Pricing Two Trees
When Trees and Investors are Heterogeneous

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Motivation

- **Basic Lucas tree** equilibrium model (Lucas 1978)
  - only one asset (’the market portfolio’) in positive net supply
  - representative agent
  - results (assuming log-utility and infinite horizon)
    - constant \( p/d \) ratio
    - no excess volatility
    - constant risk-free rate (term structure is flat)

- Extension by **Cochrane, Longstaff, Santa-Clara** (CLS 2008)
  - two trees, structurally identical
  - representative agent
  - results (again assuming log-utility and infinite horizon)
    - \( p/d \) ratio for individual trees depends on dividend share
    - return volatility differs from dividend volatility
Motivation

- This paper: **heterogeneous assets**
  - focus on information about assets
    - asset is characterized by expected dividend growth rate (drift) and uncertainty of dividend growth rate (volatility)
    - well-known asset: drift and volatility of dividend are known
    - unfamiliar asset: drift is stochastic and maybe unobservable, volatility is known
  - main question: resulting differences between stocks

- Furthermore: **heterogeneous agents**
  - focus on agents’ information and learning process
    - irrational investors: use only information from dividend process
    - rational investors: use more information (signal) for learning
  - main questions
    - impact of information quality on equilibrium outcome
    - trading between investors, long-run survival
Results

- **Stochastic dividend drift ...**
  - return volatilities of both stocks increase
  - return correlations change from positive to negative

+ **Learning (stochastic drift is not observable) ...**
  - volatility and expected excess return
    - increase for unfamiliar asset
    - decrease for well-known asset
  - effect is the larger the less precise the signal is
  - (mostly) negative return correlation between stocks

+ **Heterogeneous investors (rational vs. irrational) ...**
  - large trading volume in the unfamiliar stock (when small)
  - trading volume increases in difference in beliefs and information precision
  - irrational investors are driven out of the market rather fast
Model Setup: Economy

- Lucas tree economy with two independent **dividend** processes

  Blue chip: \[ dB_t = \mu_B B_t \, dt + \sigma_B B_t \, dW_{B,t} \]
  Young firm: \[ dY_t = \mu_{Y,t} Y_t \, dt + \sigma_Y Y_t \, dW_{Y,t} \]

- **Expected dividend growth rate** for \( Y \) is stochastic

  \[ d\mu_{Y,t} = \lambda (\bar{\mu}_Y - \mu_{Y,t}) \, dt + \sigma_\mu dW_{\mu,t} \]

  - uncertain dividend growth rate reflects 'young' firm with additional source of uncertainty
  - **structural difference**, not just more or less volatile dividends

- **Signal**: may contain information about dividend growth rate

  \[ \frac{ds_t}{s_t} = \alpha_t \, dt + \sigma_s dW_{s,t} \]

- \( W_B, W_Y, W_\mu, W_s \): independent standard Brownian motions
Model Setup: Investors

- Investors: identical preferences . . .
  - derive utility from consumption
  - infinite planning horizon
  - log-utility, time-discount rate $\rho$

$$E \left[ \int_{0}^{\infty} e^{-\rho t} \ln c_t \, dt \right]$$

- . . . but maybe different information
Model Setup: Heterogeneous Information

- In general: dividend drift $\mu_{Y,t}$ cannot be observed
- Investor $j$ learns about dividend drift $\mu_{Y,t}$
  - current estimate: $\mu_{Y,t}^j := \mathbb{E}_j[\mu_{Y,t}|\mathcal{F}_t^j]$
  - variance of estimation error: $\gamma_{t}^j := \mathbb{E}_j[(\mu_{Y,t}^j - \mu_{Y,t})^2|\mathcal{F}_t^j]$
- Sources of information: observed dividend $Y$, maybe signal
- **Different priors about structure of model**
  - Irrational investors assume signal to be pure noise: $\alpha_t = \text{noise}$
    → base inference on observed dividend only
  - Rational investors assume signal to contain information: $\alpha_t = \mu_{Y,t}$
    → base inference on observed dividend and signal
    limiting case: perfect signal with $\sigma_s = 0$
    → dividend drift observable
- Even with the same information, investors do not agree . . .
  - research reports in lifesciences → different conclusions
  - recommendations of analysts → differ from each other
Homogeneous Economy

- **Homogeneous economy**: benchmark case
  - all investors are equal
  - difference to CLS: stochastic $Y$-drift, learning

- Equilibrium interest rate in type $j$ economy

\[
r_j^t = \rho + z_t \mu_B + (1 - z_t) \mu_{Y,t}^j - z_t^2 \sigma_B^2 - (1 - z_t)^2 \sigma_Y^2
\]

- $z$: dividend share, $z_t = \frac{B_t}{B_t + Y_t}$
- structurally equal in all homogeneous economies (CLS, II, RI)

- Market prices of risk

\[
\theta_{B,t}^j = z_t \sigma_B \quad \theta_{Y,t}^j = (1 - z_t) \sigma_Y
\]

\[
\theta_{s,t}^j = 0
\]

- identical in all homogeneous economies (CLS, II, RI)
- signal shocks: not priced (no impact on consumption)
Homogeneous Economy

• Price of **market portfolio** with dividend $D = B + Y$:
  \[ P_{M,t} = \frac{D_t}{\rho} \]
  - does not depend on uncertainty about dividend drift
    (due to log-utility, investor is myopic)

• Prices of **individual stocks**
  \[ P_{B,t} = (B_t + Y_t) \mathbb{E}_t \left[ \int_0^\infty e^{-\rho \tau} \frac{B_{t+\tau}}{B_{t+\tau} + Y_{t+\tau}} d\tau \right] \]
  \[ P_{Y,t} = (B_t + Y_t) \mathbb{E}_t \left[ \int_0^\infty e^{-\rho \tau} \frac{Y_{t+\tau}}{B_{t+\tau} + Y_{t+\tau}} d\tau \right] \]
  - depend on type of economy
  - remark on calculation: prices calculated by Fourier inversion
Numerical Example: Price-Dividend Ratios

- **Fundamental effect on p/d ratio (CLS):** desire to diversify, increases price of stock with small dividend share
- **Stochastic drift** increases p/d ratio for stock with small dividend share further
- Impact of **information quality** is negligible
- Trees are symmetric: no differences between stocks

**Price-dividend ratio B-stock**

**Price-dividend ratio Y-stock**
Numerical Example: Return Volatilities

- **Stochastic drift** increases volatilities of both assets
- **Learning** (\(Y\)-drift is not observable)
  - **\(B\)-stock**: lower volatility
    (smaller reaction to less precise signal, small exposure to \(Y\)-risk increases)
  - **\(Y\)-stock**: higher volatility
    (exposure to signal decreases, large exposure to \(Y\)-risk increases further)

Effect is the larger the less precise the signal is
Numerical Example: Return Correlations

- **Stochastic drift**
  - $\mu_{Y,t}$ increases: $Y$-stock more attractive relative to $B$-stock
  - Correlations change from positive (CLS) to negative

- **Learning** ($Y$-drift is not observable)
  - Negative correlation increases in absolute terms (large $Y$)
  - Effect is the larger the less precise the signal is
Numerical Example: Risk Premia

- Market prices of risk (structurally) equal in all four economies
- **Stochastic drift**: negligible impact
- **Learning** induces additional exposure to $W_Y$
  - $B$-stock: negative exposure $\rightarrow$ negative risk premium
  - $Y$-stock: large positive exposure $\rightarrow$ higher risk premium
  
  Effect is the larger the less precise the signal is
Heterogeneous Economy

- **Heterogeneous Economy**
  - both rational and irrational investors (both have log-utility)
  - complete market

- Equilibrium interest rate
  - weighted average of interest rates in homogeneous economy
  - weights: consumption shares

- Market prices of risk
  - similar to homogeneous economy
  - market price of $W_Y$ depends on relative optimism/pessimism
  - signal risk: not priced

- Asset prices
  - weighted averages
  - return volatilities, correlations, expected excess returns: between values in homogeneous economies in most cases
Optimal Portfolios (Scenario: RI More Optimistic)

- Initial positions: equal shares of stocks, no 'signal-risk'-asset
- Investors agree on $W_B$ risk: little trading volume in $B$-stock
- Positions in $Y$-stock depend on relative optimism/pessimism
  - optimistic investor (here: rational) holds larger position
  - high leverage when $Y$-stock is small
- Pure 'signal-risk'-asset eliminates exposure to signal risk
- Trading volume increases in differences in beliefs and in signal quality
Long-Run Survival

- Irrational investors will ultimately be driven out of the market (because they use an incorrect model structure)
- Important question: How long does it take?
  - previous papers: very long (several hundred years ...)
  - our paper: roughly 20-30 years until irrational investors have lost half of their initial wealth
- Reasons for fast natural selection
  - log-utility
  - modeling choice for irrationality

Expected wealth share of rational investors under true measure

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Pricing Two Trees When Trees and Investors are Heterogeneous
Economy with **heterogeneous assets** ...  
- dividend drift of one asset is stochastic (but observable)  
  - return volatilities of both assets increases  
  - return correlation changes from positive to negative  
- investors learn about stochastic dividend drift  
  - well-known asset: lower volatility, lower risk premium  
  - unfamiliar asset: higher volatility, higher risk premium  
  - large negative return correlation when unfamiliar asset is large

... and **heterogeneous investors**  
- prices are weighted average of homogeneous economies  
- **portfolio holdings** depend on differences in beliefs and on information precision  
- moderate trading volume in well known stock, **high trading volume in small unfamiliar stock**  
- irrational investors driven out of the market rather fast