Unawareness of Theorems*

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Abstract

This paper provides a set-theoretic model of knowledge and unawareness. A new property called Awareness Leads to Knowledge shows that unawareness of theorems not only constrains an agent's knowledge, but also, can impair his reasoning about what other agents know. For example, in contrast to Li (2006), Heifetz et al. (2006) and the standard model of knowledge, it is possible that two agents disagree on whether another agent knows a particular event. The model follows Aumann (1976) in defining common knowledge and characterizing it in terms of a self evident event, but departs in showing that no-trade theorems do not hold.

1 Introduction

1.1 Motivation and outline

A common assumption in economics is that agents who participate in a model perceive the "world" the same way the analyst does. This means that they understand how the model works, they know all the relevant theorems and they do not miss any dimension of the problem they are facing. In essence, agents are as educated and as intelligent as the analyst and they can make the best decision, given their information and preferences.

Modeling unawareness aims at relaxing this assumption, so that agents may perceive a more simplified version of the world. Intuitively, there are many instances where agents of different perception coexist in the same market. In the stock market, for example, one can find investors who are highly educated about how the market and the economy work, together with investors whose understanding is much more limited.

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One way in which we might hope to capture differences between these two types of investors is by attributing differences to asymmetric information. The standard model of knowledge and asymmetric information was introduced into economics by Aumann (1976). Its simplicity and the fact that it was purely set-theoretic led to many economic applications.[1] However, Dekel et al. (1998) showed that this model cannot accommodate unawareness. Moreover, it can be criticized on the grounds that it only models a highly sophisticated and rational agent, who is aware of everything, knows all the possible theorems that can be derived and has no constraints on the number of calculations he can perform.

This paper provides a model of knowledge and awareness, where agents may not know some of the relevant theorems, may be unaware of some of the dimensions of the world and thus can make mistakes. Moreover, the paper does not depart from the set-theoretic approach of Aumann (1976) and its advantages, while aiming at giving a better insight into the connection between awareness and knowledge.

Consider the following comparative statics exercise, where an agent gains awareness. He becomes aware of new events, and some of these he may subsequently know. This effect of awareness on knowledge is well described by other papers. The second, less immediate connection is that more awareness can lead to awareness of new theorems, which connect answers to different questions. As a result, more awareness can lead to knowing an event that the agent previously was aware of but did not know. Or equivalently, what one *is unaware of*, may constrain his knowledge about events he *is aware of.* This less immediate connection is not accommodated in the other papers that model unawareness - it is expressed in this model by the property Awareness Leads to Knowledge.

The implications of this property in a multi agent setting can be stark. The unaware agent 1 may falsely conclude that agent 2 does not know an event, when in fact agent 2 knows it, because he knows a theorem beyond 1's awareness.

It is worthwhile noting that these mistakes in reasoning about others (due to unawareness of theorems) can be accommodated by the standard model of knowledge or the extensions discussed below, only if we allow for false beliefs. But allowing for false beliefs permits all kinds of mistakes. For instance, it allows for agents to make numerical mistakes. The purpose of this paper is to isolate and study this specific type of mistake due to unawareness of theorems, without relaxing the assumption that agents are otherwise rational.

In order to overcome the impossibility result of Dekel et al. (1998), the paper follows the approach of Heifetz et al. (2007a) and Li (2006) of introducing multiple state space. However, it retains the set-theoretic nature exhibited also in the standard model of knowledge and as a result, familiar notions naturally extend here. For instance, common knowledge is characterized in terms of a self evident event, just like in the standard model. Moreover, there is a well defined notion of a common state space. This is the state space that everyone is aware of and this is common knowledge. As the following discussion on no-trade theorems reveals, results that are true for the unique state space of the standard model are also true when stated for the common state space of this model, but fail to hold in general.

[1] An overview of the standard model of knowledge is given in Rubinstein (1998). A more philosophical treatment is given in Hintikka (1962).

A natural question is whether agents can agree to disagree and trade in an environment with unawareness. In the standard model of knowledge this is not possible, if we assume a common prior - Aumann (1976) shows that common knowledge of posteriors implies they are identical. In this model it is shown that the same result is true for common priors and posteriors defined on the common state space. However, an example with two agents i and j demonstrates that although the posteriors defined on the common state space are common knowledge and therefore identical, i's actual posterior is different and beyond j's reasoning because agent i is aware of a theorem that j is unaware of. As a result, the two agents can agree to disagree and trade.

Intuition for this result can be obtained if we interpret common knowledge of posteriors as the outcome of the following procedure. Initially the posteriors are different. Agent i announces his posterior and j updates his information and announces a possibly different posterior. Then, i can update and announce a different posterior, which triggers a new round of updating. Geanakoplos and Polemarchakis (1982) show in the standard model that if the state space is finite, then after finitely many steps the agents will agree on their posteriors. A necessary condition for this result is that partitions are common knowledge in the standard model. This is true in this model, but only for the common state space. Hence, updating of information due to other agents' actions or announcements still takes place in an environment with unawareness, but it is constrained by what is commonly known that everyone is aware of. As a result, agents can engage in trade when the differences in their posteriors stem from asymmetric information acquired by theorems that others are unaware of.

An example

Consider the following example, which has been cited numerous times in the literature on unawareness. Sherlock Holmes and Dr. Watson are investigating a crime where a horse was stolen from a stable and the keeper was killed. The question they want to answer is whether there was an intruder in the stable. Holmes is the highly sophisticated and intelligent agent who has already solved the mystery, while Watson struggles to keep up. Watson is unable to answer the question because he is unaware that the dog did not bark, and therefore he is also unaware of the theorem that no barking implies no intruder.

Using the example, we can distinguish three features of unawareness. The first is a restricted perception of the world, which limits the agent's reasoning and subsequently what he can potentially know, or know that he does not know. Watson does not know that the dog did not bark, and he does not know that he does not know. He also cannot reason whether Holmes knows whether or not the dog barked. The possibility of the dog not barking simply never crosses his mind - he is unaware of it.

Watson is already aware of the possibility of an intruder, but he does not know whether there was one or not. Although the information about the dog not barking is available to him, he is simply unaware of it. The second feature of unawareness is that readily available information cannot be used by the agent. In other words, what Watson is unaware of, constrains his knowledge about events he is aware of.

The third feature of unawareness is that it constrains an agent's ability to reason about the knowledge of others. Unawareness of the theorem "no barking implies no intruder" results not only in Watson not knowing whether there was an intruder, but also in reasoning that Holmes does not know. In fact, Watson may be aware of many other ways (or theorems) in which Holmes could have known (for example, because he asked a police officer), but Watson has correctly deduced that none of these ways were employed. He therefore inevitably concludes that Holmes does not know whether there was an intruder. In other words, Watson's expressive power is not rich enough to include Holmes' knowledge of no intruder through the specific theorem "no barking implies no intruder". Moreover, Watson, within the bounds of his awareness, is not making a mistake.

To conclude the example, Holmes and Watson are exposed to the same information and the standard model would model them as having the same state space and the same partition. However, Watson's reasoning is limited in three ways. First, his expressive power is poorer than Holmes', limiting the events that he knows and the events he knows that he does not know. Second, information readily available to Watson is left unexploited, because he is unaware of its existence. As a result, his knowledge about an event he is aware of is constrained by a theorem that he is unaware of. Finally, Watson incorrectly deduces that Holmes does not know whether there was an intruder. This is not a result of a logical mistake, but of Watson's constrained reasoning, due to his unawareness of the theorem "no barking implies no intruder".

Suppose Holmes pointed out to Watson that the incident of the dog is important. Once Watson becomes aware of the dog, he can collect the information of the dog not barking that was always available to him, become aware of the theorem "no barking implies no intruder", and answer the question whether there was an intruder. Increased awareness can lead to increased knowledge about questions that one was already aware of.

1.2 Related literature

Models of knowledge (and of unawareness) are either syntactic or semantic (set-theoretic). The two approaches are equivalent, but syntactic models are widely used by logicians and computer scientists, while set-theoretic ones are more common in the economics literature, following Aumann (1976). Beginning with Fagin and Halpern (1988), there has been a stream of syntactic models, namely Halpern (2001), Modica and Rustichini (1994, 1999), Halpern and Rêgo (2005) and Heifetz et al. (2007b). Applications in the context of games with unawareness have been provided by Feinberg (2004, 2005), Sadzik (2005), Copic and Galeotti (2006), Li (2006b), Heifetz et al. (2007a), Filiz (2006) and Ozbay (2006).

Geanakoplos (1989) provides one of the first set-theoretic models that deals with unawareness, by using non-partitional information structures, defined on a standard state space. However, Dekel et al. (1998) propose three intuitive properties for unawareness and show that they are incompatible with the use of a standard state space. Addressing this impossibility result has been achieved with two different approaches. The first is by arguing against one of the properties (Ely (1998)), or by relaxing them (Xiong (2007)). The second is by introducing multiple state spaces, one for each state of awareness. This approach was initiated by Li (2006) and Heifetz et al. (2006) and is being followed by the present paper.[2]

Before illustrating the differences between these models, recall that the standard model of knowledge (Aumann (1976)) specifies a unique state space Ω and a possibility correspondence P which maps states in Ω to subsets of Ω. The interpretation is that for any $\omega \in \Omega$, the set $P(\omega)$ denotes the states that the agent considers possible when ω has occurred. In contrast, modeling unawareness using multiple state spaces leads to a possibility correspondence that maps states of any possible state space to subsets of possibly different state spaces. The reason is that awareness varies with the state. For example, suppose that state $\omega \in \Omega$ specifies that the agent's awareness is different, so that if ω occurs, the agent's state space is Ω' and not Ω. Then, the set of states that the agent considers possible, $P(\omega)$, is a subset of Ω' and not of Ω. As a result, a model with unawareness has to impose axioms on the possibility correspondence P, restricting what it prescribes across different state spaces.

One of the main differences between this model and the set-theoretic models Li (2006) and Heifetz et al. (2006), is that weaker restrictions are imposed here on what the possibility correspondence P prescribes across state spaces.

Li assumes a possibility correspondence P, just as in the standard model, which maps full states to subsets of the full state space Ω^*, which in Li's terminology is the most complete state space.[3] For each full state $\omega^* \in \Omega^*$, $P(\omega^*)$ denotes the set of full states that the agent would consider possible if he were fully aware. In Li's terminology, $P(\omega^*)$ denotes the agent's factual information. If the agent is not fully aware at ω^*, so that his state space is different from the full state space, then what he actually perceives as possible is the projection of $P(\omega^*)$ onto the state space that he is aware of. Similarly, when i reasons about j's knowledge, he projects j's full state partition to i's state space. Heifetz et al. (2006) follow a similar approach. Their property "Projections Preserve Knowledge" requires that if the agent considers states in $P(\omega)$ to be possible at ω, then at the projection of ω to a more limited state space S he considers possible the projection of $P(\omega)$ to S. In essence, these two properties place a restriction on what the possibility correspondence can prescribe across different state spaces.

In order to illustrate why these two properties are restrictive, recall the Holmes example depicted in the following figure. The two relevant dimensions or questions of the problem are "Did the dog bark?" and "Was there an intruder?". Holmes is always aware of both questions, so his subjective state space is the full state space, containing the four states $(\omega_1, \omega_2, \omega_3, \omega_4)$ on the plane. At state ω_4 which specifies that there was no intruder and no barking, Holmes knows that there is no intruder because he knows that the dog did not bark and he is also aware of and knows the theorem "no barking implies no intruder". Hence, $P^H(\omega_4) = \omega_4$.

Watson is aware only of the question "Was there an intruder?". His subjective state space consists of states ω_5 and ω_6 on the horizontal axis. As modeled by Li (2006) and Heifetz et al. (2006), when Watson reasons at ω_6 about Holmes' knowledge, he projects $P^H(\omega_4) = \omega_4$ to

[2] Halpern and Rêgo (2005) and Heifetz et al. (2007b) provide syntactic foundations of Heifetz et al. (2006).

[3] Since the full state space Ω^* is the most complete state space, only an agent who is aware of all possible questions, is also aware of Ω^*. A full state ω^* is an element of the full state space.

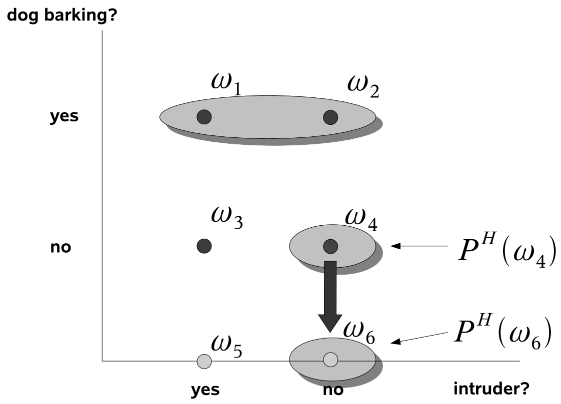

Figure 1: Projecting knowledge downwards.

his state space. Therefore, he reasons that $P^H(\omega_6) = \omega_6$ and that Holmes knows at ω_6 that there is no intruder. Although Watson is unaware of any theorem that could lead someone to know whether there is an intruder, he nevertheless is able to correctly deduce that Holmes knows at ω_6.

In order to accommodate the example so that Watson reasons that Holmes does not know whether there is an intruder, we have to abandon projections.[4] When Watson reasons about Holmes at ω_6, he is unaware of the theorem "no barking implies no intruder" and therefore he cannot reason that Holmes is aware of it. As a result, $P^H(\omega_6) = \{\omega_5, \omega_6\}$ and Watson reasons that Holmes does not know. This is depicted in the following figure.

The example suggests that unawareness can restrict Watson's reasoning about Holmes' knowledge, concerning an event that both are aware of. This is not captured in other papers that model unawareness. Moreover, Watson formally makes no mistake. It is true that with Watson's awareness, Holmes would not know that there is no intruder and Watson can reason only up to his awareness. Essentially, there are two different views of Holmes' knowledge. This is formally captured in this model by creating one knowledge operator for each state of awareness. If Watson's state space is S, then his view of Holmes' knowledge is K_S. But Holmes' state space is S', so his view of Holmes' knowledge is $K_{S'}$. Moreover, S' is "more expressive" than S. In the model this is captured by having a partial order $\preceq$ on the collection of state spaces. The relationship between the two different views about knowledge is given by the property Awareness Leads to Knowledge. It states that if S' is more expressive than S, then $K_{S'}$ gives a better description of one's knowledge than K_S. Heifetz et al. (2006) specify one knowledge operator K so that there is always one objective view of Holmes' knowledge.

[4]The only other way of accommodating the example is by allowing for false beliefs. But as was argued before, this carries the excess baggage of allowing for any kind of false beliefs, even those unrelated to unawareness.

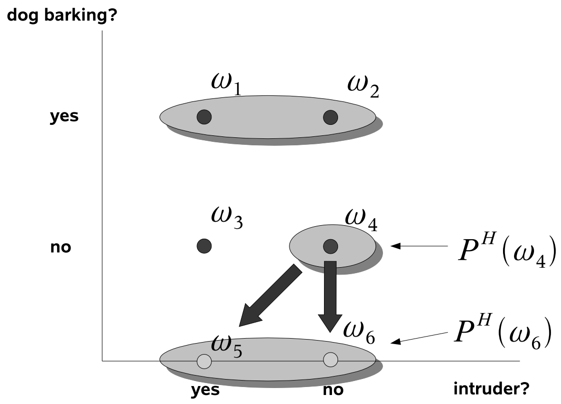

Figure 2: Allowing for unawareness of theorems.

The paper proceeds as follows. Section 2 introduces the basic single-agent model, while its main results are presented in Section 3. Section 4 describes the multi-agent model; in particular, common knowledge is defined and characterized in terms of a self evident event. Section 5 examines no-trade theorems. Proofs are contained in the Appendix.

2 The Model

Consider a complete lattice of disjoint state spaces $\mathcal{S} = \{S_a\}_{a \in A}$ and denote by $\Sigma = \cup_{a \in A} S_a$ the union of these state spaces. A state ω is an element of some state space S. Let $\preceq$ be a partial order on $\mathcal{S}$. For any S, $S' \in \mathcal{S}$, $S \preceq S'$ means that S' is more expressive than S. Moreover, there is a surjective projection $r^{S'}_S : S' \to S$. Projections are required to commute. If $S \preceq S' \preceq S''$ then $r^{S''}_S = r^{S'}_S \circ r^{S''}_{S'}$. If $\omega \in S'$, denote $\omega_S = r^{S'}_S(\omega)$ and $\omega_{S''} = (r^{S''}_{S'})^{-1}(\omega)$. If $B \subseteq S'$, denote $B_S = \{\omega_S : \omega \in B\}$ and $B_{S''} = \{\omega_{S''} : \omega \in B\}$. Let $g(S) = \{S' : S \preceq S'\}$ be the collection of state spaces that are at least as expressive as S. For a set $B \subseteq S$, denote by $B^{\uparrow} = \bigcup_{S' \in g(S)} (r^{S'}_S)^{-1}(B)$ the enlargements of B to all state spaces which are at least as expressive as S.

Consider a possibility correspondence $P : \Sigma \to 2^{\Sigma} \setminus \emptyset$ with the following properties:

(0) Confinedness: If $\omega \in S$ then $P(\omega) \subseteq S'$ for some $S' \preceq S$.

(1) Generalized Reflexivity: $\omega \in (P(\omega))^{\uparrow}$ for every $\omega \in \Sigma$.

(2) Stationarity: $\omega' \in P(\omega)$ implies $P(\omega') = P(\omega)$.

(4) Projections Preserve Awareness: If $\omega \in S'$, $\omega \in P(\omega)$ and $S \preceq S'$ then $\omega_S \in P(\omega_S)$.

(3) Projections Preserve Ignorance: If $\omega \in S'$ and $S \preceq S'$ then $(P(\omega))^{\uparrow} \subseteq (P(\omega_S))^{\uparrow}$.

(5) Projections Preserve Knowledge: If $S \preceq S' \preceq S''$, $\omega \in S''$ and $P(\omega) \subseteq S'$ then $(P(\omega))_S = P(\omega_S)$.

The setting above is identical to that of Heifetz et al. (2006). The first difference with the present model is that we drop the last axiom, Projections Preserve Knowledge (PPK). To argue against PPK, consider the example in the introduction. There are two different state spaces, $S' = S'' = \{\omega_1, \omega_2, \omega_4\}$ and $S = \{\omega_5, \omega_6\}$. At ω_4, Holmes is aware of the theorem "no barking implies no intruder" and he knows that there is no intruder. Hence, $P(\omega_4) = \omega_4$. Since the projection of ω_4 to S is ω_6, PPK implies that $P(\omega_6) = \omega_6$. As was argued in the introduction, this is restrictive. In order to allow for $P(\omega_6) = \{\omega_5, \omega_6\}$, we drop PPK.

2.1 Events, awareness and knowledge

Dropping PPK means that we need to also change the definitions of knowledge, awareness and events. In the setting of Heifetz et al. (2006), a subset $E \subseteq \Sigma$ is an event if it is of the form $B^{\uparrow}$, where $B \subseteq S$ for some state space $S \in \mathcal{S}$. Hence, an event in their setting contains states lying in different state spaces. For an event E, knowledge of E is defined to be $K(E) = \{\omega \in \Sigma : P(\omega) \subseteq E\}$. Similarly, the negation of $K(E)$ is $\neg K(E) = \{\omega \in \Sigma : P(\omega) \nsubseteq E\}$. With PPK, $K(E)$ and $\neg K(E)$ are also events, so $K^i \neg K^j K^k(E)$, for example, is a well defined event.

However, if we drop PPK then $K(E)$ and $\neg K(E)$ may not be events. To see why, consider again the example in figure 2 where, dropping PPK, we have $P(\omega_5) = P(\omega_6) = \{\omega_5, \omega_6\}$, $P(\omega_4) = \omega_4$ and $P(\omega_1) = P(\omega_2) = \{\omega_1, \omega_2\}$. Let $E = \{\omega_6, \omega_4, \omega_2\}$ be the event "there is no intruder". Then, $\neg K(E) = \{\omega_1, \omega_2, \omega_5, \omega_6\}$. But $\neg K(E)$ is not an event, because $\{\omega_5, \omega_6\}^{\uparrow} \neq \neg K(E)$, so it is not of the form $B^{\uparrow}$, where $B \subseteq S$.

As was suggested in the introduction, dropping PPK allows for differences in awareness to imply different views about knowledge. In the example, Watson's view of Holmes' knowledge is different from Holmes' view. Hence, there is not one, objective, knowledge but several subjective ones, for each state of awareness. This is formally captured by defining $K_S(E)$ for each state space $S \in \mathcal{S}$.

Moreover, allowing for different views of knowledge requires that we also change the definition of an event. The reason is that $K_S(E)$ describes "knowledge of E, with the vocabulary of state space S". Since we want $K_S(E)$ to be an event, we require that an event is a subset of some state space. Hence, contrary to Heifetz et al. (2006), an event does not contain states lying in different state spaces.

Formally, an *event* E is a subset of some (necessarily unique) state space $S \in \mathcal{S}$. The negation of E, denoted by $\neg E$, is the complement of E with respect to S. Denote the complement of S by $\emptyset_S$. Let $\mathcal{E} = \{E \subseteq S : S \in \mathcal{S}\}$ be the union of all events. For each event E, let $S(E)$ be the state space of which it is a subset. An event E "inherits" the expressiveness of the state space of which it is a subset. Hence, we can extend $\preceq$ to a partial order $\preceq_0$ on $\mathcal{E}$ in the following way: $E \preceq_0 E'$ if and only if $S(E) \preceq S(E')$. Abusing notation, we write $\preceq$ instead of $\preceq_0$.

Before defining knowledge, we need to define awareness. For any event E, for any state space S such that $E \preceq S$, define

$$A_S(E) = \{\omega \in S : E \preceq P(\omega)\}$$

to be the event which describes, with the vocabulary of S, that the agent is aware of event E. The condition $E \preceq S$ imposes that only a state space rich enough to describe E, can also describe the agent's awareness of E. The agent is aware of an event if his possibility resides in a state space that is rich enough to express event E. Unawareness is defined as the negation of awareness. More formally, the event $U_S(E)$ describes, with the vocabulary of S, that the agent is unaware of E:

$$U_S(E) = \neg A_S(E).$$

Let $\Omega : \Sigma \to \mathcal{S}$ be such that for any $\omega \in \Sigma$, $\Omega(\omega) = S$ if and only if $P(\omega) \subseteq S$. $\Omega(\omega)$ denotes the agent's state space at ω. An agent knows an event E if he is aware of it and in all the states he considers possible, E is true. Formally, for any event E and for any state space S such that $E \preceq S$, define

$$K_S(E) = \{\omega \in A_S(E) : P(\omega) \subseteq E_{\Omega(\omega)}\}.$$

3 Results

3.1 Overview of the properties of the standard model

Consider a state space Ω and a possibility correspondence $P : \Omega \to 2^{\Omega} \setminus \emptyset$. The interpretation is that when $\omega \in \Omega$ occurs, the agent reasons that one state in $P(\omega)$ has occurred. An event E is a subset of Ω. Knowledge of events is represented by the knowledge operator $K : 2^{\Omega} \to 2^{\Omega}$. In particular, for any event $E \subseteq \Omega$,

$$K(E) = \{\omega \in \Omega : P(\omega) \subseteq E\}.$$

Thus the agent knows event E at ω if in all the states he considers possible, E is true. Note that $K(E)$ is also an event, since it is a subset of Ω.

It is assumed that the possibility correspondence P satisfies the following properties:

P1 For any $\omega \in \Omega$, $\omega \in P(\omega)$.

P2 For any $\omega, \omega' \in \Omega$, $\omega' \in P(\omega)$ implies $P(\omega') \subseteq P(\omega)$.

P3 For any $\omega, \omega' \in \Omega$, $\omega' \in P(\omega)$ implies $P(\omega') \supseteq P(\omega)$.

The first property says that the agent never excludes the true state from being possible. The second property states that if the agent knows an event E at ω and he considers ω' to be possible, then he will also know E at ω'. The third property states that if the agent does not know an event E at ω and he considers ω' to be possible, then he will also not know E at ω'.

The following properties hold for the knowledge operator:

- K1 Necessitation: $K(\Omega) = \Omega$.
- K2 Monotonicity: $E \subseteq F \implies K(E) \subseteq K(F)$.
- K3 Conjunction: $K(E) \cap K(F) = K(E \cap F)$.
- K4 The Axiom of Knowledge: $K(E) \subseteq E$.
- K5 The Axiom of Transparency: $K(E) \subseteq K(K(E))$.
- K6 The Axiom of Wisdom: $\neg K(E) \subseteq K(\neg K(E))$.

Properties $K1, K2, K3$ are derived from the definition of the knowledge operator K, while property $P1$ implies $K4$, $P2$ implies $K5$ and $P3$ implies $K6$.

3.2 Results

The next property is the most important departure from other models dealing with unawareness.

Proposition 1. Awareness Leads to Knowledge
If $E \preceq S \preceq S'$ *then* $K_S(E) \subseteq (K_{S'}(E))_S \cap A_S(E)$.

The condition $E \preceq S \preceq S'$ ensures that S and S' are rich enough to describe the agent's knowledge and awareness of E, so that $K_S(E)$, $K_{S'}(E)$ and $A_S(E)$ are well defined. The property says that state spaces which are more expressive give a more complete description of the agent's knowledge. In other words, whatever we capture by describing knowledge with S, we can capture by describing knowledge with the more expressive S'. But the converse is not true.

Recall the example in the introduction. On the one hand, Holmes is aware of S' and state $\omega \in S'$ specifies that the dog did not bark, there is no intruder, and because of the theorem "no barking implies no intruder", Holmes knows event E, "there is no intruder". Hence, $\omega \in K^H_{S'}(E)$. On the other hand, Watson is aware of S, and his limited perception of the truth is ω_S, specifying that there is no intruder and that Holmes is aware of E, so $\omega_S \in A^H_S(E)$. The property allows for $\omega_S \notin K^H_S(E)$, so that according to Watson's limited view, Holmes does not know E at ω_S. As was argued in the introduction, the reason behind Watson's faulty reasoning about Holmes is Watson's unawareness of the theorem "no barking implies no intruder".

Intuitively, if a state space is more complete then it may also include more "theorems", and in effect contain more ways in which an agent can know an event. Conversely, what an agent is unaware of constrains his knowledge about events he is aware of. In the multi-agent case an agent's limited awareness may lead to incomplete reasoning about other agents' knowledge. See Section 4.1 for further discussion and illustration of this property in the multi-agent context.

This feature is new. On the one hand, the standard model assumes an agent who is aware of everything and knows all relevant theorems. On the other hand, the property Projections

Preserve Knowledge of Heifetz et al. (2006) implies that $K_S(E) = (K_{S'}(E))_S \cap A_S(E)$. Nothing is lost by describing knowledge in less expressive state spaces.

Concluding, more awareness can lead to increased knowledge of events that we were already aware of. For example, becoming aware of (and knowing) Newton's theory enabled us to explain how the planets move, a question of which we were aware since ancient times. Equivalently, what we *are unaware of* constrains our knowledge about things we *are aware of*. The reason is that information that is readily available to us (for example, the distance between the planets and their size) is left unexploited, either because we are unaware of its existence, or because we do not have the theorems that utilize this information in order to provide answers. Aragones et al. (2005) argue that these phenomena can be partly explained by computational complexity. An agent may learn something without getting new information, just by noticing certain regularities in the data he observes and by forming new theorems.

The next theorem verifies properties that have been proposed in the literature, or are generalizations of properties of the standard model.

Theorem 1. *Suppose $E, F \preceq S$. Then,*

1. **Subjective Necessitation** *For all $\omega \in S$, $\omega \in K_S(\Omega(\omega))$.*
2. **Generalized Monotonicity** $E_{S(E)\vee S(F)} \subseteq F_{S(E)\vee S(F)}$, $F \preceq E \implies K_S(E) \subseteq K_S(F)$.[5]
3. **Conjunction** $K_S(E) \cap K_S(F) = K_S\left(E_{S(E)\vee S(F)} \cap F_{S(E)\vee S(F)}\right)$.
4. **The Axiom of Knowledge** $K_S(E) \subseteq E_S$.
5. **The Axiom of Transparency** $\omega \in K_S(E) \iff \omega \in K_S(K_{\Omega(\omega)}(E))$.
6. **The Axiom of Wisdom** $\omega \in A_S(E) \cap \neg K_S(E) \iff \omega \in K_S(A_{\Omega(\omega)}(E) \cap \neg K_{\Omega(\omega)}(E))$.
7. **Plausibility** $U_S(E) \subseteq \neg K_S(E) \cap \neg K_S(\neg K_S(E))$.
8. **Strong Plausibility** $U_S(E) \subseteq \neg K_S(E) \cap \neg K_S(\neg K_S(E)) \cap \ldots \cap \neg K_S(\neg K_S(\ldots \neg K_S(E)))$.
9. **AU Introspection** $U_S(E) \subseteq U_S(U_S(E))$.
10. **KU Introspection** $K_S(U_S(E)) = \emptyset_S$.[6]
11. **Symmetry** $U_S(E) = U_S(\neg E)$.
12. **AA-Self Reflection** $\omega \in A_S(E) \iff \omega \in A_S(A_{\Omega(\omega)}(E))$.[7]
13. **AK-Self Reflection** $\omega \in A_S(E) \iff \omega \in A_S(K_{\Omega(\omega)}(E))$.[8]

[5] A variant of this property states that if $\omega \in K_S(E)$, $F \preceq \Omega(\omega)$ and $E_{\mathcal{V}_E \cup \mathcal{V}_F} \subseteq F_{\mathcal{V}_E \cup \mathcal{V}_F}$, then $\omega \in K_S(F)$.
[6] A variant of this property is $\omega \notin K_S(U_{\Omega(\omega)}(E))$ for all $\omega \in S$.
[7] A variant of this property is $\omega \in A_S(E) \iff \omega_{\Omega(\omega)} \in A_{\Omega(\omega)}(A_{\Omega(\omega)}(E))$.
[8] A variant of this property is $\omega \in A_S(E) \iff \omega_{\Omega(\omega)} \in A_{\Omega(\omega)}(K_{\Omega(\omega)}(E))$.

14. **A-Introspection** $\omega \in A_S(E) \iff \omega \in K_S(A_{\Omega(\omega)}(E))$.[9]

The first six properties are generalizations of the six properties of the standard model, cited in Section 3.1. Some of these generalizations are proposed by Li (2006). Plausibility, Strong Plausibility, AU Introspection and KU Introspection are proposed by Dekel et al. (1998). Symmetry, AA-Self Reflection, AK-Self Reflection and A-Introspection are proposed by Modica and Rustichini (1999) and Halpern (2001).

Subjective necessitation states that at any state ω, the agent knows his state space, which is $\Omega(\omega)$. Generalized monotonicity says that if at ω the agent knows event E, he is aware of F and E implies F, then he knows F. These two events may be subsets of different state spaces, so the usual notion of implication, $E \subseteq F$, is not defined. Li (2006) has proposed a generalized version of implication: The event E implies the event F if the enlargement of E to the join of spaces $S(E)$ and $S(F)$ is a subset of the respective enlargement of F. Conjunction states that the agent knows events E and F if and only if he knows that E and F have occurred. If E and F are subsets of different state spaces then their conjunction is take to be the intersection of their enlargements to the meet of state spaces $S(E)$ and $S(F)$.

The Axiom of Knowledge specifies that whenever an agent knows an event, then this event is true. The next two properties generalize the axioms of transparency and wisdom. The Axiom of Transparency states that the agent knows an event E at ω if and only if he knows that he knows it at ω. Note that $K_{\Omega(\omega)}(E)$ is the event "the agent knows event E", expressed in the awareness of the agent at ω. The Axiom of Wisdom is similar. The agent is aware of but does not know event E if and only if he knows that he is aware of and does not know it.

Plausibility states that if the agent is unaware of an event, then he does not know it, and he does not know that he does not know it. Strong Plausibility extends the result for any higher order of not knowing that he does not know. AU Introspection specifies that if the agent is unaware of an event, then he is unaware that he is unaware of it. KU Introspection states that the agent cannot know that he is unaware of an event E.

Symmetry states that if an agent is unaware of an event, then he is also unaware of its negation. Properties AA-Self Reflection, AK-Self Reflection and A-Introspection say that equivalent conditions for an agent to be aware of an event is that he is aware that he is aware of it, he is aware that he knows it and he knows that he is aware of it.

4 Multi-agent model

4.1 Unawareness and reasoning about others

In a multi-agent context, the property Awareness Leads to Knowledge implies that i's limited awareness may impair his reasoning about j's knowledge. For example, it may be that while i is aware of E, he wrongly deduces that j does not know it, exactly because i is unaware of the theorem that led j to know E. This clearly distinguishes the present approach from that

[9] A variant of this property is $\omega \in A_S(E) \iff \{\omega\}_{\Omega(\omega)} \in K_{\Omega(\omega)}(A_{\Omega(\omega)}(E))$.

of Li (2006) and Heifetz et al. (2006), which do not allow for such information processing errors.

To illustrate, suppose that agent i's state space is S^i, while agent j's state space is S^j and $S^j \preceq S^i$, so that i is more aware than j. They are both reasoning whether agent k knows event E. Suppose that both i and j are informed that the true state has occurred. That is, i is informed that ω has occurred, while j is informed that ω_{S^j} has occurred, which is the projection of ω to the more limited state space. Moreover, suppose that $\omega \in K^k_{S^i}(E)$ but $\omega_{S^j} \notin K^k_{S^j}(E)$, which is permitted by the Awareness Leads to Knowledge property. Since i knows that ω has occurred and j knows that ω_{S^j} has occurred, it is the case that i knows that k knows event E, while j knows that k does not know event E! Agents i and j disagree on what k knows.

It is important to emphasize that j's information processing error about k's knowledge is due to j's unawareness, not due to j's logical mistakes. Agent j is not excluding the true state, he merely perceives a limited version of the truth.

The standard model of knowledge excludes the possibility of two agents disagreeing about what a third agent knows. To be more precise, it can never be that i knows that k knows an event, while j knows that k does not know this event. Clearly, if this were to happen then one agent would be wrong, and the Axiom of Knowledge would be violated. Li (2006) and Heifetz et al. (2006) also exclude such a possibility, because they assume that i's view of j's knowledge is the projection of P^j to i's state space. On the contrary, the present model allows for such a possibility without violating the Axiom of Knowledge, because knowledge is defined "locally", for each state space.

Consider the following example which illustrates how two agents can disagree on what a third agent knows. Suppose that agent k is inside a basement with no windows, and that it is raining. Agent j is informed that k is inside the basement, so he reasons that because k cannot see what is happening outside, he does not know that it is raining, and j knows that this is the case. On the other hand, agent i is aware of and knows the existence of a computer in the basement, connected with a camera outside the building. If he is informed that k is also aware of and knows this, then he can reason that k can see whether it is raining by checking the computer. Moreover, he knows that this is the case. Concluding, the more aware agent i knows that k knows that it is raining, while the less aware agent j knows that k does not know whether it is raining.

It is worth emphasizing that the source of the two agents' *disagreement stems from their different awareness, not from their different information.* Had j been aware of the possibility of a computer in the basement, even if he did not know whether it is connected with a camera or whether k was aware of it, would enable him to say that he did not know whether k knows that it is raining. In that case, i and j would not disagree, but i would have more information. It is precisely the fact that j is unaware of the possibility of the computer that makes him know that k does not know that it is raining. Moreover, j is not making any mistakes, because it is true that with this limited awareness, k would not know whether it rained. Finally, this disagreement can only occur if what one agent is unaware of, constrains his knowledge about what he is aware of, so that the " Awareness Leads to

Knowledge" property is necessary.

As an epilogue to this example, suppose that agent k performs a specific action if and only if he knows that it is raining. Moreover, suppose that agent j knows this and k's action is visible to him. Since j knows that k does not know that it is raining, he reasons that k should not perform this action. Nevertheless, he observes him performing it. If agent j excludes the possibility that he has made some mistake in his reasoning, then he can only conclude that k is aware of something that j is not aware of, that led him to know that it is raining. In other words, *agent j understands that he is unaware of something that he cannot specify.*

4.2 Common knowledge

An event is common knowledge if everyone knows it, everyone knows that everyone knows it and so on, ad infinitum. The extra complication that arises when defining common knowledge in the present setting is that every agent has possibly different awareness. The property Awareness Leads to Knowledge shows that differences in awareness imply differences in reasoning about knowledge. Hence, an agent has to reason about other agents' awareness, before reasoning about their knowledge.

To give an example, suppose $\omega \in S$ specifies that $P^i(\omega) \subseteq \Omega^i(\omega)$, so agent i is aware of state space $\Omega^i(\omega)$. Agent i's reasoning about j's knowledge of event E is represented by event $K^j_{\Omega^i(\omega)}(E)$, because $\Omega^i(\omega)$ is the most complete state space that i is aware of.[10] When i reasons about j's reasoning about k's knowledge of E, i first has to specify what is j's most complete state space. If we denote this by $\Omega^{ij}(\omega)$, then i's view of j's view of k's knowledge of E is the event $K^k_{\Omega^{ij}(\omega)}(E)$.

What remains to be determined is $\Omega^{ij}(\omega)$, i's view of j's most complete state space at ω. Note that state $\omega' \in \Omega^i(\omega)$ specifies that j's state space is $\Omega^j(\omega')$. But i does not necessarily know what state has occurred - he only knows that one state in $P^i(\omega)$ has occurred. $\Omega^{ij}(\omega)$ is therefore defined to be the most complete state space that i *knows* that j is aware of, at ω:

$$\Omega^{ij}(\omega) = \bigwedge_{\omega' \in P^i(\omega)} \Omega^j(\omega').$$

This is the meet of all (most complete) state spaces that, according to i's knowledge, j could be aware of.

At ω, agent i's view of the event "j knows that k knows E" is the set $K^j_{\Omega^i(\omega)}K^k_{\Omega^{ij}(\omega)}(E)$. The event "$i$ knows $K^j_{\Omega^i(\omega)}K^k_{\Omega^{ij}(\omega)}(E)$" is the set

$$K^i_S K^j_{\Omega^i(\omega)} K^k_{\Omega^{ij}(\omega)}(E).$$

Note that $K^i_S K^j_{\Omega^i(\omega)} K^k_{\Omega^{ij}(\omega)}(E)$ is a subset of S. This event depends on ω, which determines i's awareness and knowledge, and therefore i's view of j's awareness.

[10] Recall that the Awareness Leads to Knowledge shows that more complete state spaces give a better description of one's knowledge.

Before expanding the sequence of agents we need to define, for any event E and any agent i, the set $P^i(E)$, which denotes the set of states that agent i considers possible when the true state lies in E.

Definition 1. *For any event E,*

$$P^i(E) = \bigcup_{\omega \in E} (P^i(\omega))_{S_E^i},$$

where $S_E^i = \bigwedge_{\omega \in E} \Omega^i(\omega)$.

This is analogous to the definition of $P^i(E) = \bigcup_{\omega \in E} P^i(\omega)$ in the standard model.[11] The extra complication that arises in the present model is that different states in E may describe different awareness for agent i, and therefore the sets $P^i(\omega)$ and $P^i(\omega')$ for $\omega, \omega' \in E$ may be subsets of different state spaces. S_E^i denotes the meet of these different state spaces and we project $P^i(\omega)$ and $P^i(\omega')$ to that state space.

We can now define the state space that at ω, i knows that j knows that k is aware of to be

$$\Omega^{ijk}(\omega) = \bigwedge_{\omega' \in P^j(P^i(\omega))} \Omega^k(\omega').$$

The event "at ω, i knows that j knows that k knows that l knows event E" is

$$K_S^i K_{\Omega^i(\omega)}^j K_{\Omega^{ij}(\omega)}^k K_{\Omega^{ijk}(\omega)}^l(E).$$

Adding more agents to the sequence can easily be accommodated. For any $k \geq 2$, define

$$\Omega^{i_1 i_2 \ldots i_k}(\omega) = \bigwedge_{\omega' \in P^{i_{k-1}}(\ldots(P^{i_2}(P^{i_1}(\omega))))} \Omega^{i_k}(\omega')$$

to be the state space that i_1 knows that i_2 knows that ... that i_k is aware of at ω.

The following definition of common knowledge is analogous to that of the standard model. The only difference is that each knowledge operator K is expressed in a particular state space.

Definition 2. *Event $E \preceq S$ is common knowledge among agents $i = 1, \ldots, I$ at $\omega \in S$ if and only if for any $n \in \mathbb{N}$ and any sequence of agents $i_1, \ldots, i_n$,*
$\omega \in K_S^{i_1} K_{\Omega^{i_1}(\omega)}^{i_2} K_{\Omega^{i_1 i_2}(\omega)}^{i_3} \ldots K_{\Omega^{i_1 i_2 \cdots i_{n-1}}(\omega)}^{i_n}(E)$.[12]

Let $CK_S(E)$ be the subset of S describing that E is common knowledge. If S is the uppermost state space then $CK_S(E)$ is the analyst's or fully aware agent's perception of

[11] For details, see Geanakoplos (1992).

[12] $K_S^{i_1} K_{\Omega^{i_1}(\omega)}^{i_2} K_{\Omega^{i_1 i_2}(\omega)}^{i_3} \ldots K_{\Omega^{i_1 i_2 \cdots i_{n-1}}(\omega)}^{i_n}(E)$ is defined if $E \preceq \Omega^{i_1 i_2 \ldots i_{n-1}}(\omega)$. By definition, for any $n \geq 2$, $\Omega^{i_1 i_2 \ldots i_n}(\omega) \preceq \Omega^{i_1 i_2 \ldots i_{n-1}}(\omega)$.

common knowledge.[13] If S is not the uppermost state space then we get the agent's view of common knowledge. One of the main results of the paper is that more complete state spaces give a better description of one's knowledge. Similarly, they give a better description of common knowledge. This property is expressed in the following Lemma.

Lemma 1. *If $E \preceq S \preceq S'$ then $CK_S(E) \subseteq (CK_{S'}(E))_S$.*

Just as in the standard model, there is an equivalent definition of common knowledge, which employs the possibility correspondences P^i, instead of the knowledge operators K^i.

Proposition 2. *Event E is common knowledge among agents $i = 1, \ldots, I$ at ω if and only if for any $n \in \mathbb{N}$ and any sequence of agents $i_1, \ldots, i_n$, $E \preceq \Omega^{i_1 \cdots i_n}(\omega)$ and*

$$P^{i_n} \ldots P^{i_2} P^{i_1}(\omega) \subseteq E_{\Omega^{i_1 \cdots i_n}(\omega)}.$$

4.3 Common knowledge of awareness

Define $\Omega^\wedge(\omega)$ to be the meet of all state spaces $\Omega^{i_1 \cdots i_n}(\omega)$, for any sequence $i_1, i_2, \ldots, i_n$, $n \in \mathbb{N}$:

$$\Omega^\wedge(\omega) = \bigwedge_{\substack{i_1 \cdots i_n \\ n \in \mathbb{N}}} \Omega^{i_1 \cdots i_n}(\omega).$$

Lemma 2. *$\Omega^\wedge(\omega)$ is common knowledge at $\omega \in S$. Moreover, if $E \in \mathcal{E}$ is common knowledge at ω then $E \preceq \Omega^\wedge(\omega)$.*

The Lemma states that each state ω specifies a "common" state space $\Omega^\wedge(\omega)$, that every agent is aware of and this fact is common knowledge. Moreover, $\Omega^\wedge(\omega)$ is the most complete state space with this property, because any event E that is common knowledge at ω can be expressed within the vocabulary of $\Omega^\wedge(\omega)$.

4.4 Characterizing common knowledge

In the standard model an event E^* is common knowledge at ω if and only if there is an event E which is self evident for all agents, it contains ω and is a subset of E^*. The following two theorems provide a similar characterization of common knowledge in an environment with unawareness. The definition of a self evident event is given below, and it is a direct analog of the standard definition. Recall that if E is an event then $S(E)$ is the state space of which it is a subset.

Definition 3. *Event E is self evident for $i \in I$ if $E \subseteq K^i_{S(E)}(E)$. If E is self evident for all $i \in I$, then it is called public.*

[13] Such a space exists because $\mathcal{S}$ is assumed to be a complete lattice.

An event E is self evident for agent $i \in I$ if whenever it happens, the agent knows it. It is public if everyone knows it.

The following theorem provides sufficient conditions for event E^* to be common knowledge. In particular, suppose that there is a public event E, whose enlargement to state space S contains $\omega \in S$, it is more expressive than E^* ($E^* \preceq E$) and it is a subset of the enlargement of E^* to S. Then E^* is common knowledge at ω.

Theorem 2. *Suppose event E is public and there is another event E^* such that $E^* \preceq E \preceq S$ and $\omega \in E_S \subseteq E^*_S$. Then, E^* is common knowledge at ω.*

Although in the standard model the existence of a public event with the aforementioned properties is also a necessary condition for common knowledge, this is not true here. However, there is a necessary and sufficient condition for the existence of such a public event. Let $S_\omega = \{S \in \mathcal{S} : S = \Omega^{i_1 \cdots i_n}(\omega)$ for some sequence $i_i, \ldots, i_n\}$ be the collection of state spaces that agents "reach" when reasoning about their awareness and knowledge. Consider the following property.

Definition 4. *Suppose that $E \preceq \Omega^\wedge(\omega)$.[14] Moreover, for any sequence $i_1, \ldots, i_n$, for any $S \in S_\omega$, we have*

$$\omega_S \in K_S^{i_1} K_{\Omega^{i_1}(\omega_S)}^{i_2} \ldots K_{\Omega^{i_1 \cdots i_{n-1}}(\omega_S)}^{i_n}(E).$$

Then, $\omega_{\Omega^\wedge(\omega)} \in K_{\Omega^\wedge(\omega)}^{i_1} \ldots K_{\Omega^\wedge(\omega)}^{i_n}(E)$ for any sequence $i_1, \ldots, i_n$.

This is effectively a continuity property. Every $S \in S_\omega$ is more expressive than $\Omega^\wedge(\omega)$, and elements in S_ω get arbitrarily close to $\Omega^\wedge(\omega)$. The property states that if ω_S specifies that "i_1 knows that $\ldots i_n$ knows E" for every such S, then the same is true at the limit, which is $\Omega^\wedge(\omega)$. Say that *knowledge of E is continuous at ω* if this condition is satisfied. Lemma 6 in the appendix shows that PPK implies continuity for any event E and any ω. The following theorem shows that this continuity is necessary and sufficient for the existence of a public event with the two needed properties.

Theorem 3. *Suppose E^* is common knowledge at $\omega \in S$. Then, there exists a public event E such that $E^* \preceq E \preceq S$ and $\omega \in E_S \subseteq E^*_S$ if and only if knowledge of E^* is continuous at ω.*

Li (2006) also gives a characterization of common knowledge, under two assumptions that are not needed here.[15] The first is that what the agent knows about one question does not depend on the answers of any other question; that is, there is no correlation between answers of different questions. The second assumption is that if two full states specify the same answer for a particular question, then the agent will either be aware of that question in both states, or unaware of it in both states. This means that there can be no correlation between the awareness of a particular question and the answers of other questions.

[14] This condition ensures that the event $K_{\Omega^\wedge(\omega)}^{i_1} \ldots K_{\Omega^\wedge(\omega)}^{i_n}(E)$ is well defined.

[15] The primitives in Li's model are questions and their respective answers.

5 No-trade theorems

The standard model of knowledge specifies that asymmetric information alone cannot explain trade. In this section we provide, with an example and a theorem, an explanation of why agents with asymmetric information *and* asymmetric awareness can engage in trade.

The literature on no-trade theorems stems from the well known result of Aumann (1976) that if agents have common priors and their posteriors about an event are common knowledge, then these posteriors must be identical. This section shows that in an environment with unawareness the same result is true only for common priors and posteriors which are defined on the "common" state space, which is the state space that not only everyone is aware of, but it is also common knowledge that everyone is aware of. However, as the property Awareness Leads to Knowledge suggests, state spaces which carry more awareness give a more complete description of one's knowledge and posteriors. An example with two agents shows that although the posteriors defined on this "common" state space are common knowledge and therefore identical, there still can be trade because one agent's higher awareness implies that his actual posterior is different and beyond the other agent's reasoning.

Recall that in the standard model without unawareness and given a prior μ on the unique state space Ω we can define i's posterior of event E at $\omega' \in \Omega$ using Bayes' law:

$$q^i(E)(\omega') = \frac{\mu(P^i(\omega') \cap E)}{\mu(P^i(\omega'))}. \tag{1}$$

Every state in Ω specifies a posterior about event E for each agent. The posteriors are common knowledge at a state ω if an event specifying a single posterior for each agent is common knowledge at ω.

Translating the above in an environment with unawareness gives rise to the following complications. The first is that agents typically have different subjective state spaces, since they have different awareness. Because we need to specify a prior for each state space, the definition of a common prior needs to be generalized. In what follows, we impose a common prior μ on the common state space S and require that if S' is more expressive than S, $(S \preceq S')$ then the marginal of prior μ' (defined on S') is μ. Note that given μ, there are many μ' that satisfy this condition.

The second complication is that the multiplicity of state spaces implies that each state of any state space defines a possibly different posterior for each agent. However, as Lemma 2 shows, posteriors can be common knowledge only if they are defined on the "common" state space or a less complete state space. Since state spaces generated by more awareness give a more complete specification of an agent's posterior, it is meaningful to talk about posteriors being common knowledge only if they are defined on this common state space.

Let $I = \{i, j\}$ and suppose μ is a prior on $\Omega^\wedge(\omega)$, the most complete state space that it is commonly known at ω that both agents are aware of. Let $E \subseteq \Omega^\wedge(\omega)$ be an event. Agent i's posterior about E at $\omega' \in \Omega^\wedge(\omega)$ is given by equation (1).[16] Event E^* specifies that both

[16]This definition requires that the agent is aware of E at ω'. In the theorem below this condition is satisfied for self evident events, so we do not need to specify the more complicated definition of a posterior.

agents are aware of E and that i's posterior is q^i, while j's posterior is q^j:

$$E^* = \left\{\omega' \in \Omega^\wedge(\omega) : q^i(E)(\omega') = q^i, q^j(E)(\omega') = q^j\right\}.$$

The following theorem states that if E^* is common knowledge, then $q^i = q^j$. There is an added assumption which was discussed in the previous section where common knowledge was characterized in terms of a self evident event.

Theorem 4. *Suppose that E^* is common knowledge at ω, so that it is commonly known that i's posterior is q^i and j's posterior is q^j. If knowledge of E^* is continuous at ω, then $q^i = q^j$.*

Theorem 4 states that if the posteriors defined on the common state space are common knowledge, they are identical. The following example shows that if an agent's awareness is bigger than the common one, then his actual posterior may be different and beyond the other agent's reasoning. Hence, agents can agree to disagree and trade.

Example

Recall the example in the introduction, depicted in the figure below.

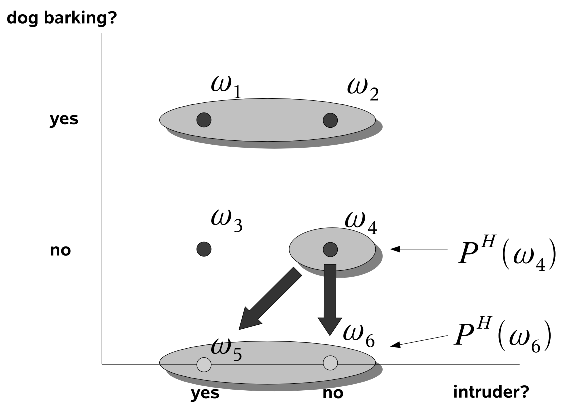

Figure 2

There are two agents, Holmes and Watson. There are two state spaces, $S' = \{\omega_1, \omega_2, \omega_4\}$ and $S = \{\omega_5, \omega_6\}$.[17] The union of S and S' is Σ. Watson is always unaware of the extra dimension and his possibility correspondence is such that $P^W(\omega) = \{\omega_5, \omega_6\}$, for any state $\omega \in \Sigma$. Holmes' possibility correspondence is as follows:

$$P^H(\omega_1) = P^H(\omega_2) = \{\omega_1, \omega_2\},$$

$$P^H(\omega_4) = \{\omega_4\},$$

[17]State ω_3 is impossible so we do not include it in S'.

$$P^H(\omega_5) = P^H(\omega_6) = \{\omega_5, \omega_6\}.$$

At ω_4, Holmes is aware of both dimensions and because he knows the theorem "no barking implies no intruder" and he receives information about the dog not barking, he is able to deduce that there is no intruder. However, Watson, being unaware of the dog, reasons that Holmes' possibility correspondence is $P^H(\omega_5) = P^H(\omega_6) = \{\omega_5, \omega_6\}$.

The common state space is S. Let μ be the common prior defined on S, such that $\mu(\omega_5) = \mu(\omega_6) = 1/2$. Suppose that Holmes and Watson bet on whether there is an intruder, that is, on the occurrence of event $E = \{\omega_6\}$. The posterior of Holmes about E at ω is $p^H(\omega)$, while Watson's is $p^W(\omega)$. Note that different state spaces give different descriptions of posteriors.

As discussed in section 4.3, an event can be common knowledge only if it is expressed in S.[18] At ω_4, event $\{\omega_5, \omega_6\}$ is common knowledge, specifying that Holmes' posterior is $p^H(\omega_5) = p^H(\omega_6)$, while Watson's posterior is $p^W(\omega_5) = p^W(\omega_6)$. In accordance with Theorem 4, $p^H(\omega_5) = p^H(\omega_6) = p^W(\omega_5) = p^W(\omega_6)$.

For Watson this is the end of the story, since he is unaware of the extra dimension of the dog not barking. However, Holmes is more aware. At ω_4, he knows that there is no intruder and hence he is willing to bet. His posterior about E at ω_4 is 1. Hence, although the posteriors described in S are common knowledge and equal, Holmes' "actual" posterior is different.[19]

Discussion

Theorem 4 shows that whenever the posteriors defined on the common state space are common knowledge, they are identical. Nevertheless, the example showed that if Holmes is more aware, his true posterior may be different and beyond the other agent's reasoning. Hence, agents can agree to disagree and trade. Note that we could have easily specified that also Watson was more aware in other dimensions that Holmes is unaware of. In that case, his true posterior would also be beyond Holmes' reasoning. But this was not necessary in order to have trade.

Intuition for this result can be obtained if we interpret the equality of the posteriors as the outcome of the following procedure, described in the context of the standard model of knowledge by Geanakoplos and Polemarchakis (1982). Suppose that initially Holmes and Watson have different posteriors about E, and in particular Holmes has a posterior above a half and wants to buy, while Watson has a posterior below a half, and wants to sell. Suppose that they meet and they announce their posteriors and their willingness to trade. Holmes can then use Watson's announcement in order to further refine what he knows, by taking the intersection of his own information with the set of states that describe a posterior below a half for Watson. Holmes can now announce a possibly different posterior which reflects his new information, while Watson can use Holmes' announcement to further refine his own

[18] Or a less complete state space, which does not exist in this example.

[19] Note that we have not specified Holmes' prior on S'. The result is true for any prior that assigns positive probability to ω_4.

information. Geanakoplos and Polemarchakis (1982) shows that the agents will eventually agree on the posteriors.

A necessary condition for this result is that partitions are common knowledge, which is true in the standard model. It is also true in this model but only for state spaces commonly known that everyone is aware of. Therefore, the updating of the posteriors that was described above can only refer to such a common state space. If Holmes is more aware, then announcing his "true" posterior or his willingness to buy will be of no value to Watson, because he is simply unaware of the states that would enable Holmes to make these announcements. As a result, Watson cannot further refine his own knowledge. Updating of information due to other agents' actions or announcements still takes place in an environment with unawareness, but it is constrained by what is commonly known that everyone is aware of. Hence, agents can engage in trade when the differences in their posteriors stem from asymmetric information acquired by theorems that others are unaware of.

Concluding, we need to emphasize that the purpose of the example is not to show that there *can* be trade. This can easily be shown within the framework of the standard model, by assuming that agents have different priors, or that they make mistakes - and unawareness is a form of mistake. The purpose of the example is to isolate a particular type of mistake (unawareness of theorems) and use it to provide an interesting or plausible story of why (otherwise rational) agents might trade.

Heifetz et al. (2006) and Heifetz et al. (2007a) also provide alternative examples of speculative trade in an environment with unawareness. In their setting, an owner contemplates selling his firm to a potential buyer. The "common" state space specifies that the value of the firm can be either 100 or 80. The owner is aware of a possible lawsuit that could decrease the firm's value by 20, but not of a possible novelty that could increase its value by 20. The potential buyer is aware of the novelty but not of the lawsuit. It is shown that "there is common certainty of preference to trade, but each player strictly prefers to trade".

Both the example in Heifetz et al. (2007a) and that in the present paper specify a "common" state space where agents are indifferent between trading or not, but are willing to trade in their respective, more complete, state spaces.[20] But the reason is different. In the example of Heifetz et al. (2007a), differences in awareness imply differences in the perception of the actual payoff, and wrong reasoning about the other agent's perception of his payoff. For instance, the owner wrongly deduces that the firm's value can be 60, 80 or 100 and that the buyer thinks that the value can be 80 or 100. In the example of the present paper, differences in awareness imply differences in posteriors about events of the common state space, and wrong inferences about the other agent's perception of his posterior. For instance, Watson's posterior about the event "there is no intruder" is 1/2 and he wrongly deduces that Holmes' posterior about the same event is also 1/2.

[20] In the example of this paper only one agent's state space is more expressive than the common state space. This can easily be extended to an example where this is true for both agents.

A Appendix

Proof of Proposition 1.

First we prove that if $S \preceq S'$, then $(K_S(E))_{S'} \subseteq K_{S'}(E)$. Suppose $\omega \in (K_S(E))_{S'}$. Then, $\omega_S \in K_S(E)$, which implies that $E \preceq P(\omega_S)$ and $P(\omega_S) \subseteq E_{\Omega(\omega_S)}$. Projections Preserve Ignorance implies that $E \preceq P(\omega_S) \preceq P(\omega)$ and $P(\omega) \subseteq (P(\omega_S))_{\Omega(\omega)} \subseteq E_{\Omega(\omega)}$. Hence, $\omega \in K_{S'}(E)$. Finally, $(K_S(E))_{S'} \subseteq K_{S'}(E)$ implies $K_S(E) \subseteq (K_{S'}(E))_S$. Also, note that $K_S(E) \subseteq A_S(E)$. That the other direction is not necessarily true is shown by the main example.

□

Proof of Theorem 1.

1. **Subjective Necessitation** Suppose $\omega \in S$. Confinedness implies that $P(\omega) \subseteq S'$ for some $S' \preceq S$. Since $\Omega(\omega) = S'$ we have $\omega \in K_S(\Omega(\omega))$.

2. **Generalized Monotonicity** Suppose $\omega \in K_S(E)$. Then, $E \preceq P(\omega)$ and $P(\omega) \subseteq E_{\Omega(\omega)}$. Also, $F \preceq P(\omega)$ which implies $S(E) \vee S(F) \preceq \Omega(\omega)$ and $E_{\Omega(\omega)} \subseteq F_{\Omega(\omega)}$. Therefore, $\omega \in K_S(F)$.

3. **Conjunction** We have that $E \preceq P(\omega)$ and $F \preceq P(\omega)$ if and only if $S(E) \vee S(F) \preceq P(\omega)$. Also, $P(\omega) \subseteq E_{\Omega(\omega)}$ and $P(\omega) \subseteq F_{\Omega(\omega)}$ if and only if $P(\omega) \subseteq E_{\Omega(\omega)} \cap F_{\Omega(\omega)} = (E_{S(E)\vee S(F)} \cap F_{S(E)\vee S(F)})_{\Omega(\omega)}$. The latter equality follows because $\omega_1 \in (E_{S(E)\vee S(F)} \cap F_{S(E)\vee S(F)})_{\Omega(\omega)} \iff \{\omega_1\}_{S(E)\vee S(F)} \in E_{S(E)\vee S(F)} \cap F_{S(E)\vee S(F)} \iff \omega_1 \in E_{\Omega(\omega)} \cap F_{\Omega(\omega)}$.

4. **The Axiom of Knowledge** $\omega \in K_S(E)$ implies $E \preceq P(\omega)$ and $P(\omega) \subseteq E_{\Omega(\omega)}$. Generalized Reflexivity implies $\omega_{\Omega(\omega)} \in P(\omega)$. Hence, $\omega_{\Omega(\omega)} \in E_{\Omega(\omega)}$, which implies $\omega \in E_S$.

5. **The Axiom of Transparency** Suppose $\omega \in K_S(E)$. Then, $E \preceq P(\omega)$ and $P(\omega) \subseteq E_{\Omega(\omega)}$. We have to show that $P(\omega) \subseteq K_{\Omega(\omega)}(E)$, or that $\omega_1 \in P(\omega)$ implies $E \preceq P(\omega_1)$ and $P(\omega_1) \subseteq E_{\Omega(\omega_1)}$. From Stationarity we have that $\omega_1 \in P(\omega)$ implies $P(\omega_1) = P(\omega)$ and $\Omega(\omega) = \Omega(\omega_1)$. Hence, $E \preceq P(\omega_1)$ and $P(\omega_1) \subseteq E_{\Omega(\omega_1)}$. Suppose $\omega \in K_S K_{\Omega(\omega)}(E)$. Then, $P(\omega) \subseteq K_{\Omega(\omega)}(E)$. From Generalized Reflexivity we have $\omega_{\Omega(\omega)} \in P(\omega)$ and from the proof of Proposition 1 we have $(K_{\Omega(\omega)}(E))_S \subseteq K_S(E)$. Therefore, $\omega \in K_S(E)$.

6. **The Axiom of Wisdom** Suppose $\omega \in A_S(E) \cap \neg K_S(E)$. Then, $E \preceq P(\omega)$ and $P(\omega) \nsubseteq E_{\Omega(\omega)}$. We need to show that $P(\omega) \subseteq A_{\Omega(\omega)}(E) \cap \neg K_{\Omega(\omega)}(E)$. Suppose $\omega_1 \in P(\omega)$. Stationarity implies that $P(\omega_1) = P(\omega)$. Hence, $E \preceq P(\omega_1)$ and $P(\omega_1) \nsubseteq E_{\Omega(\omega_1)}$, which imply that $\omega_1 \in A_{\Omega(\omega)}(E) \cap \neg K_{\Omega(\omega)}(E)$.

   Suppose $\omega \in K_S(A_{\Omega(\omega)}(E) \cap \neg K_{\Omega(\omega)}(E))$. Then, $P(\omega) \subseteq A_{\Omega(\omega)}(E) \cap \neg K_{\Omega(\omega)}(E)$. Since $A_{\Omega(\omega)}(E)$ is defined only if $E \preceq \Omega(\omega)$, we have that $\omega \in A_S(E)$. It remains to show that $\omega \in \neg K_S(E)$, or that $P(\omega) \nsubseteq E_{\Omega(\omega)}$. We know that for all $\omega_1 \in P(\omega)$, $\omega_1 \in \neg K_{\Omega(\omega)}(E)$, which implies that $P(\omega_1) \nsubseteq E_{\Omega(\omega)}$. Since $P(\omega) = P(\omega_1)$, we have that $P(\omega) \nsubseteq E_{\Omega(\omega)}$.

8. **Strong Plausibility** Suppose $\omega \in U_S(E)$. By definition, we have $E \preceq S$ and $E \not\preceq P(\omega)$ which imply $S \not\preceq P(\omega)$. Hence, $\omega \in \neg K_S(E) \cap \neg K_S(\neg K_S(E)) \cap \ldots \cap \neg K_S(\neg K_S(\ldots \neg K_S(E)))$.

9. **AU Introspection** Suppose $\omega \in U_S(E)$. By definition, we have $E \preceq S$ and $E \not\preceq P(\omega)$ which imply $S \not\preceq P(\omega)$ and $\omega \in U_S(U_S(E))$.

10. **KU Introspection** Suppose $\omega \in K_S(U_S(E))$. Then, $S \preceq P(\omega)$ and from Confinedness and $\omega \in S$ we have $P(\omega) \preceq S$ and $P(\omega) \subseteq U_S(E)$. Generalized Reflexivity implies that $\omega \in U_S(E)$, which implies $E \not\preceq P(\omega)$. But this contradicts that $E \preceq S$. The proof of footnote 6 is identical.

11. **Symmetry** Follows since by definition $E \preceq \neg E$ if and only if $\neg E \preceq E$.

12. **AA-Self Reflection** $\omega \in A_S(E)$ implies $E \preceq S$ and $E \preceq P(\omega)$. Therefore, $A_S(A_{\Omega(\omega)}(E))$ is well defined and $\Omega(\omega) \preceq P(\omega)$ implies $\omega \in A_S(A_{\Omega(\omega)}(E))$. For the other direction, suppose that $\omega \in A_S(A_{\Omega(\omega)}(E))$. Since $A_{\Omega(\omega)}(E)$ is defined only if $E \preceq \Omega(\omega)$, we have that $\omega \in A_S(E)$.

13. **AK-Self Reflection** The proof is similar.

14. **A-Introspection** $\omega \in A_S(E)$ implies $E \preceq S$ and $E \preceq P(\omega)$, so we just have to show that $P(\omega) \subseteq A_{\Omega(\omega)}(E)$. Suppose that $\omega_1 \in P(\omega)$. Stationarity implies $P(\omega) = P(\omega_1)$, so we have $E \preceq P(\omega_1)$ and $\omega_1 \in A_{\Omega(\omega)}(E)$. For the other direction, suppose that $\omega \in K_S(A_{\Omega(\omega)}(E))$. This implies that $\omega \in A_S(A_{\Omega(\omega)}(E))$ and $\omega \in A_S(E)$ follows from AA-Self Reflection.

□

Lemma 3. *Suppose $E \preceq S$. For any sequence $i_1, \ldots, i_n$, $\Omega^{i_1\cdots i_n}(\omega_S) \preceq \Omega^{i_1\cdots i_n}(\omega)$ and $(P^{i_n} \ldots P^{i_1}(\omega))_{\Omega^{i_1\cdots i_n}(\omega_S)} \subseteq P^{i_n} \ldots P^{i_1}(\omega_S)$.*

Proof. For $n = 1$, PPI implies that $(P^{i_1}(\omega))^{\uparrow} \subseteq (P^{i_1}(\omega_S))^{\uparrow}$. Suppose $\omega' \in P^{i_1}(\omega)$. If $\Omega^{i_1}(\omega_S) \not\preceq \Omega^{i_1}(\omega)$ then $\omega' \notin (P^{i_1}(\omega_S))^{\uparrow}$, a contradiction. Moreover, $\omega' \in (P^{i_1}(\omega_S))^{\uparrow}$ implies $\omega'_{\Omega^{i_1}(\omega_S)} \in P^{i_1}(\omega_S)$.

Suppose the claim is true for $n = k$. Recall that,

$$\Omega^{i_1\cdots i_{k+1}}(\omega) = \bigwedge_{\omega' \in P^{i_k}\ldots P^{i_1}(\omega)} \Omega^{i_{k+1}}(\omega'),$$

$$P^{i_{k+1}} \ldots P^{i_1}(\omega) = \bigcup_{\omega' \in P^{i_k}\ldots P^{i_1}(\omega)} (P^{i_{k+1}}(\omega'))_{\Omega^{i_1\cdots i_{k+1}}(\omega)}.$$

From the induction hypothesis we know that $(P^{i_k} \ldots P^{i_1}(\omega))_{\Omega^{i_1\cdots i_k}(\omega_S)} \subseteq P^{i_k} \ldots P^{i_1}(\omega_S)$. Hence, for any $\omega' \in P^{i_k} \ldots P^{i_1}(\omega)$ we have $\omega'_{\Omega^{i_1\cdots i_k}(\omega_S)} \in P^{i_k} \ldots P^{i_1}(\omega_S)$. From PPI, we have that $\Omega^{i_{k+1}}(\omega'_{\Omega^{i_1\cdots i_k}(\omega_S)}) \preceq \Omega^{i_{k+1}}(\omega')$ and that $(P^{i_{k+1}}(\omega'))_{\Omega^{i_1\cdots i_{k+1}}(\omega_S)} \subseteq (P^{i_{k+1}}(\omega'_{\Omega^{i_1\cdots i_k}(\omega_S)}))_{\Omega^{i_1\cdots i_{k+1}}(\omega_S)}$. Hence, the result follows. □

Proof of Lemma 1. We first show that $(CK_S(E))_{S'} \subseteq CK_{S'}(E)$. Suppose that $\omega \in (CK_S(E))_{S'}$. Then, $\omega_S \in CK_S(E)$ which implies that for any sequence $i_1, \dots i_n$ we have $E \preceq \Omega^{i_1 \cdots i_n}(\omega_S)$ and $P^{i_n} \dots P^{i_1}(\omega_S) \subseteq E_{\Omega^{i_1 \cdots i_n}(\omega_S)}$. Applying Lemma 3 we have $E \preceq \Omega^{i_1 \cdots i_n}(\omega)$ and $P^{i_n} \dots P^{i_1}(\omega) \subseteq E_{\Omega^{i_1 \cdots i_n}(\omega)}$, which implies, from Proposition 3, that $\omega \in CK_{S'}(E)$. Finally, $(CK_S(E))_{S'} \subseteq CK_{S'}(E)$ implies $CK_S(E) \subseteq (CK_{S'}(E))_S$. That the other direction is not necessarily true is shown by the following example. There are two agents, equipped with possibility correspondences identical to that of Holmes, of the original example. Then, $\{\omega_6\}$ is common knowledge at ω_4, but not at ω_6. □

The proof of Proposition 2 is an immediate consequence of the following Proposition.

Proposition 3. *For any sequence $i_1 \dots i_n$, the following two statements are equivalent:*

- $E \preceq \Omega^{i_1 \cdots i_n}(\omega)$ *and* $P^{i_n}(\dots(P^{i_2}(P^{i_1}(\omega)))) \subseteq E_{\Omega^{i_1 \cdots i_n}(\omega)}$,
- $\omega \in K_S^{i_1}(K_{\Omega^{i_1}(\omega)}^{i_2}(\dots K_{\Omega^{i_1 \cdots i_{n-1}}(\omega)}^{i_n}(E)))$.

Proof. Note that $P^{i_n}(\dots(P^{i_2}(P^{i_1}(\omega)))) \subseteq \Omega^{i_1 \cdots i_n}(\omega)$ for all $n \geq 1$ because $S^{i_n}_{P^{i_{n-1}}(\dots(P^{i_2}(P^{i_1}(\omega))))} = \Omega^{i_1 \cdots i_n}(\omega)$. This implies that $\Omega^{i_1 \cdots i_n}(\omega) \preceq \Omega^{i_1 \cdots i_{n-1}}(\omega)$ for all $n \geq 2$.

Suppose that $E \preceq \Omega^{i_1 \cdots i_n}(\omega)$ and $P^{i_n}(\dots(P^{i_2}(P^{i_1}(\omega)))) \subseteq E_{\Omega^{i_1 \cdots i_n}(\omega)}$. Then, $E \preceq \Omega^{i_1 \cdots i_{n-1}}(\omega)$ and $K_S^{i_1}(K_{\Omega^{i_1}(\omega)}^{i_2}(\dots K_{\Omega^{i_1 \cdots i_{n-1}}(\omega)}^{i_n}(E)))$ is well defined.

The rest of the proof is by induction:

- For $n = 1$ we have that $\omega \in K_S^{i_1}(E)$ if and only if $E \preceq \Omega^{i_1}(\omega)$ and $P^{i_1}(\omega) \subseteq E_{\Omega^{i_1}(\omega)}$.
- For $n = k$, suppose that $E \preceq \Omega^{i_1 \cdots i_k}(\omega)$ and $P^{i_k}(\dots(P^{i_1}(\omega))) \subseteq E_{\Omega^{i_1 \cdots i_k}(\omega)}$ if and only if $\omega \in K_S^{i_1}(\dots K_{\Omega^{i_1 \cdots i_{k-1}}(\omega)}^{i_k}(E))$.
- For $n = k + 1$, we need to show that $E \preceq \Omega^{i_1 \cdots i_{k+1}}(\omega)$ and $P^{i_{k+1}}(\dots(P^{i_1}(\omega))) \subseteq E_{\Omega^{i_1 \cdots i_{k+1}}(\omega)}$ if and only if $\omega \in K_S^{i_1}(\dots K_{\Omega^{i_1 \cdots i_k}(\omega)}^{i_{k+1}}(E))$.

Suppose that $E \preceq \Omega^{i_1 \cdots i_{k+1}}(\omega)$ and $P^{i_{k+1}}(\dots(P^{i_1}(\omega))) \subseteq E_{\Omega^{i_1 \cdots i_{k+1}}(\omega)}$. This implies that $(P^{i_{k+1}}(\omega'))_{\Omega^{i_1 \cdots i_{k+1}}(\omega)} \subseteq E_{\Omega^{i_1 \cdots i_{k+1}}(\omega)}$, for all $\omega' \in P^{i_k}(\dots(P^{i_1}(\omega)))$. We want to show that $P^{i_k}(\dots(P^{i_1}(\omega))) \subseteq K_{\Omega^{i_1 \cdots i_k}(\omega)}^{i_{k+1}}(E)$ which from the induction hypothesis implies that $\omega \in K_S^{i_1}(\dots(K_{\Omega^{i_1 \cdots i_{k-1}}(\omega)}^{i_k}(K_{\Omega^{i_1 \cdots i_k}(\omega)}^{i_{k+1}}(E))))$. Suppose that $\omega' \in P^{i_k}(\dots(P^{i_1}(\omega)))$. Then, $(P^{i_{k+1}}(\omega'))_{\Omega^{i_1 \cdots i_{k+1}}(\omega)} \subseteq E_{\Omega^{i_1 \cdots i_{k+1}}(\omega)}$. Since $E \preceq \Omega^{i_1 \cdots i_{k+1}}(\omega) \preceq \Omega^{i_{k+1}}(\omega')$, we also have $P^{i_{k+1}}(\omega') \subseteq E_{\Omega^{i_{k+1}}(\omega')}$. Together, they imply $\omega' \in K_{\Omega^{i_1 \cdots i_k}(\omega)}^{i_{k+1}}(E)$.

For the other direction, suppose that $\omega \in K_S^{i_1} \dots K_{\Omega^{i_1 \cdots i_k}(\omega)}^{i_{k+1}}(E)$. The induction hypothesis implies that $P^{i_k}(\dots(P^{i_1}(\omega))) \subseteq K_{\Omega^{i_1 \cdots i_k}(\omega)}^{i_{k+1}}(E)$. Hence, for all $\omega' \in P^{i_k}(\dots(P^{i_1}(\omega)))$ we have $E \preceq \Omega^{i_{k+1}}(\omega')$, $P^{i_{k+1}}(\omega') \subseteq E_{\Omega^{i_{k+1}}(\omega')}$ and $(P^{i_{k+1}}(\omega'))_{\Omega^{i_1 \cdots i_{k+1}}(\omega)} \subseteq E_{\Omega^{i_1 \cdots i_{k+1}}(\omega)}$. Therefore,

$$E \preceq \bigwedge_{\omega' \in P^{i_k}(\dots(P^{i_1}(\omega)))} \Omega^{i_{k+1}}(\omega') = \Omega^{i_1 \cdots i_{k+1}}(\omega) \text{ and}$$

$$P^{i_{k+1}}(\dots(P^{i_1}(\omega))) = \bigcup_{\omega' \in P^{i_k}(\dots(P^{i_1}(\omega)))} \left(P^{i_{k+1}}(\omega')\right)_{\Omega^{i_1\cdots i_{k+1}}(\omega)} \subseteq E_{\Omega^{i_1\cdots i_{k+1}}(\omega)}.$$

□

Proof of Lemma 2. Take any sequence $i_1, \dots i_n$. By definition, $\Omega^\wedge(\omega) \preceq \Omega^{i_1\cdots i_n}(\omega)$ and since $P^{i_n}(\dots(P^{i_1}(\omega))) \subseteq \Omega^{i_1\cdots i_n}(\omega)$, we also have $P^{i_n}(\dots(P^{i_1}(\omega))) \subseteq \left(\Omega^\wedge(\omega)\right)_{\Omega^{i_1\cdots i_n}(\omega)}$. Applying Proposition 2 we have the result. For the second claim, suppose that $E \in \mathcal{E}$ is common knowledge at ω. By Proposition 2, $E \preceq \Omega^{i_1\cdots i_n}(\omega)$ for any sequence $i_1, \dots i_n$. Hence, $E \preceq \Omega^\wedge(\omega)$. □

Proof of Theorem 2. First, we prove the following Lemma.

Lemma 4. *Event E is common knowledge at ω.*

Proof. Using Proposition 2, we just need to show that for any sequence $i_1 \dots i_n$ of agents, $E \preceq \Omega^{i_1\cdots i_n}(\omega)$ and $P^{i_n}(\dots(P^{i_1}(\omega))) \subseteq E_{\Omega^{i_1\cdots i_n}(\omega)}$. The proof is by induction:

- For $n = 1$, since E is self evident for i_1 and from the proof of Property 1 we have $\omega \in E_S \subseteq \left(K^{i_1}_{S(E)}(E)\right)_S \subseteq K^{i_1}_S(E)$. Hence, $E \preceq \Omega^{i_1}(\omega)$ and $P^{i_1}(\omega) \subseteq E_{\Omega^{i_1}(\omega)}$.
- Suppose that for $n = k$, $E \preceq \Omega^{i_1\cdots i_k}(\omega)$ and $P^{i_k}(\dots(P^{i_1}(\omega))) \subseteq E_{\Omega^{i_1\cdots i_k}(\omega)}$.
- For $n = k+1$, we need to show that $E \preceq \Omega^{i_1\cdots i_{k+1}}(\omega)$ and $P^{i_{k+1}}(\dots(P^{i_1}(\omega))) \subseteq E_{\Omega^{i_1\cdots i_{k+1}}(\omega)}$. By definition,

$$P^{i_{k+1}}(\dots(P^{i_1}(\omega))) = \bigcup_{\omega' \in P^{i_k}(\dots(P^{i_1}(\omega)))} \left(P^{i_{k+1}}(\omega')\right)_{\Omega^{i_1\cdots i_{k+1}}(\omega)}.$$

From the induction hypothesis, for any $\omega' \in P^{i_k}(\dots(P^{i_1}(\omega)))$ we have

$$\omega' \in E_{\Omega^{i_1\cdots i_k}(\omega)} \subseteq \left(K^{i_{k+1}}_{S(E)}(E)\right)_{\Omega^{i_1\cdots i_k}(\omega)} \subseteq K^{i_{k+1}}_{\Omega^{i_1\cdots i_k}(\omega)}(E).$$

Hence, $E \preceq \Omega^{i_{k+1}}(\omega')$ and $P^{i_{k+1}}(\omega') \subseteq E_{\Omega^{i_{k+1}}(\omega')}$. Therefore, $E \preceq \Omega^{i_1\cdots i_{k+1}}(\omega)$, and $P^{i_{k+1}}(\dots(P^{i_1}(\omega))) \subseteq E_{\Omega^{i_1\cdots i_{k+1}}(\omega)}$.

□

Since $E^* \preceq E$ and $E_S \subseteq E^*_S$, we have that $E \subseteq E^*_{S(E)}$. Fix a sequence $i_1 \dots i_n$ of agents. From Generalized Monotonicity and the fact that $E^* \preceq E \preceq \Omega^{i_1\cdots i_{n-1}}(\omega)$ we have $K^{i_n}_{\Omega^{i_1\cdots i_{n-1}}(\omega)}(E) \subseteq K^{i_n}_{\Omega^{i_1\cdots i_{n-1}}(\omega)}(E^*_{S(E)}) \subseteq K^{i_n}_{\Omega^{i_1\cdots i_{n-1}}(\omega)}(E^*)$. By applying Generalized Monotonicity recursively we have that $K^{i_1}_S(\dots(K^{i_n}_{\Omega^{i_1\cdots i_{n-1}}(\omega)}(E))) \subseteq K^{i_1}_S(\dots(K^{i_n}_{\Omega^{i_1\cdots i_{n-1}}(\omega)}(E^*)))$. Therefore, $\omega \in K^{i_1}_S(\dots(K^{i_n}_{\Omega^{i_1\cdots i_{n-1}}(\omega)}(E^*)))$ and since this holds for all sequences $i_1, \dots, i_n$, E^* is common knowledge at ω.

□

Lemma 5. *Fix a sequence $j_i, \dots, j_l$ and let $S = \Omega^{j_1 \cdots j_l}(\omega)$. Then, for any sequence $i_1, \dots i_n$, $\Omega^{j_1 \cdots j_l i_i \cdots i_n}(\omega) \preceq \Omega^{i_i \cdots i_n}(\omega_S)$ and $P^{i_n} \dots P^{i_1}(\omega_S) \subseteq (P^{i_n} \dots P^{i_1} P^{j_l} \dots P^{j_1}(\omega))_{\Omega^{i_i \cdots i_n}(\omega_S)}$.*

Proof. For $n = 1$, by Reflexivity we have that $\omega_S \in P^{j_l} \dots P^{j_1}(\omega)$. Hence, $P^{i_1}(\omega_S) \subseteq (P^{i_1} P^{j_l} \dots P^{j_1}(\omega))_{\Omega^{i_1}(\omega_S)}$ and $\Omega^{j_1 \cdots j_l i_1}(\omega) \preceq \Omega^{i_1}(\omega_S)$. Suppose that for $n = k$ we have that

$$P^{i_k} \dots P^{i_1}(\omega_S) \subseteq (P^{i_k} \dots P^{i_1} P^{j_l} \dots P^{j_1}(\omega))_{\Omega^{i_1 \cdots i_k}(\omega_S)},$$

$$\Omega^{j_1 \cdots j_l i_1 \dots i_k}(\omega) \preceq \Omega^{i_1 \dots i_k}(\omega_S).$$

Recall that,

$$P^{i_{k+1}} \dots P^{i_1} P^{j_l} \dots P^{j_1}(\omega) = \bigcup_{\omega' \in P^{i_k} \dots P^{i_1} P^{j_l} \dots P^{j_1}(\omega)} (P^{i_{k+1}}(\omega'))_{\Omega^{j_1 \cdots j_l i_1 \cdots i_{k+1}}(\omega)},$$

$$P^{i_{k+1}} \dots P^{i_1}(\omega_S) = \bigcup_{\omega' \in P^{i_k} \dots P^{i_1}(\omega_S)} (P^{i_{k+1}}(\omega'))_{\Omega^{i_1 \cdots i_{k+1}}(\omega_S)}.$$

Take $\omega' \in P^{i_k} \dots P^{i_1}(\omega_S)$. From the induction hypothesis we have that $\omega'_{\Omega^{j_1 \cdots j_l i_1 \cdots i_k}(\omega)} \in P^{i_k} \dots P^{i_1} P^{j_l} \dots P^{j_1}(\omega)$, which implies (from PPI) that $\Omega^{i_{k+1}}(\omega'_{\Omega^{j_1 \cdots j_l i_1 \cdots i_k}(\omega)}) \preceq \Omega^{i_{k+1}}(\omega')$. Since this holds for all such ω', we have,

$$\Omega^{j_1 \dots j_l i_1 \dots i_{k+1}}(\omega) \preceq \Omega^{i_1 \dots i_{k+1}}(\omega_S),$$

$$(P^{i_{k+1}}(\omega'))_{\Omega^{i_1 \cdots i_{k+1}}(\omega_S)} \subseteq (P^{i_{k+1}}(\omega'_{\Omega^{j_1 \cdots j_l i_1 \cdots i_k}(\omega)}))_{\Omega^{i_1 \cdots i_{k+1}}(\omega_S)},$$

which implies,

$$P^{i_{k+1}} \dots P^{i_1}(\omega_S) \subseteq (P^{i_k} \dots P^{i_1} P^{j_l} \dots P^{j_1}(\omega))_{\Omega^{i_1 \cdots i_{k+1}}(\omega_S)}.$$

□

Lemma 6. *PPK implies that knowledge of E is continuous at ω, for any E and any ω.*

Proof. Let $S = \Omega^{j_1 \cdots j_l}(\omega)$. For $n = 1$, we will show that $\Omega^{i_1}(\omega_{\Omega^\wedge(\omega)}) = \Omega^\wedge(\omega)$, $\omega_{\Omega^\wedge(\omega)} \in K^{i_1}_{\Omega^\wedge(\omega)}(E)$ and $P^{i_1}(\omega_{\Omega^\wedge(\omega)}) \subseteq (P^{i_1}(\omega_S))_{\Omega^\wedge(\omega)}$. From Lemma 5 we have that $E \preceq \Omega^\wedge(\omega) \preceq \Omega^{j_i \cdots j_l i_1}(\omega) \preceq \Omega^{i_1}(\omega_S)$. We know that $\omega_S \in K^{i_1}_S(E)$, which implies $P^{i_1}(\omega_S) \subseteq E_{\Omega^{i_1}(\omega_S)}$. PPK implies that $P^{i_1}(\omega_{\Omega^\wedge(\omega)}) = (P^{i_1}(\omega_S))_{\Omega^\wedge(\omega)} \subseteq E_{\Omega^\wedge(\omega)}$. But then, $\Omega^{i_1}(\omega_{\Omega^\wedge(\omega)}) = \Omega^\wedge(\omega)$ and $\omega_{\Omega^\wedge(\omega)} \in K^{i_1}_{\Omega^\wedge(\omega)}(E)$. For $n = k$, suppose that $\Omega^{i_1 \dots i_k}(\omega_{\Omega^\wedge(\omega)}) = \Omega^\wedge(\omega)$, $\omega_{\Omega^\wedge(\omega)} \in K^{i_1}_{\Omega^\wedge(\omega)} \dots K^{i_k}_{\Omega^\wedge(\omega)}(E)$ and $P^{i_k} \dots P^{i_1}(\omega_{\Omega^\wedge(\omega)}) \subseteq (P^{i_k} \dots P^{i_1}(\omega_S))_{\Omega^\wedge(\omega)}$. Let $\omega' \in P^{i_k} \dots P^{i_1}(\omega_{\Omega^\wedge(\omega)})$. Then, there is ω'' such that $\omega''_{\Omega^\wedge(\omega)} = \omega'$ and $\omega'' \in P^{i_k} \dots P^{i_1}(\omega_S)$. Because $\omega_S \in K^{i_1}_S \dots K^{i_{k+1}}_{\Omega^{i_1 \cdots i_k}(\omega_S)}(E)$, we have that $(P^{i_{k+1}}(\omega''))_{\Omega^{i_1 \cdots i_{k+1}}(\omega_S)} \subseteq E_{\Omega^{i_1 \cdots i_{k+1}}(\omega_S)}$. Because $\Omega^{i_{k+1}}(\omega') \preceq \Omega^{i_1 \dots i_{k+1}}(\omega_S) \preceq \Omega^{i_{k+1}}(\omega'')$, PPK implies $P^{i_{k+1}}(\omega') \subseteq (P^{i_{k+1}}(\omega''))_{\Omega^\wedge(\omega)} \subseteq E_{\Omega^\wedge(\omega)}$ and $\Omega^{i_{k+1}}(\omega') = \Omega^\wedge(\omega)$. Hence, $\Omega^{i_1 \dots i_{k+1}}(\omega_{\Omega^\wedge(\omega)}) = \Omega^\wedge(\omega)$, $\omega_{\Omega^\wedge(\omega)} \in K^{i_1}_{\Omega^\wedge(\omega)} \dots K^{i_{k+1}}_{\Omega^\wedge(\omega)}(E)$ and $P^{i_{k+1}} \dots P^{i_1}(\omega_{\Omega^\wedge(\omega)}) \subseteq (P^{i_{k+1}} \dots P^{i_1}(\omega_S))_{\Omega^\wedge(\omega)}$.

□

Proof of Theorem 3. Suppose that knowledge of E^* is continuous at ω. Since E^* is common knowledge at ω we have that for any sequence $i_1 \dots i_n$, $E^* \preceq \Omega^{i_1 \cdots i_n}(\omega)$. Therefore, $E^* \preceq \Omega^\wedge(\omega)$ and the following event is well defined:

$$E = \bigcap_{\substack{i_1,\dots,i_n \\ n \in \mathbb{N}}} K^{i_1}_{\Omega^\wedge(\omega)} K^{i_2}_{\Omega^\wedge(\omega)} \dots K^{i_n}_{\Omega^\wedge(\omega)}(E^*).$$

Since $\Omega^\wedge(\omega) \preceq E$, we have that $E^* \preceq E$. It remains to show that $\omega \in E_S \subseteq E^*_S$ and that E is a public event.

- $\omega \in E_S$

  Fix a sequence $i_1, \dots, i_n$ and a state space $S' \in S_\omega$, where $S' = \Omega^{j_1 \cdots j_k}(\omega)$. Since E^* is common knowledge at ω, we have that $\omega \in K^{j_1}_S \dots K^{j_k}_{\Omega^{j_1 \cdots j_{k-1}}(\omega)} K^{i_1}_{\Omega^{j_1 \cdots j_k}(\omega)} \dots K^{i_n}_{\Omega^{j_1 \cdots j_k i_i \cdots i_{n-1}}(\omega)}(E^*)$. From Lemma 5 and Proposition 3 we have that $\omega_{S'} \in K^{i_1}_{S'} \dots K^{i_n}_{\Omega^{i_1 \cdots i_n}(\omega_{S'})}(E^*)$. Since this is true for any sequence $i_1, \dots, i_n$ and any $S' \in S_\omega$, and knowledge of E^* is continuous at ω we have that $\omega_{\Omega^\wedge(\omega)} \in K^{i_1}_{\Omega^\wedge(\omega)} \dots K^{i_n}_{\Omega^\wedge(\omega)}(E)$. Hence, we have $\omega_{\Omega^\wedge(\omega)} \in E$, which implies $\omega \in E_S$.

- $E_S \subseteq E^*_S$

  Take any sequence $i_1 \dots i_n$. The Axiom of Knowledge implies that $K^{i_n}_{\Omega^\wedge(\omega)}(E^*) \subseteq E^*_{\Omega^\wedge(\omega)}$. From Generalized Monotonicity and the Axiom of Knowledge we have that $K^{i_{n-1}}_{\Omega^\wedge(\omega)} K^{i_n}_{\Omega^\wedge(\omega)}(E^*) \subseteq K^{i_{n-1}}_{\Omega^\wedge(\omega)}(E^*_{\Omega^\wedge(\omega)}) \subseteq K^{i_{n-1}}_{\Omega^\wedge(\omega)}(E^*) \subseteq E^*_{\Omega^\wedge(\omega)}$. Continuing recursively we have that $K^{i_1}_{\Omega^\wedge(\omega)} K^{i_2}_{\Omega^\wedge(\omega)} \dots K^{i_n}_{\Omega^\wedge(\omega)}(E^*) \subseteq E^*_{\Omega^\wedge(\omega)}$. Since this holds for any sequence $i_1 \dots i_n$, we have that $E \subseteq E^*_{\Omega^\wedge(\omega)}$. Hence, $E_S \subseteq E^*_S$.

- E is a public event.

  By Conjunction,

$$K^i_{\Omega^\wedge(\omega)}(E) = \bigcap_{\substack{i_1 \dots i_n \\ n \in \mathbb{N}}} K^i_{\Omega^\wedge(\omega)} K^{i_1}_{\Omega^\wedge(\omega)} K^{i_2}_{\Omega^\wedge(\omega)} \dots K^{i_n}_{\Omega^\wedge(\omega)}(E^*) = E.$$

Suppose that E^* is common knowledge at $\omega \in S$ and there exists a public event $E \subseteq S'$ such that $E^* \preceq E \preceq S$, $\omega \in E_S \subseteq E^*_S$. Generalized monotonicity implies that for any sequence $i_1, \dots, i_n$ we have $\omega_{S'} \in E \subseteq K^{i_1}_{S'} \dots K^{i_n}_{S'}(E)$. We want to show that $K^{i_1}_{S'} \dots K^{i_n}_{S'}(E) \subseteq (K^{i_1}_{\Omega^\wedge(\omega)} \dots K^{i_n}_{\Omega^\wedge(\omega)}(E))_{S'}$.[21] For $n = 1$ this is true because of the property ALTK. Suppose that for $n = k$ we have $K^{i_1}_{S'} \dots K^{i_k}_{S'}(E) \subseteq (K^{i_1}_{\Omega^\wedge(\omega)} \dots K^{i_k}_{\Omega^\wedge(\omega)}(E))_{S'}$. Since $K^{i_{k+1}}_{S'}(E) \subseteq (K^{i_{k+1}}_{\Omega^\wedge(\omega)}(E))_{S'}$ we also have,

$$K^{i_1}_{S'} \dots K^{i_k}_{S'} K^{i_{k+1}}_{S'}(E) \subseteq (K^{i_1}_{\Omega^\wedge(\omega)} \dots K^{i_k}_{\Omega^\wedge(\omega)} K^{i_{k+1}}_{S'}(E))_{S'} \subseteq (K^{i_1}_{\Omega^\wedge(\omega)} \dots K^{i_k}_{\Omega^\wedge(\omega)} (K^{i_{k+1}}_{\Omega^\wedge(\omega)}(E))_{S'})_{S'}.$$

[21] Note that since E is common knowledge at ω, from Lemma 2 we have $S' \preceq \Omega^\wedge(\omega)$.

From the definition of knowledge, $K^{i_k}_{\Omega^\wedge(\omega)}((K^{i_{k+1}}_{\Omega^\wedge(\omega)}(E))_{S'}) \subseteq K^{i_k}_{\Omega^\wedge(\omega)}K^{i_{k+1}}_{\Omega^\wedge(\omega)}(E)$. Hence, we have that $K^{i_1}_{S'} \dots K^{i_{k+1}}_{S'}(E) \subseteq (K^{i_1}_{\Omega^\wedge(\omega)} \dots K^{i_{k+1}}_{\Omega^\wedge(\omega)}(E))_{S'}$. This implies that for any sequence $i_1, \dots, i_n$, we have $\omega_{\Omega^\wedge(\omega)} \in (K^{i_1}_{\Omega^\wedge(\omega)} \dots K^{i_n}_{\Omega^\wedge(\omega)}(E))_{S'}$

Since $E_S \subseteq E^*_S$ and $E^* \preceq E$ we also have that for any sequence $i_1, \dots, i_n$, $K^{i_1}_{\Omega^\wedge(\omega)} \dots K^{i_n}_{\Omega^\wedge(\omega)}(E) \subseteq K^{i_1}_{\Omega^\wedge(\omega)} \dots K^{i_n}_{\Omega^\wedge(\omega)}(E^*)$. Hence, for any sequence $i_1, \dots, i_n$ we have $\omega_{\Omega^\wedge(\omega)} \in K^{i_1}_{\Omega^\wedge(\omega)} \dots K^{i_n}_{\Omega^\wedge(\omega)}(E^*)$ and knowledge of E^* is continuous at ω.

□

Proof of Theorem 4. From Theorem 3, there exists a public event E' such that $E^* \preceq E'$ and $\omega \in E'_S \subseteq E^*_S$. Its proof also shows that $E' \subseteq \Omega^\wedge(\omega)$, which implies that $E' \subseteq E^*$. We need to show that $E' = \bigcup_{\omega \in E'} P^i(\omega)$. Generalized Reflexivity implies $E' \subseteq \bigcup_{\omega \in E'} P^i(\omega)$. For the opposite direction, since E' is a public event, $\omega \in E'$ implies $P^i(\omega) \subseteq E'$. Therefore, $E' = \bigcup_{\omega \in E'} P^i(\omega)$, and by symmetry $E' = \bigcup_{\omega \in E'} P^j(\omega)$.

The next step is to show that E' is partitioned by P^i. First, since E' is public, for any $\omega' \in E'$, $\Omega^\wedge(\omega) \preceq \Omega^i(\omega') \preceq \Omega^\wedge(\omega)$. Generalized Reflexivity and Stationarity imply that if $\omega', \omega'' \in E'$ then either $P^i(\omega') = P^i(\omega'')$ or $P^i(\omega') \cap P^i(\omega'') = \emptyset$. The rest of the proof is identical to that of Aumann (1976).

Agent i's posterior at $\omega' \in E'$ is

$$q^i(\omega') = \frac{\mu(P^i(\omega') \cap E)}{\mu(P^i(\omega'))}.$$

Since $q^i(\omega') = q^i$ for all $\omega' \in E'$ we can sum over the disjoint partition cells of E' and derive $\mu(E')q^i = \mu(E' \cap E)$. Similarly for agent j we have $\mu(E')q^j = \mu(E' \cap E)$ and therefore $q^i = q^j$.

□

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