A Simple Boundary Model of Dynamic Covariance

Julian Williams
Christos Ioannidis
School of Management
University of Bath, Claverton Down, BA2 7AY

First Draft Jan 2007

Introduction and Literature

- The first model of this type was introduced in Bollerslev, Engle and Woodridge (1988) this algorithm utilizes the *vech* matrix operator and specifies the model dynamics as a V-ARMA type process.
- From this foundation a myriad of potential dynamic models have been proposed, from a very basic constant correlation model such as that proposes in Bollerslev (1990),
- Factor models, Alexander (2001),
- Dynamic correlations Engle (2000), Engle and Sheppard (2002).
- Two commonly utilized specifications of the conditional covariance process are the MARCH model of Engle and Kim (2002) and the BEKK model of Engle and Kroner (1994).

Aims and Objectives

- In this article we provide some stylized evidence to suggest that modelling the time evolution of the conditional covariance matrix for equity returns as a matrix autoregressive process does no fully reflect the dynamics of the system.
- Furthermore the distribution of the largest eigenvalues of the instantaneous covariance matrix suggest that a finite mixture type model would be more appropriate. We then present a simple model of regime switching covariance, which is tractable even for very high-variate systems.

Some Observable Empirical Evidence

- Consider the following simple model, where $\mathbf{y}_t$ is a vector of returns from the S&P 500 cross section, a simple univariate ARX(p) model for first stage filtration is,

$$y_{i.t} = f\left(y_{i,t-1}, y_{i,t-2}, ..., y_{i,t-p}\right) + g\left(x_t, z_t\right) + u_{i,t} \tag{0.0}$$

- where $u_{i,t} \in \mathbf{u}_t$ and $\mathbf{u}_t$ is an n length column vector of residuals and x_t is the market return at time t and z_t an appropriate risk free rate and $t \in [1, ..., \tau]$.
- The estimated residuals $\hat{\mathbf{u}}_t$ from each model are collected, forming the data matrix $\mathbf{U} = \left[\mathbf{u}_{t=1}, \mathbf{u}_{t=2}, ..., \mathbf{u}_{t=\tau}\right]^{\mathrm{T}}$, the estimate of the unconditional covariance matrix is therefore $\hat{\mathbf{\Sigma}} = \frac{1}{\tau}\mathbf{U}^{\mathrm{T}}\mathbf{U}$.

Eigenvalues

From figure 1 we observe that there are substantial differences (of order of magnitude >10) between the eigenvalues of the unconditional covariance matrix, indicating that the matrix is non-sparse and that the structure is dominated by the last few largest eigenvalues.

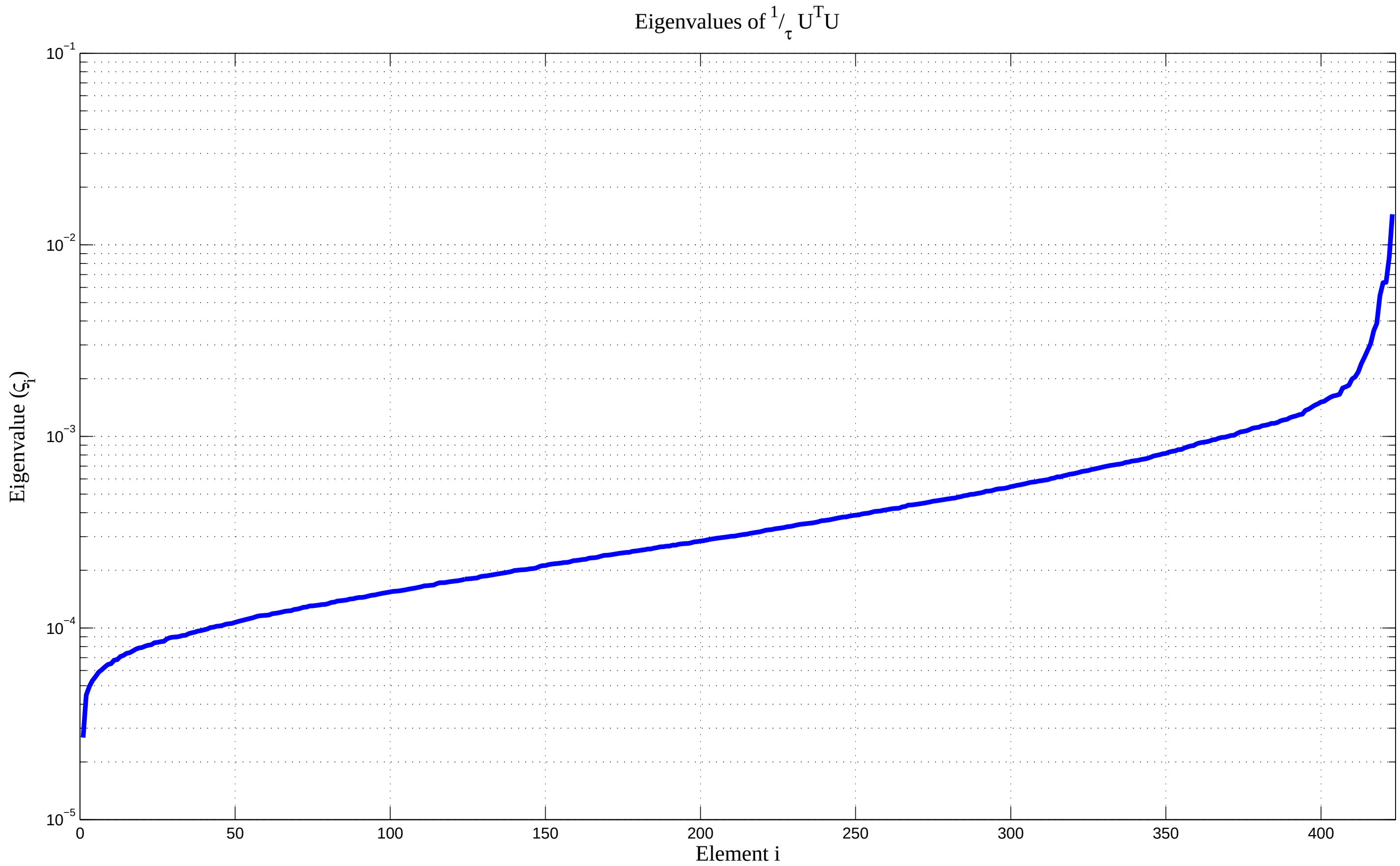

FIGURE 1. A plot of the Eigenvalues of the sample covariance matrix, $\frac{1}{\tau}\mathbf{U}^{\mathrm{T}}\mathbf{U}$, from 423 firms out of the S&P 500, for daily data over 20 years.

Time Evolution of the Eigenvalues of the Instantaneous Quadratic Covariation $u_t u_t^{\mathrm{T}}$

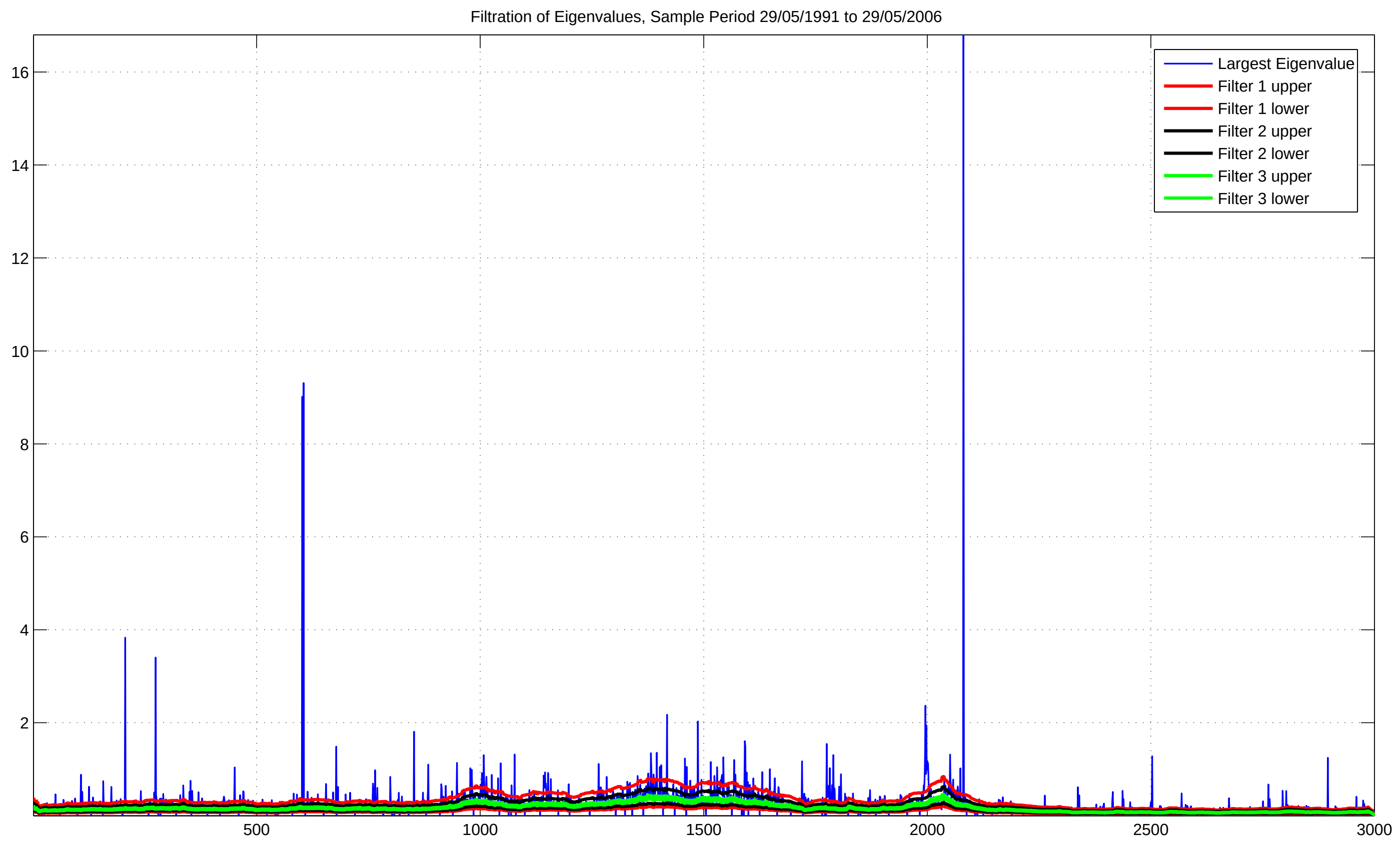

FIGURE 2. Evolution of the Eigenvalues of $\mathbf{u}_t \mathbf{u}_t^{\mathrm{T}}$, over 3000 days, original stock price data source: DataStream$^{\mathrm{TM}}$.

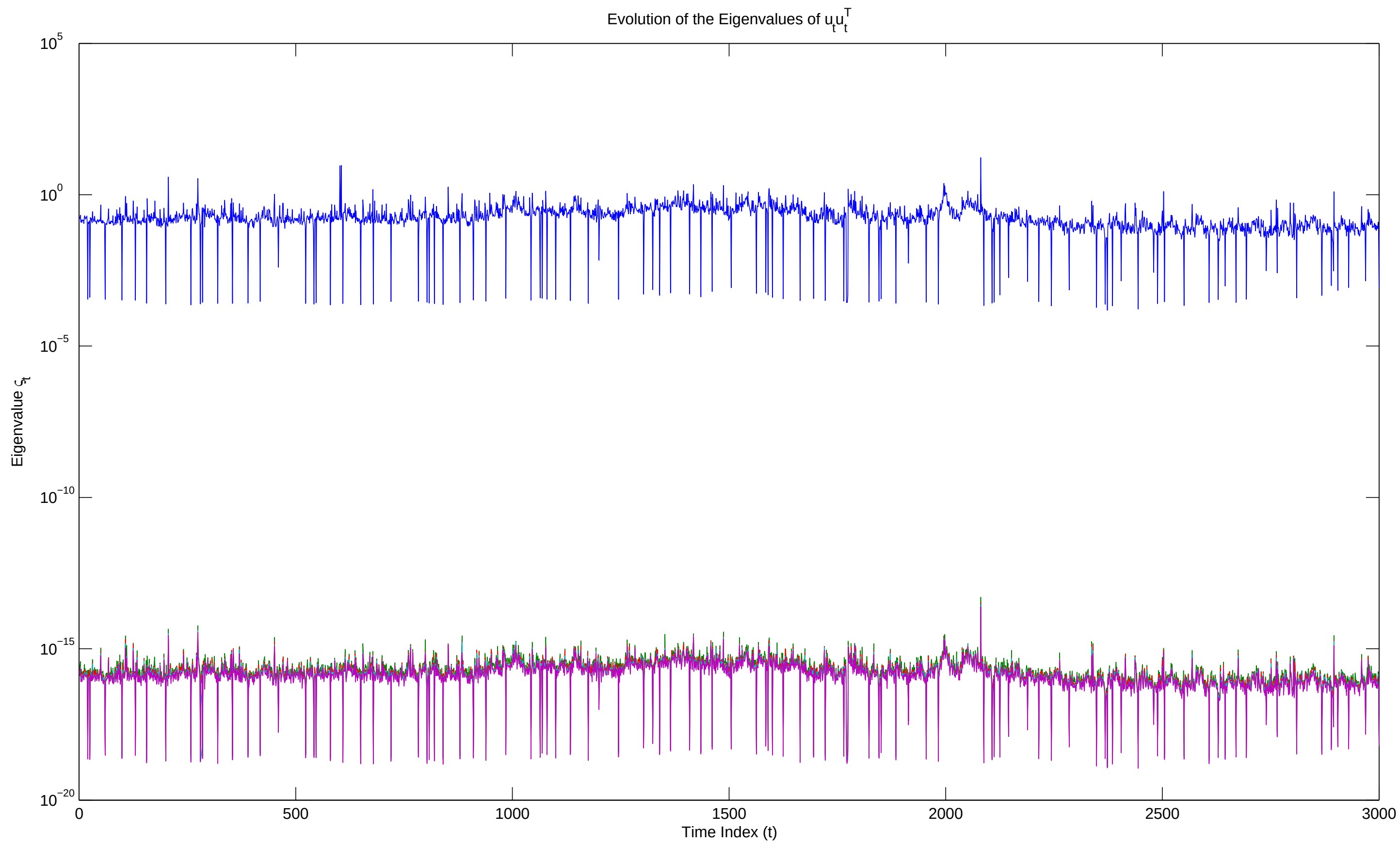

FIGURE 3. Evolution of the Log Eigenvalues of $\mathbf{u}_t\mathbf{u}_t^{\mathrm{T}}$, over 3000 days, original stock price data source: DataStream$^{\mathrm{TM}}$.

- In figure 3 we plot the time evolution of the largest eigenvalue and the set of the remaining ones of $\mathbf{u_t}\mathbf{u_t}^{\mathrm{T}}$.
- Visual inspection suggests that the distribution of the largest eigenvalue does not appear matrix normal, and it fluctuates between two states 'high' and low' although the frequency at which it resides in each state varies considerably between the two.

Distribution of Eigenvalues

To supplement this preliminary evidence we proceed by comparing the sample distribution of the largest eigenvalue to the a distribution generated via simulation.

$$\hat{\mathbf{U}} = \mathbf{E}\left(\hat{\mathbf{\Sigma}}^{\frac{1}{2}}\right)^{\mathrm{T}} \tag{0.0}$$

where,

$$\mathbf{E} = \left[\varepsilon_{t=1}^{\mathrm{T}}, \varepsilon_{t=2}^{\mathrm{T}}, ..., \varepsilon_{t=\eta}^{\mathrm{T}}\right]^{\mathrm{T}} \tag{0.1}$$

$$\varepsilon_t \sim N\left(\mathbf{0}, \mathbf{I}\right) \tag{0.2}$$

and we present them in figures 4 and 5.

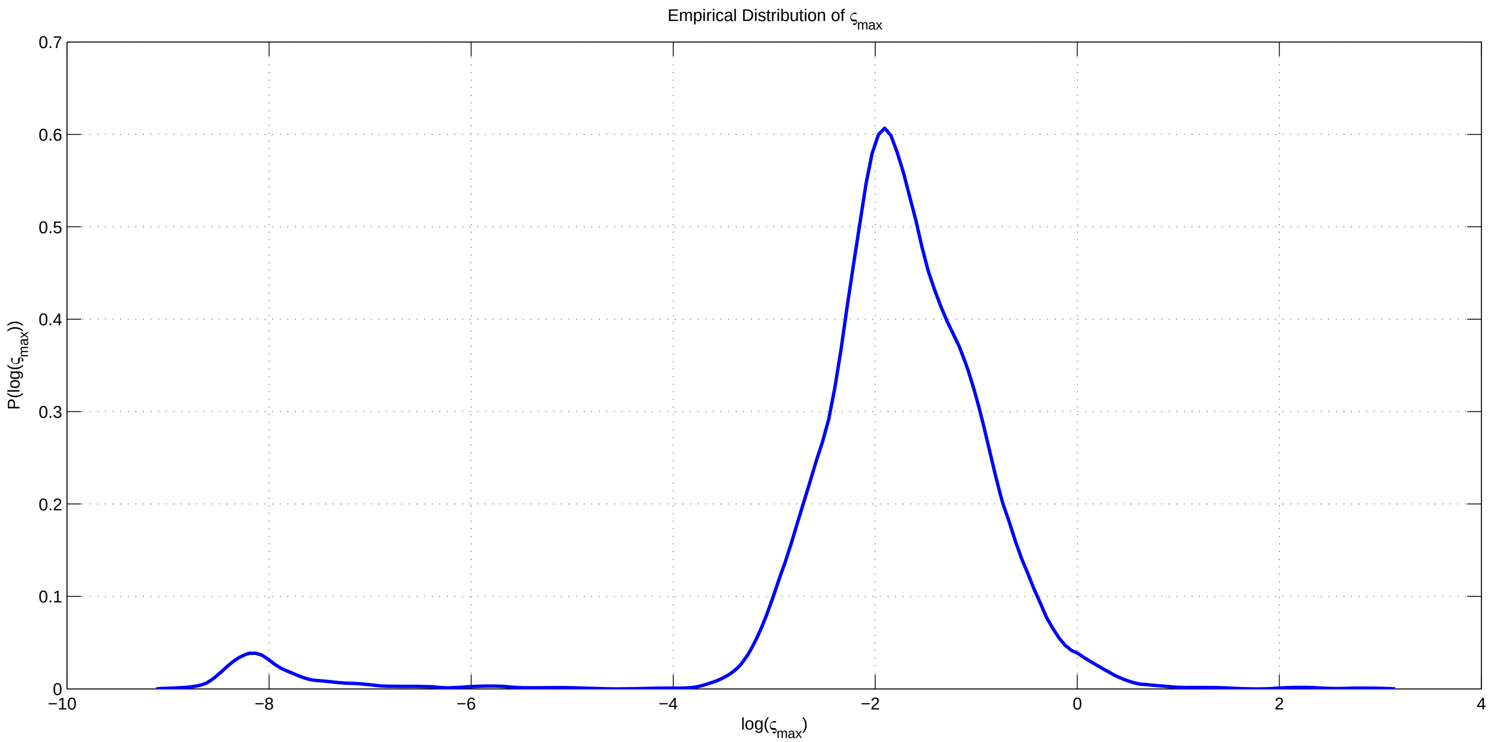

FIGURE 4. Empirical distribution of the largest eigenvalue, φ_{max}, of $\mathbf{u}_t\mathbf{u}_t^{\mathrm{T}}$

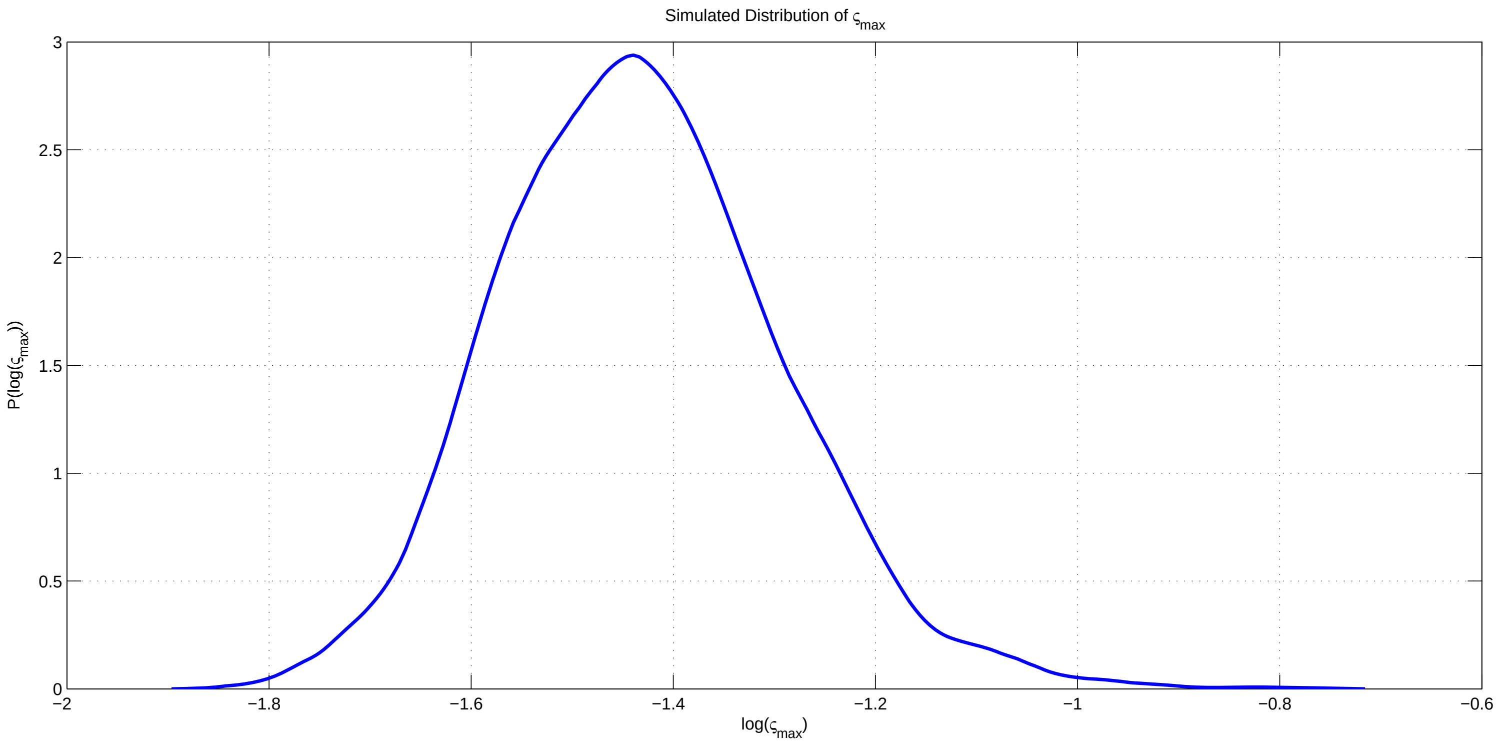

FIGURE 5. Simulated distribution of the largest eigenvalue of $\hat{\mathbf{u}}_t\hat{\mathbf{u}}_t^{\mathrm{T}}$, where $\mathbf{u}_t \sim N\left(\mathbf{0}, \hat{\boldsymbol{\Sigma}}\right)$. It is immediately apparent that for large covariance matrices the distribution of the largest eigenvalue for draws from a zero entered multi-normal distribution are far more tightly distributed than the distribution observed from 20 years of data from the S&P 500.

Inference

- Empirical evidence from filtered equity return data suggests that the multivariate distribution of the residuals from the first stage filtration rejects the unconditional normality assumption.
- A possible solution is to model $\mathbf{u}_t \sim N(\mathbf{0}, \mathbf{\Sigma}_t)$, where $\mathbf{\Sigma}_t$, is a matrix process and that this process is conditioned on some information set $\mathcal{I}_t$ and as such,

$$\mathbf{\Sigma}_t = \Psi\left(\mathcal{I}_{t|t-1,t-2,\ldots,t=1}\right) \tag{0.2}$$

- where Ψ is a matrix function operating over $\mathcal{I}_t$, that yields the non-negative definite matrix $\mathbf{\Sigma}_t$. In general this is normally a matricized vector process, alternative specifications treat the process as a full Matrix Autoregressive process.

The Boundary Model

As an alternative to the matrix autoregressive approach we suggest a regime switching model, whereby the dynamics are essentially driven by a single underlying process, $\phi_t \in \mathbb{R}$, which manifests itself via a process $\psi_t \in [0,1]$, that in effect bounces the system between two boundary states. Consider a simple multivariate linear filtration model, with n-dependent variables and m-explanatory variables,

$$\underset{n\times 1}{\mathbf{y}_t} = \underset{n\times m}{\mathbf{\Pi}^{\mathrm{T}}} \underset{m\times 1}{\mathbf{x}_t} + \underset{n\times 1}{\mathbf{u}_t} \tag{0.2}$$

Where the disturbances $\mathbf{u}_t$, are drawn from,

$$\mathbf{u}_t = \mathbf{\Sigma}_t^{\frac{1}{2}} \varepsilon_t \tag{0.3}$$

$$\varepsilon_t \sim N\left(\mathbf{0}, \underset{n\times n}{\mathbf{I}}\right) \tag{0.4}$$

Here, $\mathbf{\Sigma}_t$ is a matrix process and $\mathbf{\Sigma}_t^{\frac{1}{2}}$, is the Cholesky or upper triangular factorization, where $\mathbf{\Sigma}_t \in \mathbb{C}^{n\times n}$ and is defined as non-negative definite Hermitian, i.e. symmetric with non-zero eigenvalues.

Modelling the Covariance States

One method of modelling the covariance states is to have a regime switching vector, for example,

$$\boldsymbol{\Sigma}_t = \boldsymbol{\Psi}_t \cdot \boldsymbol{\Sigma}_u + \left(\mathbf{e}\mathbf{e}^{\mathrm{T}} - \boldsymbol{\Psi}_t\right) \cdot \boldsymbol{\Sigma}_d \tag{0.5}$$

$$\boldsymbol{\Psi}_t = \psi_t \psi_t^{\mathrm{T}} \tag{0.6}$$

However this specification cannot guarantee that the resultant conditional covariance matrix $\boldsymbol{\Sigma}_t$, will be non-negative definite.

A very simple decomposition of $\boldsymbol{\Sigma}_t$ suggests a second moment model with two boundary states, designated by the subscripts u and d,

$$\boldsymbol{\Sigma}_t = \psi_t \boldsymbol{\Sigma}_u + (1 - \psi_t)\, \boldsymbol{\Sigma}_d \tag{0.6}$$

Where the boundary matrices $\boldsymbol{\Sigma}_u$ and $\boldsymbol{\Sigma}_d$ are both non-negative definite Hermitian matrices and $\mathbf{a}^{\mathrm{T}} \boldsymbol{\Sigma}_u \mathbf{a} > \mathbf{a}^{\mathrm{T}} \boldsymbol{\Sigma}_d \mathbf{a}$, where $\mathbf{a} \in \mathbb{R}^n$.

Modelling The Conditional State of the System

- The scalar process ψ_t is some form of dynamic process constrained to the unit field.
- Now consider a function $\xi\left(\phi_t; \omega\right)$ with parameter set ω, where ϕ_t is some d dimensional process such as $\phi_t \in \mathbb{R}^d$.
- To be a valid switching function, the following must hold,

$$\xi : \mathbb{R}^d \longrightarrow [0, 1]$$

,

- For example if the dimension of the state, $d = 1$, the logistic specification is a useful transition function,

$$\xi\left(\phi_t; \alpha, \beta, \delta\right) = \left(1 + \exp\left(-\alpha\left(\phi_t + \beta\right)^{\delta}\right)\right) \tag{0.6}$$

Modelling ϕ_t

For this example we choose the following autoregressive quadratic form,

$$\phi_t = \sum_{i=1}^{p} \lambda_i^{\mathrm{T}} \left(\mathbf{u}_{t-i} \mathbf{u}_{t-i}^{\mathrm{T}} \right) \lambda_i + \sum_{j=1}^{q} \gamma_j^{\mathrm{T}} \mathbf{\Sigma}_{t-j} \gamma_j \tag{0.6}$$

Where, λ_i and γ_i are parameters vectors. The model parameter vector θ is therefore defined as follows,

$$\begin{aligned} \mathbf{\Lambda} &= [\lambda_1, ..., \lambda_p] && (0.7) \\ \mathbf{\Gamma} &= [\gamma_1, ..., \gamma_q] && (0.8) \\ \theta &= \left[\alpha, \beta, \delta, (vec\mathbf{\Lambda})^{\mathrm{T}}, (vec\mathbf{\Gamma})^{\mathrm{T}} \right]^{\mathrm{T}} && (0.9) \end{aligned}$$

Parameters

The parameters domains are as follows,

$$
\begin{aligned}
\delta &\in \mathbb{N}_+ \\
\beta &\in \mathbb{R} \\
\alpha &\in \mathbb{R} \\
\lambda_i &\in \mathbb{R}_+^{(n\times 1)} \\
\gamma_i &\in \mathbb{R}_+^{(n\times 1)} \\
\mathbf{\Sigma}_u &\in \mathbb{C}^{n\times n} \\
\mathbf{\Sigma}_d &\in \mathbb{C}^{n\times n}
\end{aligned} \tag{0.10}
$$

Where $\mathbb{C}^{n\times n}$ is the set of all non-negative $n \times n$ definite hermitian matrices and $\mathbb{N}_+$, is the set of positive natural integers.

Model Properties

- The model offers an extremely simple representation of dynamic covariation and several attractive properties are immediately apparent.
- The first interesting aspect is that $\mathbf{\Sigma}_t$, will always be non-negative hermitian if the boundary matrices $\mathbf{\Sigma}_u$ and $\mathbf{\Sigma}_d$ are non-negative definite.
- The model can capture smooth adjustments between the boundary matrices $\mathbf{\Sigma}_u$ and $\mathbf{\Sigma}_d$, or as a straightforward switching model.
- The model has two main operational modes, first when α is very large, either by parameterization or through *a priori* specification the model has the effect of generating rapid switches between the boundary matrices.
- The alternative specification restricts the size of α and the conditional covariance is characterized by a continuum of matrices between $\mathbf{\Sigma}_u$ and $\mathbf{\Sigma}_d$, the speed of this adjustment is dependent on all the other parameters, $\theta = \left[\lambda^{\mathrm{T}}, \gamma^{\mathrm{T}}, \alpha, \beta, \delta\right]^{\mathrm{T}}$.

Extensions to the Basic Model

The model is extensible to encompass correlation dynamics by separating the volatility and correlation dynamics, utilizing the standard decomposition.

$$\mathbf{\Sigma}_t = \mathbf{R}_t \circ \mathbf{H}_t \tag{0.4}$$

The boundary covariance matrices maybe estimated in the same manner as previously, the volatility (driven by the diagonal elements) and the correlation (driven by the off-diagonal elements), are then modelled separately, as follows, first decompose the boundary matrices, using the standard notation,

$$
\begin{aligned}
&\boldsymbol{\Sigma}_u = \mathbf{R}_u \circ \mathbf{H}_u && \boldsymbol{\Sigma}_d = \mathbf{R}_d \circ \mathbf{H}_d \\
&\boldsymbol{\Sigma}_u = \begin{cases} \left[\sigma^2_{i,u}\right]_{i=j} \\ \left[\sigma_{i,j,u}\right]_{i\neq j} \end{cases} && \boldsymbol{\Sigma}_d = \begin{cases} \left[\sigma^2_{i,d}\right]_{i=j} \\ \left[\sigma_{i,j,d}\right]_{i\neq j} \end{cases} \\
&\mathbf{R}_u = \begin{cases} \left[\rho_{i,j,u}\right]_{i=j} = 1 \\ \left[\rho_{i,j,u}\right]_{i\neq j} = \frac{\sigma_{i,j,u}}{\sigma_{i,u}\sigma_{j,u}} \end{cases} && \mathbf{R}_d = \begin{cases} \left[\rho_{i,j,d}\right]_{i=j} = 1 \\ \left[\rho_{i,j,d}\right]_{i\neq j} = \frac{\sigma_{i,j,d}}{\sigma_{i,d}\sigma_{j,d}} \end{cases} \\
&\mathbf{H}_u = \begin{cases} \left[h_{i,j,u}\right]_{i=j} = \sigma^2_{i,u} \\ \left[h_{i,j,u}\right]_{i\neq j} = \sigma_{i,u}\sigma_{j,u} \end{cases} && \mathbf{H}_d = \begin{cases} \left[h_{i,j,d}\right]_{i=j} = \sigma^2_{i,d} \\ \left[h_{i,j,d}\right]_{i\neq j} = \sigma_{i,d}\sigma_{j,d} \end{cases}
\end{aligned}
\tag{0.4}
$$

then specify two separate dynamics, $\psi_{\mathbf{R},t}$ and $\psi_{\mathbf{H},t}$, for the correlation and volatility processes respectively. For simplicity if only the first order $p = 1, q = 1$ version is specified we obtain,

$$\psi_{\mathbf{R},t} = \xi\left(\phi_{\mathbf{R},t}\,|\alpha_{\mathbf{R}},\beta_{\mathbf{R}},\delta_{\mathbf{R}}\right) \tag{0.5}$$

$$\psi_{\mathbf{H},t} = \xi\left(\phi_{\mathbf{H},t}\,|\alpha_{\mathbf{H}},\beta_{\mathbf{H}},\delta_{\mathbf{H}}\right) \tag{0.6}$$

$$\phi_{\mathbf{R},t} = \lambda_{\mathbf{R}}^{\mathrm{T}}\epsilon_{t-1}\epsilon_{t-1}^{\mathrm{T}}\lambda_{\mathbf{R}} + \gamma_{\mathbf{R}}^{\mathrm{T}}\mathbf{R}_{t-1}\gamma_{\mathbf{R}} \tag{0.7}$$

$$\phi_{\mathbf{H},t} = \lambda_{\mathbf{R}}^{\mathrm{T}}\mathbf{u}_{t-1}\mathbf{u}_{t-1}^{\mathrm{T}}\lambda_{\mathbf{R}} + \gamma_{\mathbf{R}}^{\mathrm{T}}\mathbf{H}_{t-1}\gamma_{\mathbf{R}} \tag{0.8}$$

where, $\epsilon_{i,t} \in \epsilon_t$ is the normalized residual, i.e. $\epsilon_{i,t} = u_{i,t}\sigma_{i,t}^{-1}$. As in the previous model there are two main model dynamics depending on the constraints placed upon the parameter vector.

Estimating the BSM Specification

- Estimation of the model is carried out in two stages.
- The first stage involves identifying the existence and estimation of the boundary matrices.
- Once the boundary matrices have been identified, the second stage seeks to estimate the remaining parameter vector, θ.

Boundary Matrices.

- To establish the existence of the two boundary matrices we proceed as follows, for a given filtration, which identifies two data subsets, $\mathbf{X}_1$ and $\mathbf{X}_2$ from a sample data matrix $\mathbf{X}$, we test the following hypothesis,
-

$$H_0 : \boldsymbol{\Sigma}_1 = \boldsymbol{\Sigma}_2 \equiv \boldsymbol{\Sigma} \tag{0.9}$$
$$H_1 : \boldsymbol{\Sigma}_1 \neq \boldsymbol{\Sigma}_2 \tag{0.10}$$

-

$$\vartheta = \max_{a,b \in \mathbb{R}^n} \left(\frac{\left(\mathbf{a}^{\mathrm{T}}\hat{\boldsymbol{\Sigma}}_1\mathbf{b}\right)\left(\mathbf{a}^{\mathrm{T}}\hat{\boldsymbol{\Sigma}}_2\mathbf{b}\right)}{\left(\left(\tau_1^{-1} + \tau_2^{-1}\right)\left(\mathbf{a}^{\mathrm{T}}\hat{\boldsymbol{\Sigma}}\mathbf{a}\right)\left(\mathbf{b}^{\mathrm{T}}\hat{\boldsymbol{\Sigma}}\mathbf{b}\right) + \left(\mathbf{a}^{\mathrm{T}}\hat{\boldsymbol{\Sigma}}\mathbf{b}\right)^2\right)^{\frac{1}{2}}} \right) \tag{0.10}$$

Empirical Example of the Regime Switching Model

- The BSM model is demonstrated using the filtered residuals, see figure 7 from 100 shares randomly selected from the S&P 500 dataset optimally analyzed in the model in .
- We then form the instantaneous variance covariance matrices per time period and these are filtered using the maximum eigenvalue approach.
- The subset data-matrices are then collected and the Takemura and Kuriki matrix equality test (MEQT) is then applied, to the sub sample covariance matrices.
- A Savitzky-Golay Smoothing Filter is used, which allows smoothing without general loss of resolution. A Variety of filter specificationsare presented in the reverse order of the proximity to the smoothed central eigenvalue.

Table 1. Table of MEQT statistics from a variety of filtration specification. Using the empirical distribution of the largest eigenvalues, a high-band filter is used to create two data matrices $\mathbf{U}_1$ and $\mathbf{U}_2$ for MEQT stage.

Filtration	MEQT statistic	Degrees of Freedom ν	Threshold (95%)
High Band 1	283.5858	100	124.3421
High Band 2	238.3220	100	124.3421
High Band 3	97.3510	100	124.3421

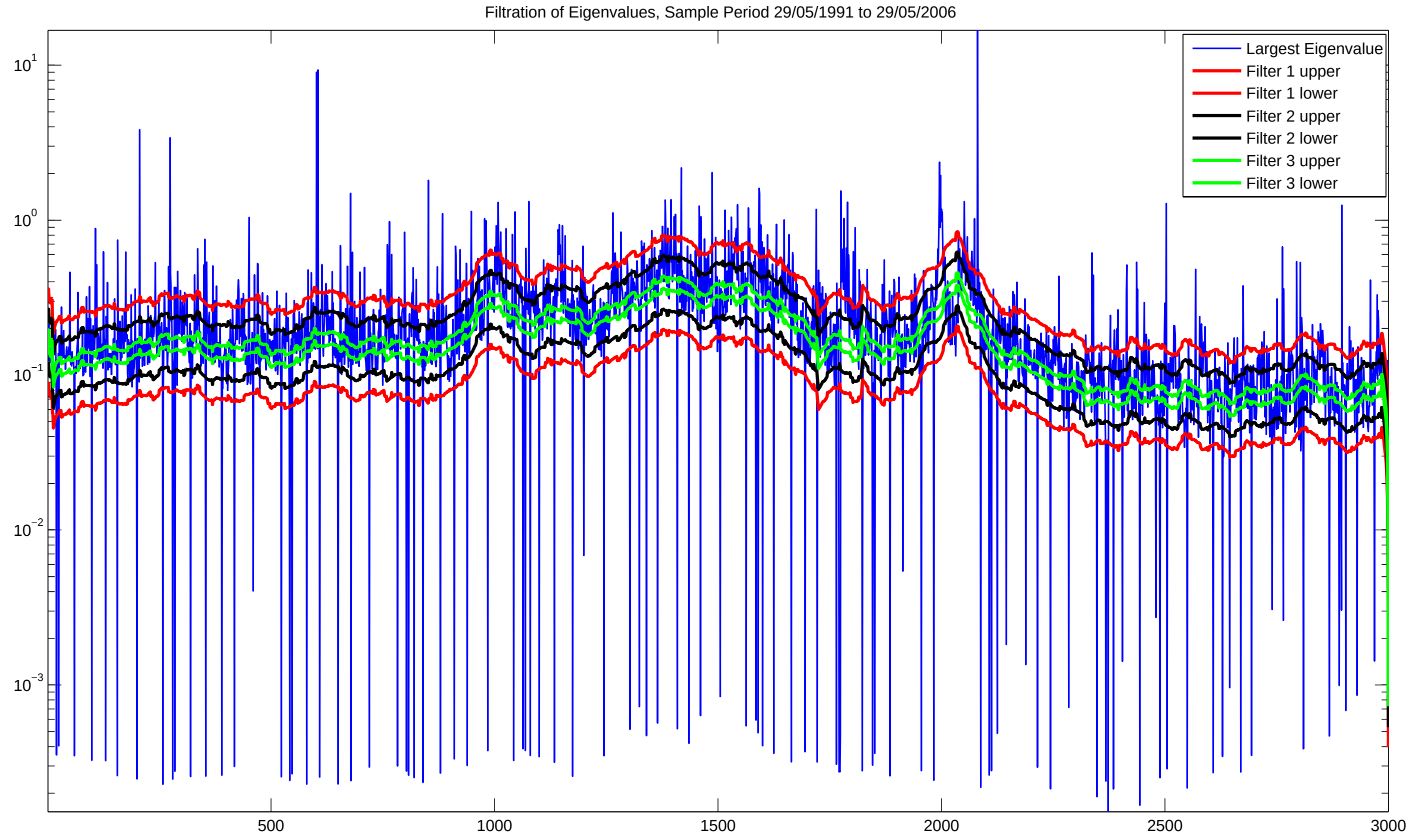

FIGURE 6. Eigen Filtration

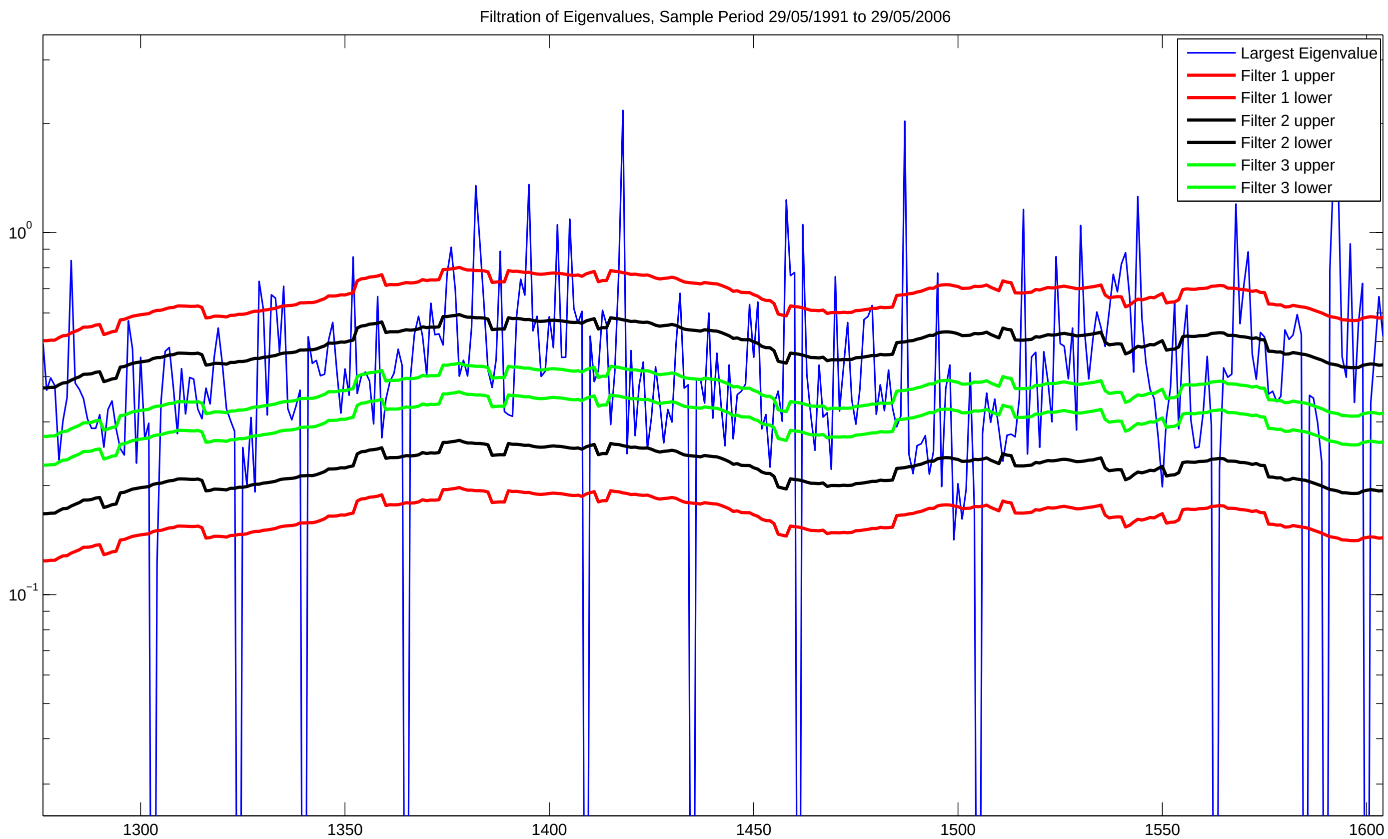

FIGURE 7. Eigen Filtration Closeup

Table 2. Estimation Results from the estimation of θ, using the BSM model structure.

Model Specification	Parameters	Log - Likelihood	LR - Ratio versus Mod1	$\chi^2(\nu)$
Mod1 $q=1, p=1$	203	-437827.9359	N/A	
Mod2 $q=2, p=1$	303	-437798.3894	29.5465	124.3421
Mod3 $q=1, p=2$	303	-437650.2673	177.6686	124.3421
Mod4 $q=2, p=2$	403	-437648.9149	179.0210	233.9943

Table 3. Parameter vector θ, with standard errors.

Parameter	$\hat{\theta}$	$stdev\left(\hat{\theta}\right)$
α	24.8747	5.0001
β	-0.7	0.154

Switching Parameters

System State

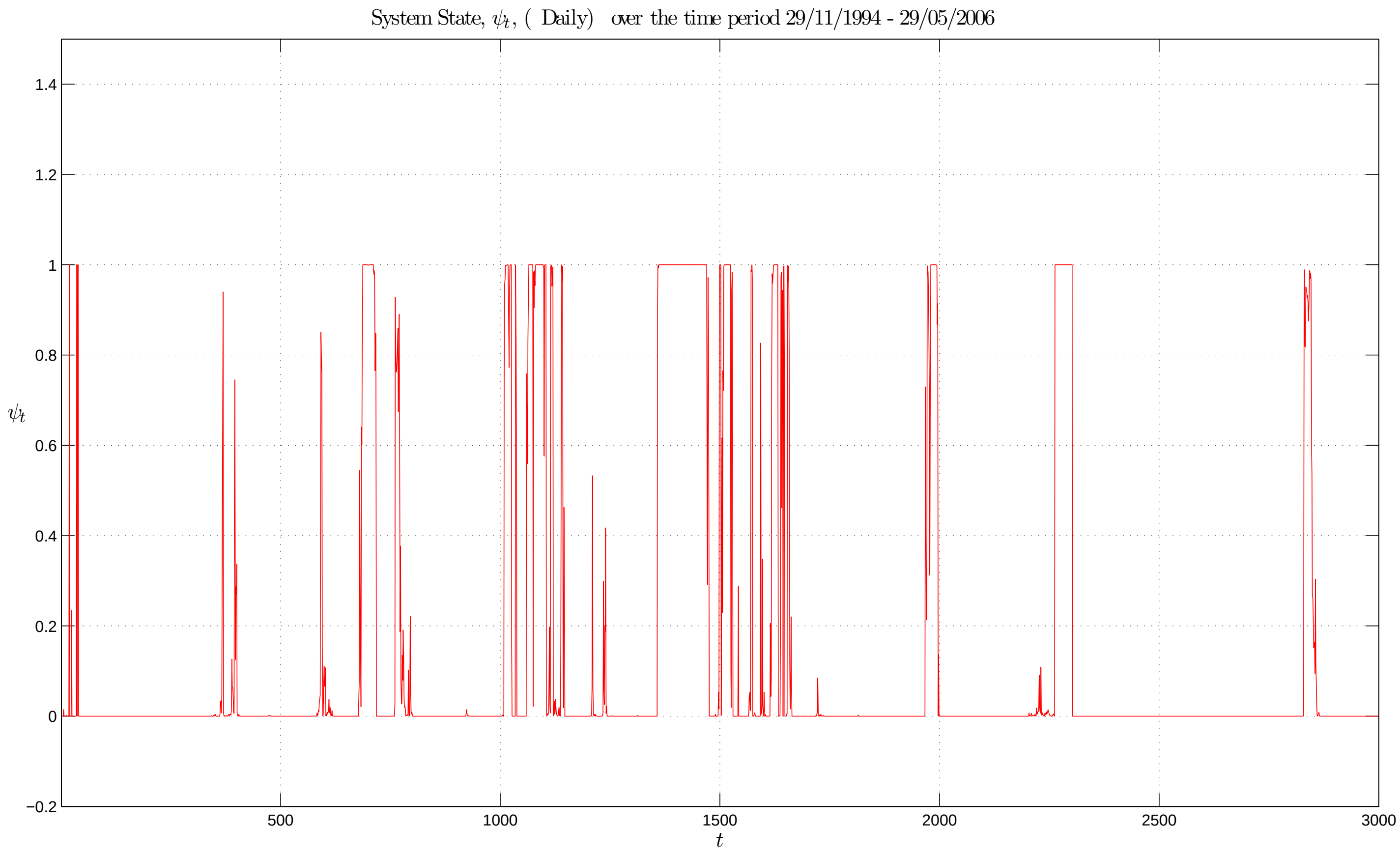

FIGURE 8. the time evolution of ψ_t, from the parameter estimates using High-Band filter 1 and $p = 2$ and $q = 1$

Dynamic Correlation Predictions

The BSM appears to capture the direction of the dynamics of the conditional correlation between individual stocks and shows promising forecasting performance as changes in the conditional correlation are anticipated as regime switches.

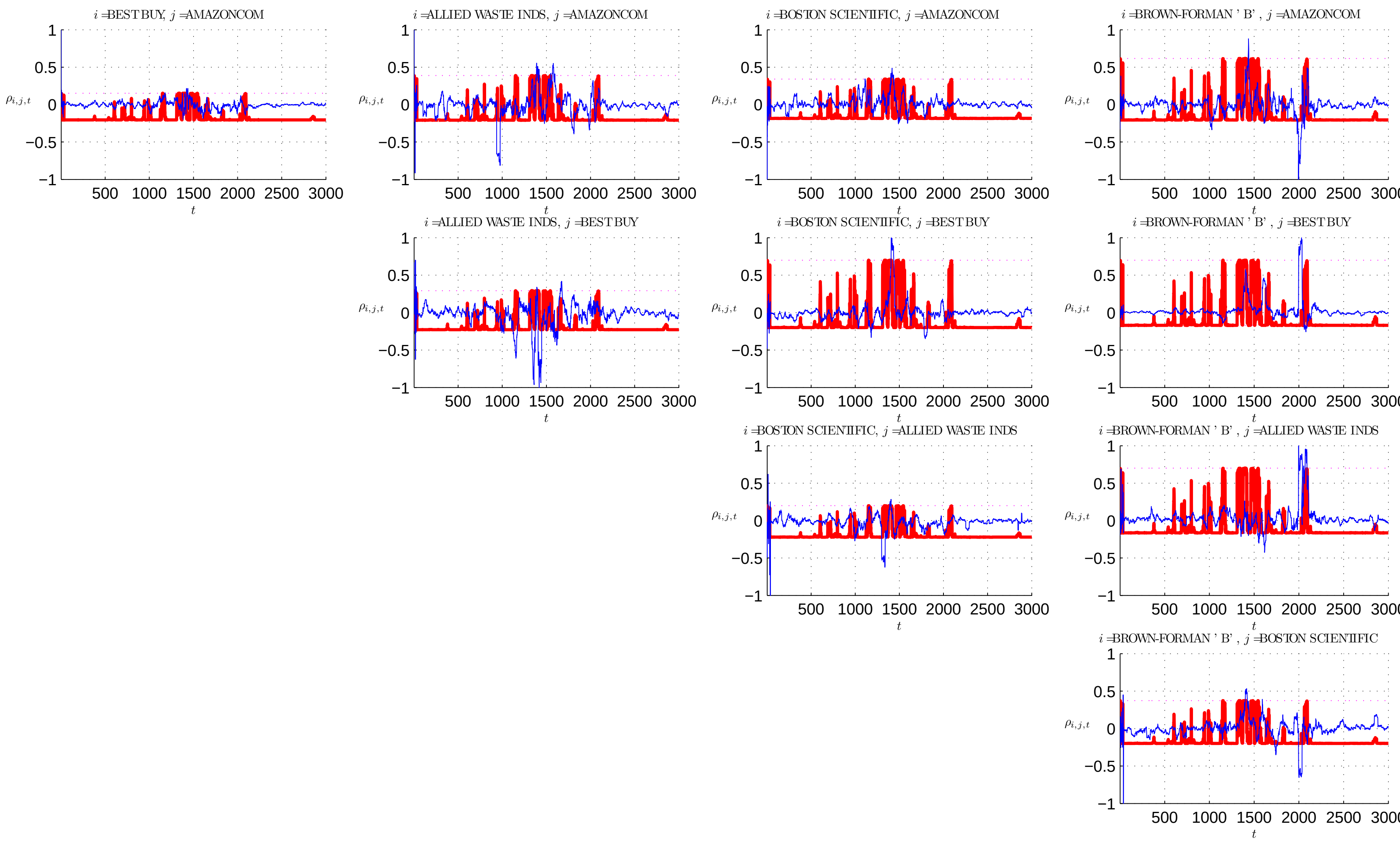

FIGURE 9. The Dynamic correlations of 5 random assets from the 100 d

Concluding Remarks

- This paper has presented a new method of estimating evolution of the multivariate second moments for high-variate models.
- The model in its basic form is shown to result in a positive-definite covariance matrix and is tractable in it's estimation, for high-variate systems.
- The model is parsimonious in the use of parameters and unlike the matrix autoregressive models, i.e. the MARCH and the BEKK, the number of parameters increases in linear proportion to the dimensionality of the system, as opposed to a quadratic increase in parameters.
- The basic model is not only applicable to modelling the dynamic interdependencies in the equity market, but could also be used in evaluating forward correlations in interest rates and the dynamic dependencies in factors relating to credit markets.

Final Remarks

In conclusion this model appears to offer a solution to the middle ground between the fully functional MV-ARCH models with their associated problems, regarding parameters and topology of the objective function and the totally ad-hoc methods such as the RiskMetrics™ smoother and the exponentially weighted correlation model.