + \operatorname{corr}(\xi_{d,t}, \xi_{f,t})\,\sigma(\xi_{d,t})\,\sigma(\xi_{f,t})}{\sigma^2(m^*_{d,t}) + \sigma^2(m^*_{f,t}) + \sigma^2(\xi_{d,t}) + \sigma^2(\xi_{f,t})}. \tag{25}$$

Although the overall risk-sharing index λ_ξ increases with the correlation between nonspanned risks and decreases with additional units of nonspanned risks, the relation is not so clear cut if the interest lies on the ratio λ_ξ/λ. The latter exceeds one only if the domestic and foreign nonspanned risks display extreme positive correlation given that, otherwise, the marginal utility growth will move less closely than the discount factors.

Table 5 documents the values of the overall risk sharing index λ_ξ as a function of the variance of the domestic and foreign nonspanned risks and of their correlation. The results confirm that accounting for the nonspanned risks may result in an overall risk sharing consistent with the consumption-based estimates even if the degree of risk sharing in asset markets is high. In particular, there is no need to assume that the domestic and foreign nonspanned risks are large in magnitude as long as their correlation is positive. This indeed is the most sensible configuration in view that most international common shocks have similar effects across economies.

6 Conclusion

This paper proposes a novel estimator for Brandt et al.'s (2006) international risk-sharing index that relies on a more robust estimates of the variances of the domestic and foreign minimum-variance stochastic discount factors. Our estimator has a nonparametric flavor in that it makes no parametric assumption on preferences and on the stochastic process that governs the dynamics of asset returns. This is in contrast with Brandt et al.'s (2006) estimator that assumes that asset returns follow a geometric Brownian motion (GBM). Monte Carlo simulations confirm that our estimator outperforms theirs in finite samples.

We then revisit Brandt et al.'s (2006) empirical risk-sharing analysis between UK and US. Our MVSDF estimates reveal there is less risk sharing between UK and US than one would find under the GBM assumption, though much more than what consumption data might suggest. We grant the first discrepancy to the robustness gains that our estimator brings about with respect to assumptions on preferences and on the stochastic processes of asset prices. A simple calibration analysis moreover shows that market incompleteness suffices to explain the difference between the consumption-based estimate of the risk-sharing index and ours. We thus conclude that, indeed, risk sharing is better than we think, though there is no evidence of risk-sharing puzzle.

Appendix: Revisiting the statistical assumptions

In what follows, we consider a more general setting in which both lognormality and conditional homoskedasticity do not necessarily hold. Let $X_{i,t+1} = m_{t+1} + r_{i,t+1}$ and denote by $\mu_{i,t+1|t}$ and $\sigma^2_{i,t+1|t}$ its conditional mean and variance given the information available at time t. As in Gallant and Nychka (1987), we approximate the conditional density function in (1) using a Hermite series expansion:

$$\mathbb{E}_t[\exp(X)] = \sum_{j=0}^{\infty} \eta_X(j) \int_{-\infty}^{\infty} e^x \, H_j(x) \, \phi(x; \mu, \sigma^2) \, \mathrm{d}x,$$

where $\eta_X(j) = \frac{1}{j!} \, \mathbb{E}[H_j(X)]$, $H_j(x) = e^{\frac{x^2}{2}} \frac{\mathrm{d}^j}{\mathrm{d}x^j} \, e^{-\frac{x^2}{2}}$, and $\phi(\cdot; \mu, \sigma^2)$ denotes the normal density function with mean μ and variance σ^2.

As in practice, the approximation error is negligible for third-order Hermite expansions (see Aït-Sahalia, 2002),[5] It then follows that

$$\begin{aligned}
\mathbb{E}_t[\exp(X)] &\approx \sum_{j=0}^{3} \eta_X(j) \int_{-\infty}^{\infty} e^x \, H_j(x) \, \phi(x; \mu, \sigma^2) \, \mathrm{d}x \\
&= \exp\left(\mu + \tfrac{1}{2}\,\sigma^2\right) \left[1 + \frac{1}{2}\left(3\,\mu - \mu^{(3)}\right)\left(\mu + \sigma^2\right) + \frac{1}{2}\left(\mu^{(2)} - 1\right)^2 \left(1 + \frac{2\,\mu\sigma^2 + \sigma^4}{\mu^{(2)} - 1}\right)\right],
\end{aligned}$$

where $\mu^{(k)} = \mathbb{E}_t(X^k)$. Taking logs then gives way to $\mu_{i,t+1|t} + \delta_{i,t} = 0$, where

$$\delta_{i,t+1} = \ln\left[1 + \frac{1}{2}\left(3\,\mu_{i,t+1} - \mu^{(3)}_{i,t+1}\right)\left(\mu_{i,t+1} + \sigma^2_{i,t+1}\right) + \frac{1}{2}\left(\mu^{(2)}_{i,t+1} - 1\right)^2 \left(1 + \frac{2\,\mu_{i,t+1}\sigma^2_{i,t+1} + \sigma^4_{i,t+1}}{\mu^{(2)}_{i,t+1} - 1}\right)\right] + \frac{1}{2}\,\sigma^2_{i,t+1}.$$

To derive the consistency of our estimator, we assume that $\delta_{i,t}$ admits a stationary Wold decomposition around a constant δ_i such that $\delta_{i,t+1} - \delta_i = \sum_{j=0}^{\infty} \lambda_{i,j}\zeta_{i,t+1-j}$, with $\zeta_{i,t}$ orthogonal to $\epsilon_{i,t}$, $\sum_{j=0}^{\infty} |\lambda_{i,j}| < \infty$ and $\lambda_{i,0} = 1$ for every $i = 1, \ldots, N$. The orthogonality assumption is to some extent analogous to the absence of leverage effects in the ambit of stochastic volatility models. It then ensues that

$$r_{i,t+1} = -\, m_{t+1} - \delta_i + \underbrace{\epsilon_{i,t+1} - \sum_{j=0}^{\infty} \lambda_{i,j}\,\zeta_{i,t+1-j}}_{\xi_{i,t+1}},$$

where $\lambda_{i,0} = 1$ for every $i = 1, \ldots, N$ and $\zeta_{i,t}$ forms a sequence of moving average zero-mean innovations with variance ς^2_i. The fixed effect δ_i involves the higher-order moments of $m_{t+1} + r_{i,t+1}$, whereas the error term $\xi_{i,t+1}$ has zero mean and finite variance $\sigma^2_i + \varsigma^2_i \sum_{j=1}^{\infty} \lambda^2_{i,j}$. Under these assumptions, the within estimator given by (12) remains consistent. The only difference is that, to conduct proper inference, one must now correct the covariance matrix of the estimates by allowing

[5] Instead of truncating the Hermite expansion, one could actually demonstrate that a similar result holds by assuming that the Lagrange residual of the Taylor expansion of (8) has an infinite moving average representation.

for serial correlation as in Arellano (1987). See Hansen (2007) for the properties of Arellano's (1987) covariance matrix estimator as N and T jointly go to infinity.

References

Aït-Sahalia, Y., 1996, Testing continuous-time models of the spot interest rate, Review of Financial Studies 9, 385–426.

Aït-Sahalia, Y., 2002, Maximum-likelihood estimation of discretely-sampled diffusions: A closed-form approximation approach, Econometrica 70, 223–262.

Araujo, F., Issler, J. V., Fernandes, M., 2006, Estimating the stochastic discount factor without a utility function, Princeton University, Getulio Vargas Foundation, and Queen Mary, University of London.

Arellano, M., 1987, Computing robust standard errors for within-groups estimators, Oxford Bulletin of Economics and Statistics 49, 431–434.

Backus, D. K., Foresi, S., Telmer, C. I., 2001, Affine term structure and foward premium anomaly, Journal of Finance 56, 279–304.

Baltagi, B. H., 2001, Econometric Analysis of Panel Data, John Wiley, Chichester.

Black, F., Scholes, M., 1973, The pricing of options and corporate liabilities, Journal of Political Economy 81, 637–654.

Brandt, M. W., Cochrane, J. H., Santa-Clara, P., 2006, International risk sharing is better than you think, or exchange rates are too smooth, forthcoming in Journal of Monetary Economics.

Campbell, J. Y., 1993, Intertemporal asset pricing without consumption data, American Economic Review 83, 487–512.

Cochrane, J. H., 2001, Asset Pricing, Princeton University Press, Princeton.

Diebold, F. X., 1988, Empirical Modeling of Exchange Rates, Springer-Verlag, New York.

Drost, F. C., Nijman, T. E., 1993, Temporal aggregation of GARCH processes, Econometrica 61, 909–928.

Gallant, A. R., Nychka, D. W., 1987, Semi-nonparametric maximum likelihood estimation, Econometrica 55, 363–390.

Hansen, C. B., 2007, Asymptotic properties of a robust variance matrix estimator for panel data when T is large, forthcoming in Journal of Econometrics.

Hansen, L. P., Jagannathan, R., 1991, Implications of security market data for models of dynamic economies, Journal of Political Economy 99, 225–262.

Hansen, L. P., Richard, S. F., 1987, The role of conditioning information in deducing testable restricions implied by dynamic asset pricing models, Econometrica 55, 587–613.

Hansen, L. P., Singleton, K. J., 1983, Stochastic consumption, risk aversion and the temporal behavior of asset returns, Journal of Political Economy 91, 249–265.

Harrison, J. M., Kreps, D. M., 1979, Martingales and arbitrage in multiperiod securities markets, Journal of Economic Theory 20, 380–408.

Lettau, M., Ludvigson, S., 2001, Resurrecting the (C)CAPM: A cross-sectional test when risk premia are time-varying, Journal of Political Economy 109, 1238–1287.

Mandelbrot, B., 1963, The variation of certain speculative prices, Journal of Business 36, 394–419.

Meddahi, N., Renault, E., 2004, Temporal aggregation of volatility models, Journal of Econometrics 119, 355–379.

Newey, W. K., West, K. D., 1987, A simple, positive semi-definite, heteroskedasticity and autocorrelation consistent covariance matrix, Econometrica 55, 703–708.

Schwartz, E. S., 1997, The stochastic behavior of commodity prices: Implications of valuation and hedge, Journal of Finance 52, 923–973.

Vasicek, O., 1977, An equilibrium characterization of the term structure, Journal of Financial Economics 5, 177–188.

Wallace, H. D., Hussain, A., 1969, The use of error components model in combining cross-section and time-series data, Econometrica 37, 55–72.

Figure 1
The real total market returns in UK and in US

The plot displays monthly real returns on the UK and US stock market indexes (source: Datastream) in the period running from January 1990 to December 1999. We convert nominal into real returns using the consumer price indexes from the IFS database.

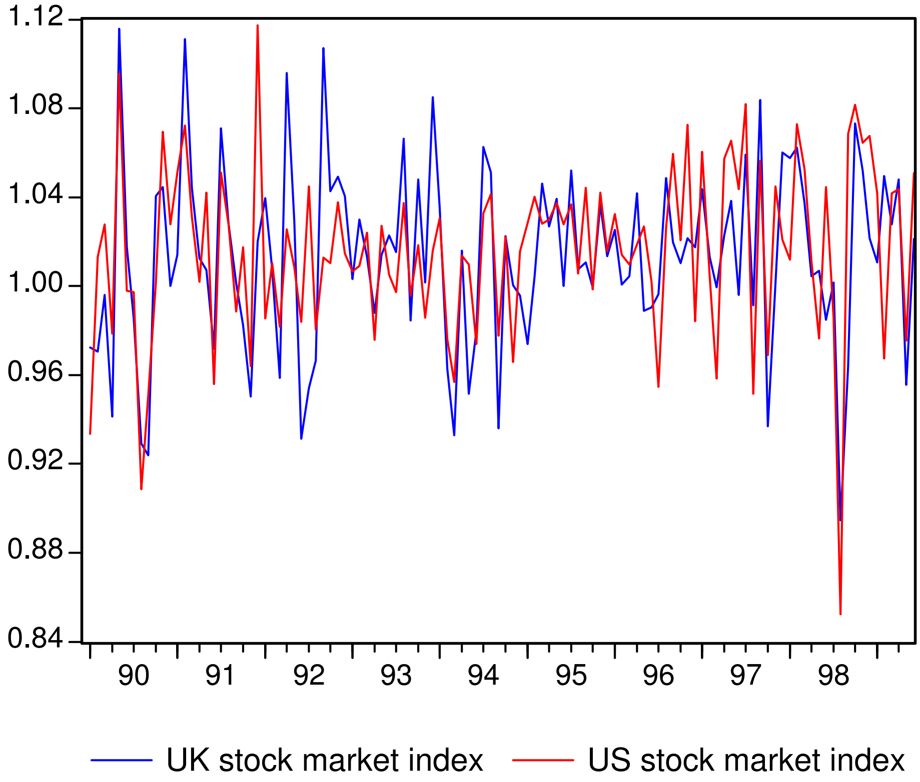

Figure 2
The MVSDF fluctuations in UK and in US

We estimate the demeaned realizations of the log-MVSDF for UK and US using the within estimators in (12), i.e., $\widehat{m_{\mathrm{UK},t} - \bar{m}_{\mathrm{UK}}}$ and $\widehat{m_{\mathrm{US},t} - \bar{m}_{\mathrm{US}}}$. The estimation sample consists of 404 monthly real returns, including stock market indexes, interest rates, and individual stocks in UK and US, in the period ranging from January 1990 to June 1999. We employ the consumer price indexes from the IFS database to convert nominal into real returns. The stock market indexes are total market returns from Datastream, whereas interest rates are one-month Eurocurrency deposits from Datastream. We also consider the 200 stocks with largest traded volume in US (source: CRSP) and the 200 stocks with highest average turnover in the London Stock Exchange (source: Datastream).

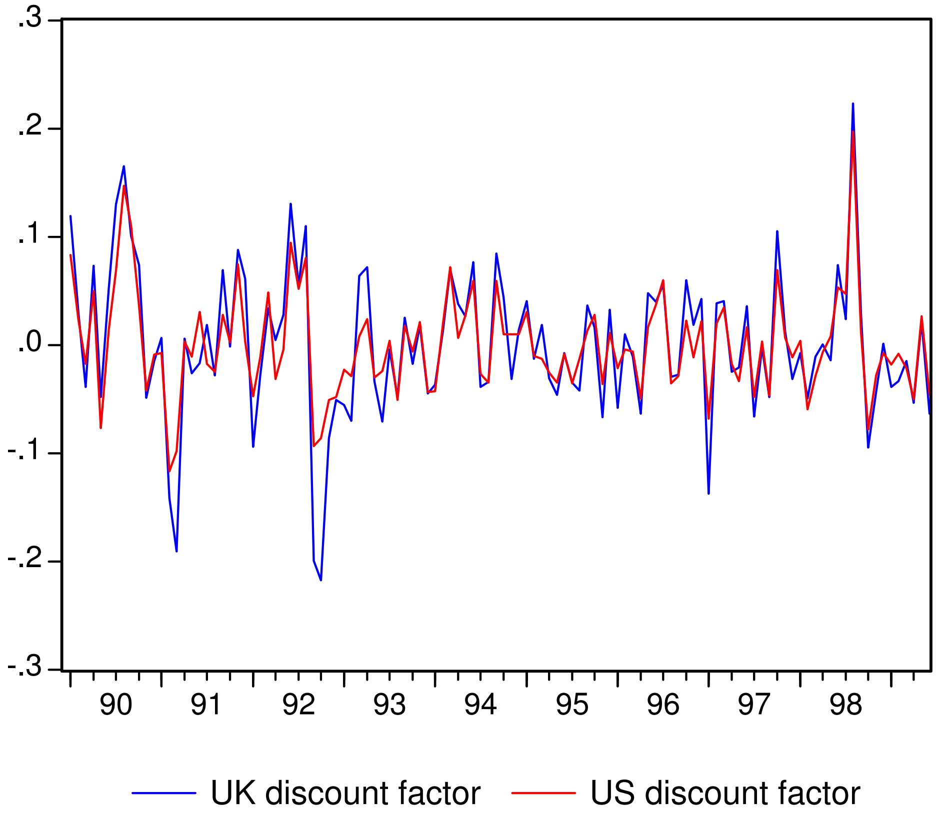

Table 1
Monte Carlo results for the geometric Brownian motion

We simulate a panel with 620 monthly time-series observations of 25 asset returns from a multivariate geometric Brownian motion. The covariance matrix is full with diagonal elements varying from 0.00025 to 0.04840, with mean of 0.22. The equity risk premia vector takes values ranging from 0.02 and 0.09, with mean of 0.07, whereas the risk-free rate of return is 10% per annum. We estimate the SDF using only the first 15 asset returns and the last 120 time-series observations. We gauge out-of-sample performance by checking whether the estimated discount factors correctly price the remaining 10 asset returns through the statistics in (19)–(22). All results hinge on 1,000 replications and the figures in parentheses refer to standard deviation across replications.

$\widehat{\text{MSE}}$	$\widetilde{\text{MSE}}$	$\text{corr}(\hat{M}_t, M_t)$	$\text{corr}(\tilde{M}_t, M_t)$	$\hat{p}$	$\tilde{p}$	$\bar{p}$
0.28	0.30	0.67	0.53	2.66×10^{-3}	0.46	0.42
(4.80×10^{-2})	(0.13)	(4.74×10^{-2})	(0.10)	(1.17×10^{-4})	(0.22)	(0.13)

Table 2
Monte Carlo results for the Ornstein-Uhlenbeck process

We simulate a panel with 620 monthly time-series observations of 25 asset returns from a multivariate Ornstein-Uhlenbeck process characterized by a long-run mean ranging from 2 to 85, with cross-sectional mean of 39, and a mean-reversion parameter varying from 0.02 and 0.09, with cross-sectional mean of 0.07. The covariance matrix is full with diagonal elements varying from 0.00025 to 0.04840, with mean of 0.22. We estimate the SDF using only the first 15 asset returns and the last 120 time-series observations. We gauge out-of-sample performance by checking whether the estimated discount factors correctly price the remaining 10 asset returns through the statistics in (19) and (20). All results hinge on 1,000 replications and the figures in parentheses refer to standard deviation across replications.

$\hat{p}$	$\tilde{p}$
3.56×10^{-6}	7.69
$\left(9.33 \times 10^{-7}\right)$	(43.61)

Table 3
Descriptive statistics for the asset real returns

The sample consists of 404 monthly real returns in the period ranging from January 1990 to June 1999. We employ the consumer price indexes from the International Monetary Fund's IFS database to convert nominal into real returns. The stock market indexes are total market returns from Datastream, whereas interest rates are one-month Eurocurrency deposits from Datastream. Finally, we also report some sample statistics for the 200 stocks with largest traded volume in US (source: CRSP) and for the 200 stocks with highest average turnover in the London Stock Exchange (source: Datastream).

	mean	median	standard deviation	skewness	kurtosis
UK					
interest rate	0.0035	0.0038	0.0046	-0.8573	6.5366
stock market index	0.0092	0.0105	0.0416	-0.4021	3.3301
individual stocks					
mean	0.0044	0.0013	0.1268	0.1902	4.7409
standard deviation	0.0114	0.0123	0.0362	0.5085	1.8490
minimum	-0.0619	-0.0397	0.0823	-2.7508	2.8310
maximum	0.0373	0.0287	0.2863	1.5770	17.9659
US					
interest rate	0.0019	0.0019	0.0020	-0.2969	2.9563
stock market index	0.0126	0.0158	0.0397	-0.9001	5.5877
individual stocks					
mean	0.0148	0.0177	0.1026	-0.2418	4.1073
standard deviation	0.0098	0.0121	0.0390	0.4743	1.8547
minimum	-0.0072	-0.0090	0.0406	-2.1877	2.2991
maximum	0.0543	0.0721	0.2258	1.5912	15.0068

Table 4

Estimation results for the risk-sharing index between UK and US

We estimate the risk sharing index is given by (7) using (14), namely,

$$\hat{\lambda}(\text{UK,US}) = 1 - \frac{\hat{\sigma}^2(\Delta e_t)}{\hat{\sigma}^2\left(\widehat{m_{\text{UK},t} - \bar{m}_{\text{UK}}}\right) + \hat{\sigma}^2\left(\widehat{m_{\text{US},t} - \bar{m}_{\text{US}}}\right)},$$

where $\widehat{m_{\text{UK},t} - \bar{m}_{\text{UK}}}$ and $\widehat{m_{\text{US},t} - \bar{m}_{\text{US}}}$ are the within estimates in (12) of the demeaned realizations of the log-MVSDF for UK and US, respectively, and Δe_t is the real return on the GBP/USD exchange rate. We compute all long-run variances $\hat{\sigma}^2(\cdot)$ using Newey and West's (1987) estimator with 8 lags.

$\hat{\sigma}^2(\Delta e_t)$	$\hat{\sigma}^2\left(\widehat{m_{\text{UK},t} - \bar{m}_{\text{UK}}}\right)$	$\hat{\sigma}^2\left(\widehat{m_{\text{US},t} - \bar{m}_{\text{US}}}\right)$	$\hat{\lambda}(\text{UK,US})$
9.44×10^{-4}	5.32×10^{-3}	2.21×10^{-3}	0.87

Table 5

Calibration results for the risk-sharing index under market incompleteness

The cells report estimated values for the risk-sharing index in (25) given the estimates of the long-run variances in Table 4, namely,

$$\hat{\lambda}_{\xi}(\text{UK,US}) = 1 - \frac{\hat{\sigma}^2(\Delta e_t) + \sigma^2(\xi_{\text{UK},t}) + \sigma^2(\xi_{\text{US},t}) - 2\,\text{cov}(\xi_{\text{UK},t}, \xi_{\text{US},t})}{\hat{\sigma}^2\left(\widehat{m_{\text{UK},t} - \bar{m}_{\text{UK}}}\right) + \hat{\sigma}^2\left(\widehat{m_{\text{US},t} - \bar{m}_{\text{US}}}\right) + \sigma^2(\xi_{\text{UK},t}) + \sigma^2(\xi_{\text{US}}, t)}.$$

The columns consider different values for the volatility of the nonspanned risks, whereas the rows consider different values for the correlation between the nonspanned risks in UK and US.

	$\sigma(\xi_{\text{UK},t}) = \sigma(\xi_{\text{US},t})$						
$\text{corr}(\xi_{\text{UK},t}, \xi_{\text{US},t})$	0.01	0.05	0.10	0.20	0.30	0.50	1.00
-1.00	0.8262	0.1270	-0.4869	-0.8385	-0.9246	-0.9721	-0.9929
-0.80	0.8314	0.2067	-0.3416	-0.6558	-0.7326	-0.7751	-0.7937
-0.40	0.8417	0.3663	-0.0511	-0.2902	-0.3487	-0.3810	-0.3952
-0.25	0.8456	0.4261	0.0578	-0.1531	-0.2048	-0.2332	-0.2457
0.00	0.8521	0.5258	0.2394	0.0753	0.0351	0.0129	0.0032
0.25	0.8585	0.6255	0.4209	0.3037	0.2751	0.2592	0.2523
0.40	0.8624	0.6854	0.5299	0.4408	0.4190	0.4070	0.4017
0.80	0.8728	0.8449	0.8204	0.8064	0.8030	0.8011	0.8002
1.00	0.8779	0.9247	0.9657	0.9892	0.9949	0.9981	0.9995