Optimal Hedging With Higher Moments*

Chris Brooks† Aleš Černý‡ Joëlle Miffre§

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Abstract: This study proposes a utility-based framework for the determination of optimal hedge ratios that can allow for the impact of higher moments on the hedging decision. The approach is applied to a set of 20 commodities that are hedged with futures contracts. We examine the entire hyperbolic absolute risk aversion (HARA) family of utilities which include quadratic, logarithmic, power and exponential utility functions. We find that for small to moderate commodity exposure, the performance of hedges constructed allowing for non-zero higher moments is only very slightly better than the performance of the much simpler OLS hedge ratio. For very high commodity exposures, higher moments do matter and their relative weights in the utility function affect the optimal decision. As one would expect, the exponential utility hedge is also affected by the higher moments, but it tends to the minimax hedge ratio, that is the ratio which minimizes the largest loss of the hedged position. We support our empirical findings by theoretical analysis of optimal hedging portfolios.

Keywords: Utility-based hedging, OLS, Non-normality risk, Commodity futures, Skewness, Kurtosis

JEL classifications: G13, C53

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†ICMA Centre, University of Reading, c.brooks@icmacentre.reading.ac.uk

‡Cass Business School, City University London, cerny@martingales.info, `http://www.martingales.info`

§EDHEC Business School, Nice, joelle.miffre@edhec.edu

1 Introduction

There is now indisputable evidence to suggest that the return distributions of risky assets depart from normality. For example, deviations from normality have been observed for emerging stock market indices (Harvey, 1995), hedge fund indices (Agarwal and Naik, 2004), individual hedge funds (Brooks and Kat, 2002), relative-strength strategies (Harvey and Siddique, 2000) and futures contracts (Christie-David and Chaudhry, 2001). Under some fairly weak assumptions concerning the shape of investor utility functions, Scott and Horvarth (1980) show that investors are concerned not just with the mean and variance of asset returns, but also with the distribution's higher moments as well. Scott and Horvarth further demonstrate that investors will have a preference for larger odd moments and smaller even moments. Importantly, Kraus and Litzenberger (1976), Harvey and Siddique (2000) and Chung et al. (2006) have made it clear that systematic risks related to skewness and kurtosis are priced by the market. To put it differently, exposure to systematic "non-normality risks" commands a risk premium.

In parallel to the analysis of Markowitz (1959), Kraus and Litzenberger (1976) have shown that it is not the total skewness of an asset that will be priced, but rather the contribution of the asset to the skewness of a well-diversified portfolio (also called systematic skewness or co-skewness). Similarly, it will only be systematic kurtosis risk, or the contribution of an asset to the kurtosis of a well-diversified portfolio, that commands a risk premium. As in Markowitz (1959), unsystematic skewness or kurtosis risk should be eliminated through diversification. Recent renewed interest in this proposition has led to a number of studies that extend existing asset pricing models to incorporate higher moments, building on the early work of Kraus and Litzenberger (1983). Examples include Chunhachinda et al. (1997) on the incorporation of moments higher than the second into the investor's portfolio decision, and Barone-Adesi et al. (2004) on incorporating co-skewness into asset pricing models. Harvey and Siddique (2000) demonstrate that conditional skewness can help to explain the cross-sectional variation in asset returns, including momentum effects. Similarly, Chung et al. (2006) have shown that the risk premia associated with size and book-to-market value are compensation for systematic exposures to a set of non-normality risks. Following this argument, it is possible that failure to incorporate higher moment considerations could help to rationalize several other widely documented asset pricing anomalies.

Another, almost entirely separate strand of finance literature has looked at the hedging decisions of risk-averse investors[1], with particular reference to hedging with futures contracts. A large number of empirical studies[2] have been concerned with estimation of the optimal hedge ratio, defined as the optimal number of futures contracts to employ per unit of the spot asset to be hedged (see, for example, Baillie and Myers, 1991; Cecchetti et al., 1988; Kroner and Sultan, 1991; Lien and Luo, 1993; Lin et al., 1994; Myers and Thompson, 1989; Park and Switzer, 1995; Strong and Dickinson, 1994). An easy way to calculate this number of futures contracts is to employ the OLS hedge ratio of Ederington (1979) and Figlewski (1984), which simply measures the hedge ratio as the slope coefficient of an OLS regression of spot returns on futures returns. This implies a static risk management strategy that involves a one-off decision on the optimal hedge and might therefore yield suboptimal hedging decisions in periods of high basis volatility. To overcome this problem, quite a

[1]It is generally accepted that privately held, owner-managed firms are risk-averse. Listed companies, too, can act as risk averters in the presence of capital market imperfections, i.e outside the Modigliani-Miller paradigm (Stulz, 1984; Smith and Stulz, 1985; Froot et al., 1993; Brown and Toft, 2002).

[2]The existing theoretical treatment of optimal hedging relies either on joint normality or log-normality (Moschini and Lapan, 1995; Brown and Toft, 2002) or on a specific decomposition, additive or multiplicative, of hedgeable and non-hedgeable risks (Benninga et al. 1983, Briys et al., 1993; Mahul, 2002.)

large literature has developed that models the optimal hedge ratio within a conditional framework, taking into account the dynamics between the spot and futures returns (see, for example, Kroner and Sultan, 1991; Brooks et al., 2002; or Miffre, 2004, to name only a few). These studies have mainly employed models from the multivariate generalized autoregressive conditionally heteroscedastic (MGARCH) family. They have reached conflicting results on the out-of-sample hedging effectiveness of conditional minimum variance hedge ratios, even before taking into account the additional costs involved with continually buying and selling futures contracts so as to rebalance the hedged portfolio when the model suggests. At best, MGARCH models have led to very modest improvements in gross hedging efficiency when evaluated on an out-of-sample basis. Hence the benefits of active risk management strategies ought to be viewed with caution.

Almost without exception, empirical studies on the determination of optimal hedge ratios at best assume that investors have two-moment (quadratic) utility functions or that the distribution of returns on the hedged portfolio is normal, so that the mean and variance alone are sufficient to determine the hedge ratio optimally[3]. In a slight generalization, Levy (1969) shows that a cubic utility function can be employed where investor preferences depend on skewness. However, it is not at all obvious, when one is released from the constraint of the mean-variance framework, why one should stop at skewness, for in addition to an aversion to negative skewness, rational investors should possess an aversion to positive excess kurtosis as well. Even less plausibly, many studies focus on minimum variance hedging, where the mean, as well as any moments of order higher than the second, are ignored. Such an assumption concerning the mean will only be appropriate if investors are infinitely risk-averse, or if the expected return is zero.

Clearly then, if return distributions depart from normality, hedging strategies that assume normality might lead to sub-optimal hedging decisions. The extant literature concerning the impact of higher moments on hedging is very sparse. Gilbert et al. (2006) derive and apply a partial equilibrium model of hedging that allows for skewness (but not kurtosis) in the hedger's utility function. They show that skewness can be important for undiversified agents, and the overall extent of speculation could either rise or fall depending upon whether there is a price bias in the forward market. However, while they allow for skewness in the utility function, they do not explicitly consider its impact on the determination of optimal hedge ratios.

The only other relevant contribution in this area is by Harris and Shen (2006), who consider cross-hedging with currencies rather than with futures. They show, using a set of daily currency exposures, that minimum variance hedging is likely to reduce the out-of-sample variance of the hedged portfolio, but the skewness and kurtosis are likely to fall and rise respectively. This result indicates that the benefit of hedging may be overstated since these higher moments move in exactly the opposite directions to those preferred by a rational utility maximizer of the form described in the theoretical literature. Similarly, Brooks and Kat (2002) observed that hedge funds, while they demonstrate impressive performance on mean-variance grounds, also typically have less desirable higher moment values than traditional asset classes.

To our knowledge, there are no previous studies that have attempted to estimate optimal hedge ratios within a utility-based framework that allows for investors to have non-zero preferences for higher moments. We propose a methodology which

1. does not employ (notoriously unreliable) *estimates* of higher moments

[3] A slightly weaker assumption than return normality is that the spot and corresponding futures returns are drawn from a multivariate elliptical distribution. In such circumstances, even if the spot returns are skewed and/or leptokurtic, the magnitude or otherwise of these higher moments is not affected by hedging with futures and thus optimally, they should not enter into the hedger's objective function.

2. does not impose parametric distribution of returns and is therefore not subject to parameter uncertainty,

3. does not require the futures market to be unbiased,

4. permits fast and reliable numerical implementation[4].

We measure, for the first time, the loss of welfare that may be incurred if one uses OLS hedge ratios in non-quadratic utility functions. We define the optimal hedge ratio as the derivative of the optimal futures position with respect to the change in the spot position. By doing so, we draw together the literatures on hedging with futures, and on utility maximization with higher moments. An important precursor to our work is Kallberg and Ziemba (1983), who study optimal equity portfolios and conclude that mean–variance portfolios differ insignificantly in welfare terms from general utility-based optimal portfolios when differences in risk aversion are properly controlled for. By contrast, Jondeau and Rockinger (2006) conclude that portfolio allocation is affected by the skewness and kurtosis of the return distribution for high levels of risk aversion. We reconcile these seemingly contradictory results within our framework. In our hedging context, the former corresponds to small commodity exposure while the latter typifies large commodity exposure.

The remainder of the article is organized as follows. Section 2 presents the theoretical underpinning of our higher moment hedge ratio and provides a numerical illustration based on an airline company hedging its fuel exposure. Section 3 introduces the dataset of 20 commodities and section 4 presents the empirical results. Finally section 5 concludes.

2 Methodology

An agent who hedges a long spot position at time t using h_t futures contracts will receive the following payoff at time $t+1$, R_{t+1}, to the hedged position

$$R_{t+1} = C_{t+1} - h_t F_{t+1}, \tag{2.1}$$

where C_{t+1} and F_{t+1} denote the changes in the cash (spot) and futures prices respectively between times t and $t+1$.

Suppose that the agent has the four-moment utility function given by

$$U_t(R_{t+1}) = E_t(R_{t+1}) - a\mathrm{Var}_t(R_{t+1}) + b\mathrm{Skew}_t(R_{t+1}) - c\mathrm{Kurt}_t(R_{t+1}) \tag{2.2}$$

where $E_t(R_{t+1})$ is the expectation formed at time t for the return during the next period, Var, Skew, and Kurt are the second, third, and fourth (standardized) moments of the distribution of expected returns respectively, and a, b, c are parameters that represent the relative desirability of the moments of the return distribution in the agent's utility function. Given the signs used to precede

[4]Our approach, based on Newton's optimization method and detailed in Appendix C, is able to deal with general non-parametric distributions of returns and apart from strict concavity it does not impose restrictions on the utility function. The method exhibits quadratic convergence, doubling the number of digits of accuracy in each iteration, and it is thus extremely fast.

Apart from computing optimal hedge ratios, our procedure is also suitable for solving optimal investment problems with a large number of assets because the computational effort grows only quadratically with the number of assets. By contrast, the use of co-skewness and co-kurtosis (see, for example, Harvey et al., 2004; Jondeau and Rockinger, 2006) makes the computational time grow with the third and fourth power of the total number of assets, respectively.

the parameters in (2.2), we would usually expect $a, b, c > 0$ (see, for example, Scott and Horvath, 1980). While the literature on determining optimal hedge ratios is now vast, traditionally, academic research has assumed that only the first two moments of the utility function are of concern to the investor, a restriction equivalent to $b = c = 0$ in (2.2). Under this assumption, and provided that the value of the hedged portfolio follows a pure martingale process, it is easy to show that the optimal hedge ratio is simply the ratio of the covariance between the cash and futures returns to the variance of the futures returns, equivalent to the OLS hedge.

In general, it is not obvious why one should stop at the fourth moment in (2.2). Thus we are compelled to adopt a more general approach based not on directly optimizing simultaneously across multiple moments, but rather on a utility function. To test robustness of our results we examine a whole family of utility functions including the logarithmic, exponential, power and quadratic utility (the so called HARA or LRT class, cf. Cass and Stiglitz, 1970; Ingersoll, 1987) as well as fourth moment polynomial approximations thereof corresponding to different choices of a, b, c in equation (2.2).

Definition 2.1 *We call $U : \mathbb{R} \to [-\infty, \infty)$ with effective domain $\mathcal{D}_U$ (i.e. the set where U is finite) a utility function if*

1. *U is at least twice differentiable on the interior of $\mathcal{D}_U$,*
2. *$U'' < 0$ on the interior of $\mathcal{D}_U$,*
3. *the maximal domain $\bar{\mathcal{D}}_U$ on which U is strictly increasing has non-empty interior,*
4. *$\lim_{v\to-\infty} U'(v) = -\infty$ or $\lim_{v\to+\infty} U'(v) \leq 0$.*

In cases when $\bar{\mathcal{D}}_U \subsetneq \mathcal{D}_U$ we define the inverse utility U^{-1} as taking values in $\bar{\mathcal{D}}_U$.

Fix a probability space $(\Omega, P, \mathcal{F})$ with finite sample space $\Omega := \{\omega_i : i = 1, \ldots, n\}, n \in \mathbb{N}$. Denote by X, Y two random variables representing the excess returns of the future contract and of the spot asset, respectively[5]. We denote their realizations concisely by X_i, Y_i with $i = 1, \ldots, n$. We assume that there is no arbitrage, i.e. there is a measure Q equivalent to P and such that $E^Q(X) = E^Q(Y) = 0$. The next theorem is technical. It states that in the absence of arbitrage the spot position is either completely unhedgeable or the optimal hedge is well-defined. This result is no longer true with infinite sample space and unbounded return distributions.

Theorem 2.2 *Consider a utility U and initial endowment $v \in \bar{\mathcal{D}}_U$ and define*

$$u(v, \eta, \vartheta) := \mathrm{E}\left(U\left(v + \eta Y + \vartheta X\right)\right) \in [-\infty, \infty).$$

The hedgeable set $I := \{\eta \in \mathbb{R} : \sup_{\vartheta\in\mathbb{R}} \mathrm{E}\left(U\left(v + \eta Y + \vartheta X\right)\right) > -\infty\}$ is an interval containing an open neighbourhood of zero. For every hedgeable spot position ($\eta \in I$) the maximizer in $\sup_{\vartheta\in\mathbb{R}} u(v, \eta, \vartheta)$ exists and is unique; we denote it by $\varphi(v, \eta)$.

Define the certainty equivalent (CE) *wealth increase in the standard way,*

$$\mathrm{CE}(v, \eta, \vartheta) := U^{-1}\left(u(v, \eta, \vartheta)\right) - v,$$

and denote its maximal value by $\widehat{\mathrm{CE}}(v, \eta) := \sup_{\vartheta\in\mathbb{R}} \mathrm{CE}(v, \eta, \varphi(v, \eta))$. For $\eta \in I$ the quantity $\widehat{\mathrm{CE}}(v, \eta) = \mathrm{CE}(v, \eta, \varphi(v, \eta))$ is finite whereas for $\eta \notin I$ we have $\widehat{\mathrm{CE}}(v, \eta) = -\infty$.

[5] We suppress time subscripts throughout this section. The random variable X corresponds to the change in the futures index F_{t+1} and Y is interpreted as the change in the cash value C_{t+1}. The expectation $E(.)$ is intepreted as the expectation at time t conditional on the information at that time.

Proof. See Černý (2003), Theorem 2. ■

Suppose the hedger is long η units of the spot asset and assume this position is hedgeable, $\eta \in I$. If the investor does not hedge, she continues to hold $\varphi(v,0)$ futures contracts. This is known as a speculative position. The literature on optimal hedging typically assumes that the futures market is unbiased, $\mathrm{E}(X)=0$, in which case the speculative position is zero by Jensen's inequality. If the investor hedges optimally, her position in the futures changes to $\varphi(v,\eta)$. One can now define the optimal hedge ratio (OHR)

$$\mathrm{OHR}(v,\eta) := -\frac{\varphi(v,\eta)-\varphi(v,0)}{\eta}. \tag{2.3}$$

We use the standard convention whereby the hedge ratio signifies the number of futures contracts the investor *shorts* as a result of being *long* one unit of the spot asset, hence the extra minus sign in equation (2.3). In the case of backwardation or contango when $\mathrm{E}(X) \neq 0$, the optimal futures position is non-zero even if the agent holds no spot assets, and therefore $\varphi(v,0)$ does not constitute a hedge in itself. In such a case, only the incremental position over and above the speculative holding $\varphi(v,0)$ should be interpreted as the hedging position, which is reflected in definitions (2.3) and (2.4).

The welfare gain (WG) from a particular (not necessarily optimal) hedge h is defined as follows[6]

$$\mathrm{WG}(v,\eta,h) = \mathrm{CE}(v,\eta,\varphi(v,0)-\eta h) - \mathrm{CE}(v,\eta,\varphi(v,0)). \tag{2.4}$$

If one wants to understand and compare optimal investment/hedging dictated by different utility functions (i.e. those whose value depends on higher moments as opposed to quadratic utility), it is important to normalize the resulting portfolio by some measure of risk aversion. This insight goes back to Arrow (1971). The most convenient normalization factor turns out to be the Arrow-Pratt coefficient of risk aversion, cf. Arrow (1963), Pratt (1964), Kallberg and Ziemba (1983), Samuelson (1970). We apply a similar normalization to the risk-adjusted performance measurement below.

Using the coefficient of local absolute risk aversion,

$$A(v) := -\frac{U''(v)}{U'(v)},$$

we define the normalized spot and futures positions

$$\lambda := A(v)\eta, \qquad \theta := A(v)\vartheta. \tag{2.5}$$

Similarly we define a normalized welfare gain which we call the *hedging potential* (HP),

$$\mathrm{HP}(v,\eta,h) = \frac{\mathrm{WG}(v,\eta,h)}{A(v)\eta^2}. \tag{2.6}$$

The normalization is performed to enable meaningful comparison of the hedging coefficients and welfare measurements across different utility functions. Essentially, we will see that results are primarily driven by the values of λ and θ and only to a lesser extent by the shape of the utility function itself. Hedging potential is *robust* in the sense that it possesses a meaningful limit as η approaches 0

[6]Our measure of the welfare loss arising from using a second-best strategy is based on the certainty equivalent as in Kallberg and Ziemba (1983) and Pulley (1983), by contrast to Simaan (1993b) who uses a compensating variation in terminal wealth.

and this limit coincides across all utility functions when the futures market is unbiased[7]. For HARA utility functions, the hedging potential is independent of the wealth level v. The role of the hedging potential is clarified in the numerical example below (see section 2.1).

To evaluate the normalized quantities, it is convenient to define a "normalized utility".

Definition 2.3 *We say that $f : \mathbb{R} \to [-\infty, \infty)$ is a normalized utility to U at initial wealth $v \in \bar{\mathcal{D}}_U$ if there are coefficients $c_1 > 0, c_2 \in \mathbb{R}$ such that*

$$f(z) = c_1 U\left(v + \frac{z}{A(v)}\right) + c_2. \tag{2.7}$$

It is well known in utility theory that the actual values of c_1 and c_2 are immaterial (see also Theorem 2.4) and hence they can be chosen with the view to eliminate dependence on v where possible. Denoting by γ the coefficient governing the shape of HARA utility, we show in Appendix B that the normalized utility can be taken as

$$f^{(\gamma)}(z) := \begin{cases} \frac{\left(1+\gamma^{-1}z\right)^{1-\gamma}-1}{1/\gamma-1} & \text{for } \gamma > 0, \gamma \neq 1, \\ \ln(1+z) & \text{for } \gamma = 1, \\ \frac{\left|1+\gamma^{-1}z\right|^{1-\gamma}-1}{1/\gamma-1} & \text{for } \gamma < 0, \\ 1 - e^{-z} & \text{for } \gamma = \pm\infty. \end{cases} \tag{2.8}$$

Conveniently, the normalized HARA utility only depends on the shape parameter γ. The literature also discusses fourth order polynomial approximations of different utility functions[8], obtained by Taylor expansion. For the HARA utility class, the corresponding polynomial normalized utility reads

$$\tilde{f}^{(\gamma)}(z) = z - \frac{z^2}{2} + \left(1 + \frac{1}{\gamma}\right)\frac{z^3}{6} - \left(1 + \frac{1}{\gamma}\right)\left(1 + \frac{2}{\gamma}\right)\frac{z^4}{24}. \tag{2.9}$$

Theorem 2.4 *Consider a utility U, initial endowment $v \in \bar{\mathcal{D}}_U$ and a corresponding normalized utility f. Define*

$$\begin{aligned} a(\lambda, \theta) &= f^{-1}\left(E\left(f\left(\lambda Y + \theta X\right)\right)\right), \\ \alpha(\lambda) &= \arg\max_{\theta \in \mathbb{R}} E\left(f\left(\lambda Y + \theta X\right)\right), \\ \hat{h}(\lambda) &= -\frac{\alpha(\lambda) - \alpha(0)}{\lambda}, \\ g(\lambda, h) &= \frac{a(\lambda, \alpha(0) - \lambda h) - a(\lambda, \alpha(0))}{\lambda^2}. \end{aligned} \tag{2.10}$$

Then η is hedgeable for utility U if and only if λ is hedgeable for utility f and

$$\mathrm{OHR}(v, \eta) = \hat{h}(\lambda), \tag{2.11}$$

$$\mathrm{HP}(v, \eta, h) = g(\lambda, h), \tag{2.12}$$

where λ is the normalized spot position from equation (2.5).

[7] In an unbiased futures market, the limiting value of the hedging potential of the optimal hedge for $\eta \to 0$ is $\rho^2(X, Y)\mathrm{Var}(Y)$ where ρ is the correlation coefficient. This holds for an arbitrary utility function.

[8] Jondeau and Rockinger (2006) use $\gamma = \infty$ in this context.

Proof. By a straightforward calculation

$$A(v)\mathrm{CE}\,(v,\eta,\vartheta) = f^{-1}\left(E\left(f\left(\lambda Y + \theta X\right)\right)\right) = a(\lambda,\theta), \tag{2.13}$$

for *any* normalized utility f, with θ given in equation (2.5). From here and (2.7) follows

$$\begin{aligned} \varphi(v,\eta) &= \alpha(\lambda)/A(v), \\ \mathrm{OHR}\,(v,\eta) &= \frac{\varphi(v,\eta)-\varphi(v,0)}{\eta} = \frac{\alpha(\lambda)-\alpha(0)}{\lambda} = \hat{h}(\lambda). \end{aligned} \tag{2.14}$$

Equation (2.14) implies $\varphi(v,0) = \alpha(0)/A(v)$. This together with (2.5) and (2.13) yields

$$\begin{aligned} \frac{\mathrm{CE}\,(v,\eta,\varphi(v,0)-h\eta)}{A(v)\eta^2} &= \frac{f^{-1}\left(E\left(f\left(\lambda Y + (\alpha(0)-h\lambda)X\right)\right)\right)}{\lambda^2}, \\ \frac{\mathrm{CE}\,(v,\eta,\varphi(v,0))}{A(v)\eta^2} &= \frac{f^{-1}\left(E\left(f\left(\lambda Y + \alpha(0)X\right)\right)\right)}{\lambda^2}, \end{aligned}$$

whereby we obtain (2.12) from (2.4) and (2.6). The existence and uniqueness of the maximizer in (2.10) was shown in Theorem 2.2. ■

2.1 Airline example

Consider a stylized example of an airline hedging its fuel exposure. Suppose that the book value of the company is \$3.5 bn and the expected net income is \$0.5 bn giving projected book value $v = \$4.0$ bn. Assume that the expected fuel bill at current prices is $\eta = \$0.8$ bn and that the fuel bill uncertainty due to price variations dominates all the other uncertainty in the airline's revenues and expenses. Assume further that the airline does not wish to pass fuel cost increases onto its passengers[9]. Finally, assume that the local relative risk aversion of the airline is moderate[10] at 5. Then the normalized exposure is $\lambda = 5 \times \frac{0.8}{4.0} = 1$. To compute the optimal hedge, we compile data on monthly jet fuel price returns[11] to obtain a histogram for Y and obtain synchronized returns for a prospective cross-hedging commodity futures (in this case light crude oil) to obtain the distribution of X. With the joint empirical distribution of X and Y, we then evaluate (2.11) with $\lambda = 1$, for different normalized utility functions shown in equation (2.8). The results are shown in Table 1. We use eight utility functions: quadratic ($\gamma = -1$) which takes into account only the first two moments, quartic ($\gamma = -3$) including also (co-)skewness and (co-)kurtosis, and exponential ($\gamma = \infty$), logarithmic ($\gamma = 1$) and fourth power hyperbolic ($\gamma = 5$) which involve all moments of the joint distribution in different proportions (column HARA). For the last three utility functions, we also examine their approximation by a fourth order polynomial (column POLY). We discuss the relationship of the OLS hedge ratio to the quadratic utility hedge ratio in Appendix A. Numerically, the OLS hedge is in this case indistinguishable from the quadratic utility hedge ($\gamma = -1$).

[Table 1 about here]

[9]Figures based on Southwest Airlines (US) financial statement for 2000 (source: SEC 10-K filing for 2001). Unlike some other major airlines, Southwest did not apply fuel surcharges to fares – see Morrell and Swan (2006).

[10]By comparison, Brown and Toft (2002) specify welfare loss of an unhedged position between 3.9% and 12.7% of *expected net income*. To obtain the same result in our example the coefficient of relative risk aversion would need to be roughly between 2.5 and 8, corresponding to normalized exposure λ between 0.5 and 1.6.

[11]Monthly returns on U.S. Gulf Coast kerosene prices in the period April 1990 to April 2007. Source: U.S. Department of Energy, Energy Information Administration.

It is evident that the optimal hedge ratios dictated by different utility functions are very similar to each other and to the OLS hedge ratio. One may nevertheless wonder about the welfare implication of using the OLS hedge when the hedger cares about higher moments of the return distribution. It might conceivably be the case that a small deviation from the optimal hedge ratio causes a large loss in the certainty equivalent wealth. In Table 2 we therefore report i) OLS HP, $g(\lambda, h_{OLS}) \times 1200$, the normalized welfare gain that results from using the second-best (i.e. the OLS) hedge ratio in each utility function, and ii) OHR HP, $g(\lambda, \hat{h}(\lambda)) \times 1200$, the welfare gain of the optimal hedge for each utility. Function g is defined via Theorem 2.4 with normalized utility given by (2.8) and (2.9). Because we use monthly data the multiplication by 1200 means we interpret the welfare gain as the percentage points increase in projected value per year.

[Table 2 about here]

What do these figures mean in our airline example? The welfare impact of hedging as opposed to no hedging is substantial and for all utility functions the hedging potential represents 440-500 basis points. However, this figure is the normalized welfare gain corresponding to a local risk aversion of unity and 100% exposure to the commodity. A simple conversion from (2.6) shows

$$\frac{\mathrm{WG}(v, \eta, h)}{v} = \mathrm{HP}(v, \eta, h) \times (A(v)v) \times \left(\frac{\eta}{v}\right)^2. \tag{2.15}$$

Our company has relative risk aversion $A(v)v = 5$ and its exposure to the commodity as a fraction of projected value is $\eta/v = 0.2$. Therefore the actual welfare gain translates to $5 \times 0.2^2 \times 500 = 100$ basis points from the projected book value of 4.0 bn, or 8% of expected net profit, which is roughly in the middle of the range [3.9%,12.7%] specified by Brown and Toft (2002). Note that as the curvature of the utility function (the deadweight loss function in Brown and Toft, 2002) decreases, $A(v) \to 0$, we move closer to the Modigliani-Miller world and the welfare gain from hedging tends to zero in equation (2.15).

Let us now evaluate the welfare loss from using an OLS hedge as opposed to using the optimal hedge ratio. The largest loss in hedging potential results for logarithmic utility and is equal to 0.1 basis points per annum. As we have argued above (see 2.15), the actual loss is 0.2 times smaller, amounting to 0.02 basis points from the projected value of $4 bn. In money terms, it represents $8,000.

The normalization above serves two purposes. Firstly, if we fix the shape of the HARA utility function (say exponential), then two different companies with the same λ will have exactly the same optimal hedge. The first company may have a smaller fuel exposure and higher risk aversion and the second conversely higher physical exposure to fuel price fluctuations but a lower degree of risk aversion. Such invariance is embedded in the definition of HARA utility and can be shown algebraically. More importantly, even if one uses different utility functions (for example quadratic vs. exponential), the optimal hedge ratios tend to be very similar, at least for economically plausible values of λ ($\lambda < 10$). This is no longer guaranteed by construction, but rather it is an empirical fact that comes out of our analysis.

A similar statement is true for the optimal hedging potential $g(\lambda, \hat{h}(\lambda))$. If we fix the shape of the HARA utility (for example as exponential), two different companies with the same λ will have identical percentage welfare gain *per unit of relative risk tolerance and per square of relative commodity exposure*. Significantly, this quantity has a non-degenerate limit as λ approaches 0 and in an unbiased futures market, the limit coincides across *all* utility functions. If we consider different utility functions (for example quadratic versus exponential), the normalized welfare gain is no longer identical across utility functions for a fixed value of λ, but it turns out empirically that for plausible values of lambda, the normalized hedging performance tends to be similar across different utility functions.

2.2 Hedgeable positions and asymptotics for large commodity exposure

Let us examine the dependence of the optimal hedge on the normalized exposure λ, for $\lambda \to \infty$. It is clear that for any utility with effective support bounded from below (i.e. utility functions, such as log utility, which take value of $-\infty$ for a finite argument) hedging will become infeasible for large enough values of λ, unless a perfect hedge is available. Hence for $0 < \gamma < \infty$ it is useful to know the range of λ values that are hedgeable. This leads to the notion of the *minimax hedge ratio* (MHR).

Definition 2.5 *Consider a random variable X with realizations $\{X_i\}_{i=1}^n$. Denote the worst case realization by $X_{\min} := \min\{X_i : i = 1, \dots, n\}$. We call $\bar{h}$ the minimax hedge ratio if it solves the problem*

$$\max_{h \in \mathbb{R}} \{(Y - hX)_{\min}\}. \tag{2.16}$$

The optimization (2.16) can be written as a linear program which admits a feasible solution under the no arbitrage assumption. Since the value function in (2.16) is bounded above by zero it follows that the minimax hedge always exists. The minimax hedge ratio itself need not be unique but the minimax return (the optimized value function) always is. We can now address the issue of hedgeable positions for HARA utility functions with $0 < \gamma < \infty$.

Proposition 2.6 *Denote by w the minimax return, $w := (Y - \bar{h}X)_{\min} \leq 0$. The normalized position λ is hedgeable for HARA utility with $0 < \gamma < \infty$ if and only if $0 < 1 + \lambda w/\gamma$.*

Proof. Available on request. ■

In the airline example, the minimax hedge ratio equals 1.3239 and the corresponding minimax return is -22.6%. Consider a firm with logarithmic HARA utility, corresponding to $\gamma = 1$. In this case, fuel price risk becomes unhedgeable if the normalized exposure exceeds $1/0.226 \approx 4.4$ which is a high but not implausible value. As λ approaches the unhedgeable threshold of 4.4 from below, the optimal hedge increasingly resembles the minimax hedge. In table 1, we have used $\lambda = 1$ which means that extreme events do not play a significant role in the hedging decision and the optimal log utility hedge does not depart significantly from the OLS hedge ratio.

We can now look at the behaviour of the optimal hedge ratio for large values of λ. As transpires from the previous discussion, this question is only interesting for $\gamma < 0$ and for $\gamma = \infty$ because for $0 < \gamma < \infty$, any large enough commodity exposure becomes unhedgeable. In Table 3, we report results for five utility functions: quadratic ($\gamma = -1$) which takes into account only the first two moments, quartic ($\gamma = -3$) including also (co-)skewness and (co-)kurtosis, and another two HARA utilities using the first 6 and the first 16 moments, respectively. Finally we employ exponential utility ($\gamma = \infty$) which involves all moments of the joint distribution.

For $\lambda < 10$ there is very little variation in the optimal hedge ratio across utility functions. Let us recall that in practice $\lambda = 10$ is a very high value corresponding to 100% exposure to the commodity and a high level of relative risk aversion (10). Nonetheless, there is no doubt that for extremely high levels of commodity exposure, higher moments do matter. Somewhat surprisingly, for exponential utility, the outcome for large λ is very close to the minimax hedge. One can view Table 3 as containing two polar cases – the OLS hedge ratio in the top left corner, and the minimax hedge ratio in the bottom right corner. (We explore the limit $\lambda \to \infty$ theoretically in proposition 2.7 below). The minimax HR is an ultra-cautious hedging strategy concerned only with the most extreme event captured by the data. On the other hand the OLS hedge ratio represents a more balanced strategy whereby one minimises the variance of hedged portfolio returns without focusing exclusively on the most extreme event in the data. The latter strategy will deliver superior performance most of the time (because it is calibrated to the entire body of data rather than to a single observation), but it may backfire if there is a temporary divergence between the spot and futures markets – Metalgesselschaft is one infamous victim of such divergence. The distance of the optimal hedge from the OLS hedge ratio is determined by two factors: firstly, how large the exposure to the commodity is, and secondly, how many moments of the return distribution one considers. From this angle, hedge ratios based on the first four moments are but an imperfect substitute to the more economically relevant minimax hedge ratio when commodity exposure is extremely high. Conversely, the choice of γ changes the balance between “ultra-conservative" and “balanced" hedging strategies. With $\gamma = -1$ one adopts the balanced approach no matter how high the exposure to the commodity. With $\gamma = \infty$ the right strategy is determined endogenously as a function of both the commodity exposure and the size of extreme events captured in the data.

[Table 3 about here]

Based on the results in table 3, one may conjecture that HARA hedge ratios have a well-defined limit for $\lambda \to \infty$. We capture the limit analytically in the following proposition.

Proposition 2.7 *1) Choose $\gamma \in (-\infty, 0)$ and set $\hat{h}^{(\gamma)}(\infty) := \lim_{\lambda\to\infty} \hat{h}^{(\gamma)}(\lambda)$. The limit is well defined and it can be obtained from the following minimization:*

$$\hat{h}^{(\gamma)}(\infty) = \arg\min_{h\in R} E(|Y - hX|^{1-\gamma}).$$

2) Suppose the minimax hedge $\bar{h}$ is unique, then the exponential utility[12] hedge ratio tends to $\bar{h}$ as $\lambda \to \infty$,

$$\hat{h}^{(\infty)}(\infty) := \lim_{\lambda\to\infty} \hat{h}^{(\infty)}(\lambda) = \bar{h}.$$

Proof. Available on request. ■

3 Data

The data, downloaded from Datastream International, comprise end-of-month spot and futures prices on 20 US commodities. This set of series was chosen on the grounds that it has been the subject of an important scrutiny in the literature (Ederington, 1979; Myers and Thompson, 1989; Baillie and Myers, 1991) and covers a wide range of commodities of interest to investors so that our results should have broad applicability. The cross-section covers 10 agricultural commodity futures (cocoa, coffee, corn, cotton, oats, soybean meal, soybean oil, soybeans, sugar and wheat), 2 energy

[12]The optimal hedge ratio will tend to the minimax hedge ratio as $\lambda \to \infty$ for any utility which is finite valued on $\mathbb{R}$, bounded from above and satisfies $\lim_{x\to-\infty} \frac{f(x)}{xf'(x)} = 0$. In the HARA class, the exponential is the only utility function with this property.

futures (heating oil and light crude oil), 5 metal futures (aluminum, copper, gold 100 oz, platinum and silver 1000 oz) and the futures on frozen pork bellies, lean hogs and lumber. To compile the time-series of futures prices, we collect the futures prices on all nearest and second nearest contracts. We hold the first nearby contract up to one month before maturity. At the end of that month, we roll our position over to the second nearest contract and hold that contract up to one month prior to maturity. Returns are then computed as the changes of these settlement prices. The procedure is then rolled forward to the next set of nearest and second nearest contracts when a new sequence of futures returns is compiled. The process is repeated throughout the dataset to generate a sequence of nearby maturity futures returns.

The characteristics of the underlying asset of the contract do not necessarily match those of the commodity that is being hedged. This is to be expected since futures contracts often amalgamate commodities with different grades or countries of origin. As a result, the return correlation between the spot asset and its corresponding futures ranges from a low of 0.27 for aluminum to 0.96 for gold with an average of 0.78.

The dataset covers the period January 31, 1979 to September 30, 2004. Note that we include in our analysis some commodity futures and spot assets that started trading after January 1979 or that were delisted before September 2004. As a result, the sample spans shorter periods for some contracts (aluminum, cocoa, copper, cotton, heating oil, lean hogs, light crude oil, lumber, silver and soybeans). Optimal hedge ratios are first constructed using the entire sample for estimation and for in-sample tests of hedging effectiveness. Then subsequently, hedging effectiveness is tested out-of-sample and for this purpose, the whole period is split into two sub-samples. The in-sample period covers approximately two-third of the dataset and is used for estimation. The out-of-sample period, used for forecasting and hedging decisions, covers the remaining one-third.

[Table 4 about here]

Table 4 presents some summary statistics for the futures returns, the spot returns, and for the hedged portfolio returns, where a time-invariant OLS hedge is employed. Most spot series are significantly leptokurtic and are positively skewed because events such as hurricanes or wars positively affect commodity prices. Hedging with futures is evidently very successful for the vast majority of the series. Compared with the spot return variance, the hedged portfolio variance is an average of 62% lower, and for gold, the reduction in variance is over 90%. However, interestingly, the skewness falls for the hedged portfolio returns in 13 of the 20 series compared with the spot skewness, falling by an average of 0.64, while the kurtosis rises for 15 of the series, by an average of 3.74. Thus, if we accept the premise that hedgers are indeed concerned with higher moments, then the effectiveness of the OLS hedge may be overstated by a consideration only of the reduction in variance.

[Table 5 about here]

The values of the minimax return (see section 2.2) for different commodities are shown in Table 5. Empirically, the smallest minimax loss (in absolute value) occurs for metals (6-11%). Most agricultural commodities have a minimax loss of about 20%; cotton is an outlier at 47%. At the other end of the agricultural spectrum we have soybean oil, soybeans and cocoa at 6.4%, 10.8% and 11.5%, respectively. Among energy futures, the minimax loss of light crude oil stands at 12.1% By contrast to 31.9% for heating oil. There appears to be no firm link between the minimax hedge and the OLS hedge ratio across different commodities. It also seems that the size of the discrepancy bears no relationship to the size of co-kurtosis and co-skewness in the data series.

4 Empirical Results

4.1 In-sample analysis

We have illustrated our higher moment hedging methodology on an airline company hedging its fuel exposure in section 2.1. We will now replicate the same computations for the 20 U.S. commodities described in the previous section. To avoid repetition, we refer the reader to section 2.1 where we have provided a thorough interpretation of the reported quantities. Table 6 measures i) OHR, the optimal hedge ratios obtained for each utility function, ii) OLS HP, $g(1, h_{OLS}) \times 1200$, the normalized welfare gain that results from using the second-best (i.e. the OLS) hedge ratio in each utility function, and iii) OHR HP $g(1, \hat{h}(1)) \times 1200$, the welfare gain of the optimal hedge for each utility. Function g is defined via Theorem 2.4 with normalized utility given by (2.8) and (2.9). The multiplication by 1200 means we interpret the welfare gain as the percentage points increase in initial wealth per year. We consider HARA utility functions with baseline risk aversion $\gamma \in \{-3, -1, 1, 5, \infty\}$ and their polynomial approximations[13] for $\gamma \in \{1, 5, \infty\}$. The framework allows us to examine a much wider range of parameters but we have found that all utility functions with $|\gamma| > 5$ essentially behave like the exponential utility, $\gamma = \infty$.

[Table 6 about here]

The results warrant two comments. First, the OLS hedge ratios only differ slightly from the utility-based hedge ratios. For example, all utility-based hedge ratios are within a range of 0.1 away from the OLS hedge ratios. This suggests that the OLS hedge ratio is a tractable and convenient first approximation of the utility-based hedge ratios. Second, overall, the welfare gain of the first-best hedge ratio only marginally exceeds, if at all, the hedging potential of the OLS hedge ratio. This tells us that per unit of risk aversion, hedgers increase their in-sample certainty equivalent wealth by only a very small amount when using the first-best hedge ratio as opposed to the OLS hedge ratio. In other words, taking higher moments into account does not substantially increase the welfare of the hedger, very much regardless of the utility function.

Take, for example, cocoa and assume HARA utility with $\gamma = 1$. The reward per unit of risk that is obtained from using the first-best hedge ratio exceeds that which is achieved with the standard OLS hedge ratio by a very marginal 0.002% per year. This suggests that using the OLS hedge ratio (of 0.853) as opposed to the OHR (of 0.873) generates a welfare loss of only 0.002% per year in a logarithmic HARA utility function. For the HARA utility with $\gamma = -3$, the welfare of the second-best is identical (to four decimal places) to the first best welfare. In other words, there is no increase in welfare that is achieved by using the utility-based hedge ratio.

Oats stands out as the commodity for which hedgers will get the maximum increase in welfare from using a utility-based hedge ratio for the majority of utility functions that we consider. Using the OLS hedge ratio (0.78) in the logarithmic HARA utility function generates a hedging potential of 4.08%. On the other hand, using the optimal hedge ratio estimated from the logarithmic HARA utility (0.87) generates an investment potential of 4.14 and thus increases welfare by, roughly, 0.06% a year. This 0.06% is the highest increase in welfare that can be achieved in-sample from using the utility-based hedge ratio as opposed to the OLS hedge ratio. By any standard, the increase in welfare is economically insignificant.

Of the eight utility functions depicted, the logarithmic HARA utility ($\gamma = 1$) is the one that generates the highest average yearly increase in welfare (0.008% across the 20 commodities). This

[13] For $\gamma \in \{-3, -1\}$ the HARA utility coincides with its polynomial approximation.

means that using the optimal hedge ratios, as opposed to the OLS hedge ratios, in the logarithmic utility function increases wealth by less than 0.01% per annum on average. The results are even less economically significant for the other utility functions, where the increase in welfare obtained with the optimal hedge relative to the standard OLS hedge is roughly 0.002%.

Intuitively, one would expect that spot series for which the OLS hedged portfolio returns show significant departures from normality (such as cotton, see Table 4), should show more considerable increases in welfare from the use of a utility-based hedge ratio estimate, but this is not true. For example, OLS-hedged oats returns show one of the lowest levels of excess kurtosis among the examined series, yet the welfare gain from departing from OLS is the highest. So it is perhaps the excess kurtosis of an unhedged position which determines the largest departure from OLS hedging. However, this hypothesis does not stand up to scrutiny either: gold and platinum exhibit high levels of non-normality in spot returns but offer virtually no gain from using the utility-based hedge. The fact that the welfare gain is most visible for logarithmic utility might suggest that extreme negative returns come into play. But this is not the case either. The OLS-hedged oats series has a lowest monthly return of -24% while frozen pork has a lowest return of -36% and yet no welfare gain is offered by using the utility-based hedge.

We could carry on this analysis by looking at the size of the co-skewness and co-kurtosis terms between spot and futures series. We could also take into account the signs of the co-skewness terms. There are many patterns one can employ, but having examined the results very carefully, we do not see a systematic way of predicting which commodity will benefit most from utility-based hedge ratios based on observed moments (or extreme returns) of the spot, futures and OLS-hedged portfolio data series.

[Table 7 about here]

Let us now look at the first four moments of the hedged portfolios depicted in Table 7. It is reasonable to expect that by comparing the first four moments of the hedged returns using OLS and optimal strategy, one will be able to identify commodities that benefit the most from incorporating higher moments. There are several commodities (cotton, light crude oil, oats) which exhibit measurable drops in kurtosis as one moves from the OLS to the optimal hedge ratio for log utility ($\gamma = 1$). These commodities indeed also produce the largest gains from switching to utility-based hedging. But this pattern is not universal. Silver exhibits a welfare gain similar to that of light crude oil but the kurtosis of the optimally hedged position actually increases compared to the OLS-hedged position. Among the commodities that do not substantially benefit in welfare terms from using utility-based hedging, we still observe measurable shifts in the first four moments. For example, the kurtosis and standard deviation of optimally hedged cocoa exposure ($\gamma = 1$) increase and this increase is compensated by a sizeable improvement in the mean return.

The main message from the above is that different choices of hedge ratios produce very subtle shifts in the first four moments. Just by looking at these shifts, it is very hard to gauge the overall impact on hedger's welfare. Our analysis is able to distinguish rigorously which of these shifts are actually welfare improving. Likewise, *a priori* it is very hard to tell by looking at the higher moments of the spot, futures and OLS-hedged return series which commodities will benefit from utility-based hedging. We provide a tool to answer this important question.

4.2 Out-of-sample analysis

4.2.1 Moderate commodity exposure

What increase in welfare can be achieved if one uses historical hedge ratios to determine appropriate hedging strategies for future time periods? Hedgers are assumed to update their information set once a month and to re-estimate their optimal hedge ratios accordingly. The new hedge ratios are then used as a basis for risk management over the following month. We calculate the resulting time series of returns according to equation (2.1). The out-of-sample (*ex post*) hedging potentials generated from different utility functions are reported in Table 8 for both the case when we inappropriately use the OLS hedge (OLS HP) and the case when we rightly use the utility-based optimal hedge (OHR HP).

[Table 8 about here]

There is no tendency *ex post* for the investment potential of the optimal hedge to exceed that of the OLS hedge. In other words, modelling the hedge ratios with the true distribution and thus taking into account higher moments does not necessarily increase the welfare of the hedger out-of-sample. To put it differently, there is no systematic loss in wealth that occurs from inappropriately using OLS hedging.

The main difference from the in-sample analysis is that out-of-sample one may end up better off by using OLS. This happens for 7 of the 20 commodities. The choice of OHR over OLS HR has no material impact on this outcome (both hedge ratios are equally good, or bad, as the case may be).

Let us take silver as an example. For all values of γ other than -1, the investment potential of the utility-based hedge ratio exceeds that of the OLS hedge ratio. In other words, adopting a more sophisticated approach to determining the HR in this case helps as it increases welfare by an incremental average return of 0.4% a year compared to the OLS hedge. At the other end of the spectrum, some commodities are better hedged with the OLS hedge ratio. Take, for example, cotton. Irrespective of the hedger's utility function, the *ex post* investment potential of the OLS hedge is higher than that of the predictive OHR. In effect, correctly modelling the optimal hedge ratio with a utility function decreases welfare by an average of 0.9%. This suggests that in this case, anything more sophisticated than OLS hedging actually hurts.

Bringing together the evidence of Table 8, it seems that there is no consistent support for the hypothesis that utility-based hedge ratios substantially increase welfare. The welfare benefits of using utility functions sensitive to higher moments are small but positive for 10 commodities (aluminium, cocoa, corn, gold, heating oil, light crude oil, oats, silver, soybean oil, and wheat). For the remaining 10 commodities, utility-based hedging is actually detrimental.

All else equal, a hedge ratio that is stable over time is preferable to one that is highly volatile in order to keep the transactions costs from rebalancing the hedged portfolio to a minimum. In order to investigate the variability of the estimated hedge ratios from the various techniques, Table 8 also reports the means and the standard deviations of the estimated 1-step ahead rolling hedge ratios. The means of the utility-based optimal hedge ratios are bigger than the means of the OLS hedge ratios for 13 of the 20 spot series hedged, while they are smaller for the remaining 7. Thus, most of the time, switching to a utility-based approach that explicitly incorporates higher moments leads to higher hedge ratios, commensurate with a more precise estimate of the risks associated with systematically leptokurtic return distributions. In almost all cases, OLS-based hedging yields HRs that have lower variances, indicating more stable hedge ratios and therefore a lower cost of hedging.

In fact, for only three series (soybean meal, lean hogs, and light crude oil), are the OLS HRs less stable than utility-based HRs out of sample.

In order to examine the relative sizes and stabilities of the estimated hedge ratios, Figures 1 to **??** plot the predictive HRs implied by OLS and various utility functions in the HARA class. The hedge ratios are estimated recursively using all in-sample data, with one observation added at each time step, for the cotton, gold, and soybean meal series respectively.[14] Figure 1 shows that in the case of cotton, the OLS hedge ratio is higher and less variable than those estimated from HARA utility functions, and in particular, logarithmic utility generates a dynamic OHR that has a lower mean but much higher variance than the others. Similarly, for gold (Figure 2), again the OLS hedge ratio is much less volatile than that of the other utility functions (although now the OLS hedge also has a lower average value). This increased variability of the utility-based hedge ratios suggests that more frequent rebalancing of the hedged portfolio would be required, which could have important consequences for the cost of implementing the hedges. Finally, Figure **??** illustrates that for some of the series, there is very little indeed to choose between the different hedge ratios, as indicated by the indistinguishability of the lines in the figure. In such cases, the temporal variation in the HRs is much more significant than the contemporaneous differences across the HRs.

[Table 9 about here]

Table 9 repeats the format of Table 7 but now for consideration of the out-of-sample rather than in-sample moments. Comparing the results with the no hedge case, both the OLS and the utility function-based hedges successfully reduce the variance of the hedged portfolio returns – sometimes moderately (for example, in the case of Soybean meal) and sometimes spectacularly (e.g., gold). Perhaps precisely because by design OLS will minimize the (in sample) variance of the hedged portfolio returns, it also results in out of sample portfolio variances that are never higher than those of the utility-based hedges. Also of interest are the impacts of hedging using the various methods on the higher moments of the hedged portfolio returns out of sample. For some series (e.g., aluminium, frozen pork, lean hogs, lumber, wheat), there is no noticeable difference between the skewness and kurtosis of the hedged portfolio returns across the OLS model and the entire HARA family. However, for other cases (e.g. corn, cotton, heating oil, silver, and especially gold), there is a marked difference. Focusing on the gold case, use of the HARA utility function with λ of 1 gives a return distribution with higher mean, lower variance, lower skewness, lower kurtosis, but a larger minimum loss than the corresponding OLS hedge.

4.2.2 Large commodity exposure

Consider a high value of normalized exposure, $\lambda = 10$. With the exception of cotton, the above conclusions remain unchanged, i.e. OLS hedge ratios are no better or worse than utility-based hedge ratios in terms of certainty equivalent wealth while being more stable over time. The cotton series shows that with large commodity exposures, paying too much attention to higher moments can be counterproductive. For example, the exponential utility hedge ratio is far inferior to the OLS hedge. This is reflected in its negative hedging potential as well as negative mean return and an almost triple standard deviation compared to the OLS hedge.

[Table 10 about here]

[14]The three figures are shown for illustration and we do not include plots for all 20 series due to space constraints, but comparable figures for every asset in our dataset are available from the authors on request.

The example is symptomatic of a more general issue that is pervasive in finance. Suppose the data generating process behind the spot and futures returns for cotton is accurately represented by the historical data. Suppose further that the manager of company A ignores extreme negative returns in the past data, arguing that such extreme events are unlikely to occur again in future. Consequently, manager A selects a hedging strategy that very much resembles an OLS hedge. By contrast, the manager of company B heeds the warning issued by the data and selects a hedging strategy leaning towards the minimax hedge ratio. The problem for manager B is that until the extreme scenario captured in the historical data repeats itself, his strategy will be extremely costly. This is exemplified in table 10 by HARA utility-based hedging strategies with $\gamma = -15, \infty$ in particular, although the loss in performance is visible for all utility-based strategies.

5 Conclusions

This study has proposed a utility-based framework for the determination of optimal hedge ratios that can allow for the impact of higher moments on the hedging decision. The approach is applied to a set of 20 commodities that are hedged with futures contracts. We find that in sample, the utilities of hedges constructed allowing for non-zero higher moments are only very slightly higher than those determined by much simpler OLS. When implemented out of sample, utility-based hedge ratios are usually less stable over time, and can make investors worse off for some assets than hedging using the traditional methods.

To the extent that using a considerably more sophisticated approach does not reliably improve upon simple OLS hedging, our results confirm those of Harris and Shen (2006). They are unable to find any consistent improvement in minimum value at risk with additional complexity. Similarly, Jondeau and Rockinger (2006) show that under many circumstances, incorporating higher moments does not affect asset allocation decisions; Post et al. (2002) reach comparable conclusions concerning the usefulness of co-skewness in explaining the cross-sectional variation in asset returns. Thus, in summary, our findings add to a growing body of very recent literature suggesting that higher moments matter in theory but not in practice.

The practical implementation of hedging strategies requires a consideration of returns on a net of transactions costs basis. We conjecture that, given the lack of welfare benefits from utility-based hedge ratio estimation even on a gross basis, this approach is likely to be less attractive still once reasonable transactions costs are accounted for. These non-parametric hedge ratios are typically less stable than those estimated using mean-variance analysis, with a consequent need for more frequent and larger rebalancing. While it may potentially be useful to determine whether our broad conclusions also hold for other hedging assets, sample periods and data frequencies, we believe that given the strength of our results and the array of commodities we considered, this is overwhelmingly likely to be the case.

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A Optimal hedging and OLS

Assuming sufficient smoothness ($f \in \mathcal{C}^2$) the quantity $\alpha(\lambda)$ is differentiable and we can think of the optimal hedge $\hat{h}(\lambda)$ as the *average* value of the marginal hedge ratios $-\alpha'(s)$ with $s \in [0, \lambda]$,

$$\hat{h}(\lambda) = -\frac{\int_0^\lambda \alpha'(s)\, ds}{\lambda}.$$

By differentiating the first order condition $E\left(X f'\left(\alpha\left(\lambda\right) X + \lambda Y\right)\right) = 0$ with respect to λ we have

$$\begin{aligned} E\left(X^2 f''\left(\lambda Y + \alpha\left(\lambda\right) X\right)\right) \alpha'\left(\lambda\right) &= -E\left(XY f''\left(\lambda Y + \alpha\left(\lambda\right) X\right)\right), \\ \alpha'\left(\lambda\right) &= -\frac{E\left(XY f''\left(\lambda Y + \alpha\left(\lambda\right) X\right)\right)}{E\left(X^2 f''\left(\lambda Y + \alpha\left(\lambda\right) X\right)\right)}. \end{aligned}$$

In the special case $f'' = \mathrm{const}$, corresponding to quadratic utility, we obtain

$$\alpha'\left(\lambda\right) = -\frac{E\left(XY\right)}{E\left(X^2\right)},$$

which means that $\hat{h}(\lambda)$ is independent of λ. If, in addition, the mean of X is zero (the futures market is unbiased) then the quadratic hedge equals the slope coefficient from the OLS regression of Y onto X and an intercept. For other utility functions, the choice of λ matters to some extent, but our numerical results show that this dependence is extremely weak for $\lambda \in [0, 1]$.

B HARA utility

Some utility functions in the HARA class already appear in Pratt (1964). The full family is described in Cass and Stiglitz (1970). We use a slight modification of the parametrization suggested in Ingersoll (1987).

Definition B.1 *The utility function*

$$U^{(\gamma)}(V;a,b) := \begin{cases} \frac{(aV/\gamma+b)^{1-\gamma}-1}{1/\gamma-1} & \text{for } \gamma > 0, \gamma \neq 1, \\ \ln(aV+b) & \text{for } \gamma = 1, \\ \frac{|aV/\gamma+b|^{1-\gamma}-1}{1/\gamma-1} & \text{for } \gamma < 0, \\ 1-e^{-aV} & \text{for } \gamma = \pm\infty, \end{cases}$$

with $a > 0$ is called the HARA (hyperbolic absolute risk-aversion) utility. We denote the corresponding effective domain by $\mathcal{D}^{(\gamma)}(a,b)$ and the maximal domain on which $U^{(\gamma)}$ is increasing by $\bar{\mathcal{D}}^{(\gamma)}(a,b)$.

The HARA utility is an infinitely differentiable utility in the sense of Definition 2.1. $U^{(\gamma)}$ is strictly increasing and unbounded from above for $\gamma \in (0,1]$; it is strictly increasing and bounded from above for $\gamma > 1$ and for $\gamma < 0$ it has a bliss point at $-\gamma b/a$. The coefficient of absolute risk aversion at $v \in \bar{\mathcal{D}}^{(\gamma)}(a,b)$ reads

$$A^{(\gamma)}(v;a,b) = \frac{1}{v/\gamma + b/a},$$

hence the acronym HARA. The HARA class has several advantages over the more frequently used power utility functions. Fixing a positive initial wealth level v one can, with an appropriate choice of a, b, make the HARA utility increasing at v even when $\gamma < 0$. Secondly, power utility ($\gamma > 0, b = 0$) produces unreasonable levels of risk aversion for large values of γ. This can be corrected in the HARA class by selecting an appropriate value of $b > 0$.

Proposition B.2 *Fix $\gamma \in \mathbb{R} \cup \{\pm\infty\}, \gamma \neq 0$, $a > 0, b \in \mathbb{R}$ and $v \in \bar{\mathcal{D}}^{(\gamma)}(a,b)$. Then $f^{(\gamma)}$ from equation (2.8) is a normalized utility to $U^{(\gamma)}$ at v, in the sense of definition 2.3. Consequently the normalized utility can be taken independent of the specific value of a, b and v.*

Proof. By direct calculation for $\gamma > 0, \gamma \neq 1$

$$\begin{aligned} c_1 U^{(\gamma)}(v + z/A^{(\gamma)}(v)) + c_2 &= c_1 \frac{(\gamma^{-1}a(v + z(\gamma^{-1}v + a^{-1}b)) + b)^{1-\gamma} - 1}{\gamma^{-1} - 1} + c_2 \\ &= c_1 \frac{(\gamma^{-1}av + b)^{1-\gamma}(1 + \gamma^{-1}z) - 1}{\gamma^{-1} - 1} + c_2, \end{aligned}$$

The value of $f^{(\gamma)}$ in equation (2.8) corresponds to the choice of constants

$$\begin{aligned} c_1 &= (av/\gamma + b)^{\gamma-1}, \\ c_2 &= \frac{c_1 - 1}{1} \end{aligned}$$

in equation (2.7). Similar calculations apply to the remaining values of γ. ■

Intuitively, $f^{(\gamma)}(z)$ in equation (2.8) is normalized to have absolute risk aversion of 1 at $z = 0$. If we restrict our attention to the economically meaningful values $v > 0$ we can obtain an alternative re-parametrization of the HARA utility by applying an affine transformation to $f^{(\gamma)}(z)$,

$$\bar{U}^{(\gamma)}(V; v, \tilde{\gamma}) := f^{(\gamma)}(\tilde{\gamma}(V/v - 1)). \tag{B.17}$$

Here $\tilde{\gamma}$ is (by construction) the coefficient of relative risk aversion of utility $\bar{U}^{(\gamma)}$ at v. While power utility forces relative risk aversion always equal to γ the HARA class allows the RRA coefficient to be chosen independently, which is particularly clear in the parametrization (B.17).

C Numerical algorithm

The problem

$$\begin{aligned} \alpha &= \arg\max_{\vartheta \in \mathbb{R}^n} E\left(f\left(Y + \vartheta X\right)\right), \\ a &= f^{-1}\left(E\left(f\left(Y + \alpha X\right)\right)\right), \end{aligned}$$

can be solved by Newton's iteration method provided that the initial guess ϑ_0 is close to the optimal portfolio α. In practice, the quadratic approximation $\vartheta_0 = -E(XY)/E(X^2)$ works very well. We define $g : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$

$$g(\vartheta) = E\left(f\left(Y + \vartheta X\right)\right)$$

and assume that in each iteration $g(\vartheta) > -\infty$. Starting at ϑ_0 we use the iteration

$$\vartheta_{k+1} = \vartheta_k - \frac{g'(\vartheta_k)}{g''(\vartheta_k)},$$

where

$$\begin{aligned} g'\left(\vartheta\right) &= E\left(X f'\left(Y + \vartheta X\right)\right), \\ g''\left(\vartheta\right) &= E\left(X^2 f''\left(Y + \vartheta X\right)\right). \end{aligned}$$

Assuming sufficient smoothness of f the Taylor expansion yields

$$|f^{-1}(g(\vartheta_k)) - f^{-1}(g(\alpha))| = -\frac{1}{2f'(f^{-1}(g(\vartheta_k)))} \frac{(g'(\vartheta_k))^2}{g''(\vartheta_k)} + o((\vartheta_k - \alpha)^2). \tag{C.18}$$

Accordingly, we stop the iteration when

$$-\frac{(g'(\vartheta_k))^2}{f'(f^{-1}(g(\vartheta_k)))g''(\vartheta_k)} < 10^{-12},$$

which in practice guarantees $|f^{-1}(g(\vartheta_k)) - f^{-1}(g(\alpha))| < 10^{-12}$. The last inequality means a very close proximity of the final iterate to the optimal hedging decision in terms of the resulting certainty equivalent. Equation (C.18) hints that $|\vartheta_k - \alpha|$ has half the number of zeros compared to the target function, i.e. in practice the final iterate satisfies $|\vartheta_k - \alpha| \approx 10^{-6}$. Rigorous proof of the quadratic convergence can be found, for example, in Dennis and Schnabel (1996).

Table 1: Optimal hedge ratios for airline fuel exposure. Illustrative example for normalized exposure $\lambda = 1$. γ determines the shape of HARA utility function. Skewness and kurtosis of the futures (spot) return distributions are respectively: 0.3, (0.2); 4.2, (5.2)

γ	-1	-3	1	1	5	5	∞	∞
			HARA	POLY	HARA	POLY	HARA	POLY
OHR	0.9511	0.9487	0.9654	0.9668	0.9520	0.9524	0.9502	0.9504

Table 2: Normalized welfare gain from using optimal and OLS hedge ratios, respectively. Illustrative example for normalized exposure $\lambda = 1$ in jet fuel.

γ	-1	-3	1	1	5	5	∞	∞
			HARA	POLY	HARA	POLY	HARA	POLY
OLS HP $\times 1200$	4.5626	4.3793	4.9474	5.0521	4.4527	4.4821	4.4045	4.4173
OHR HP $\times 1200$	4.5626	4.3793	4.9484	5.0533	4.4527	4.4821	4.4045	4.4173

Table 3: Optimal hedge ratios for jet fuel as a function of normalized fuel exposure λ. Parameter γ governs the shape of utility function – see (2.8).

$\gamma\backslash\lambda$	0	1	10	100	1000
-1	**0.9511**	0.9511	0.9511	0.9511	0.9511
-3	0.9468	0.9487	0.9696	1.0478	1.0323
-5	0.9463	0.9491	0.9849	1.1823	1.1404
-15	0.9458	0.9498	1.0093	1.3054	1.2770
∞	0.9456	0.9502	1.0275	1.3184	**1.3224**

Table 4: Summary statistics for spot and future commodity returns.

	Spot Commodity					Futures Contract					OLS Hedged Portfolio				
	Mean	Std	Skew	Kurt	JB	Mean	Std	Skew	Kurt	JB	Mean	Std	Skew	Kurt	JB
	%	%				%	%				%	%			
Aluminum	0.5	10.0	0.25	4.03*	43*	3.1	14.4	-0.12	-0.16	0.2	-0.1	9.6	0.45	5.22*	73*
Cocoa	1.6	26.3	0.72*	1.31*	39*	-8.0	29.0	0.54*	1.42*	33.0*	8.4*	8.9	-0.60*	8.19*	713*
Coffee	4.6	38.0	1.15*	4.27*	301*	-4.7	38.3	0.56*	1.57*	47.7*	8.6**	19.3	0.68*	4.38*	270*
Copper	6.8	30.7	0.50*	1.37*	37*	2.4	31.5	0.28	3.65*	74.4*	6.4	19.3	0.07	-0.32	1
Corn	2.1	25.5	0.25***	3.83*	191*	-8.0***	21.8	0.49*	4.21*	240.2*	10.4*	12.0	-1.10*	5.06*	390*
Cotton	2.0	28.4	-0.71*	9.58*	1156*	-3.0	24.8	0.14	1.10*	15.8*	4.6	18.6	-6.48*	81.38*	83762*
Frozen Pork	17.5	58.2	0.74*	2.24*	92*	-9.6	39.1	-0.12	1.16*	17.9*	26.7*	44.9	0.47*	2.96*	124*
Gold	4.0	18.8	1.11*	5.83*	500*	-3.6	18.7	0.30**	5.19*	350.1*	7.5*	5.2	1.15*	10.69*	1535*
Heating Oil	8.4	38.3	0.94*	5.16*	342*	15.8**	34.9	0.53*	2.99*	114.0*	-4.7	25.3	0.14	5.19*	306*
Lean Hogs	3.4	33.7	0.48*	2.30*	55*	3.0	25.7	-0.44*	2.20*	63.0*	0.1	24.7	0.15	1.50*	21*
Light Crude	9.7	38.9	0.93*	5.02*	308*	11.2	33.0	0.02	2.32*	57.9*	-2.3	15.9	1.15*	5.80*	419*
Lumber	10.2	39.2	0.80*	1.46*	23*	-4.4	31.5	0.40***	0.12	3.2	13.3	32.8	0.95*	1.56*	29*
Oats	4.2	29.6	1.79*	14.46*	2850*	-9.6	30.0	1.54*	11.59*	1845.3*	11.7*	18.2	-0.08	2.14*	59*
Platinum	6.4	25.8	0.35*	4.55*	272*	0.1	26.7	-0.57*	6.73*	598.6*	6.3*	7.7	0.97*	4.93*	361*
Silver	5.6	32.9	0.57*	10.88*	1537*	-13.5**	26.1	-0.07	2.39*	61.3*	11.7*	7.9	1.05*	6.39*	484*
Soybean Meal	3.4	23.6	0.47*	4.82*	196*	2.2	23.7	-0.33***	2.85*	69.4*	2.4	21.3	1.15*	9.60*	792*
Soybean Oil	3.4	28.1	0.70*	2.96*	138*	-5.5	25.6	0.36*	2.09*	62.6*	9.0*	10.0	0.40*	1.23*	28*
Soybeans	1.4	23.0	-0.30**	3.20*	136*	-5.0	22.4	-0.27***	2.26*	69.3*	6.3*	7.0	-1.51*	5.66*	528*
Sugar	7.4	37.4	0.99*	2.62*	138*	-9.8	41.9	0.35*	2.01*	58.3*	14.8*	20.4	0.00	2.96*	112*
Wheat	3.1	20.3	0.19	2.00*	53*	-8.5**	21.6	0.14	0.58**	5.4***	8.3*	15.3	0.06	2.24*	65*

Note: One, two or three stars indicate significance at 1%, 5% and 10% level, respectively.

Table 5: Comparison of OLS and MINIMAX in-sample hedge ratios.

	hedge ratio		minimax loss	(co)skewness	excess (co)kurtosis
	OLS	MINIMAX	%		
Aluminum	0.183	-0.074	8.4	-0.1 -0.1 -0.2 0.2	-0.3 0.0 -0.1 -0.5 3.6
Cocoa	0.853	0.448	11.5	0.5 0.6 0.7 0.7	1.4 1.3 1.3 1.3 1.3
Coffee	0.855	1.241	21.9	0.6 0.6 0.8 1.1	1.5 1.8 2.4 3.1 4.2
Copper	0.898	1.075	11.1	0.3 0.4 0.4 0.5	3.5 2.6 1.8 1.3 1.0
Corn	1.033	0.519	15.3	0.5 0.4 0.4 0.3	4.1 4.0 3.8 3.7 3.7
Cotton	0.862	-1.203	47.2	0.1 0.2 0.7 -0.7	1.1 1.0 1.3 -0.8 9.4
Frozen Pork	0.948	0.657	36.0	-0.1 0.2 0.4 0.7	1.1 0.5 0.4 0.7 2.2
Gold	0.966	0.587	7.0	0.3 0.6 0.8 1.1	5.1 4.9 5.0 5.3 5.7
Heating Oil	0.825	0.911	31.9	0.5 0.4 0.6 0.9	2.9 1.3 1.5 2.7 5.0
Lean Hogs	0.955	1.653	21.6	-0.2 0.1 0.3 0.5	1.1 0.5 0.5 1.0 2.2
Light Crude	1.078	0.971	12.1	0.0 0.4 0.7 0.9	2.3 2.3 2.9 3.8 4.9
Lumber	0.683	0.703	16.0	0.4 0.1 0.2 0.8	0.1 -0.1 0.0 0.3 1.3
Oats	0.777	0.006	19.9	1.5 1.6 1.7 1.8	11.4 12.1 12.8 13.4 14.2
Platinum	0.921	0.797	6.2	-0.6 -0.3 0.0 0.3	6.6 5.7 5.0 4.6 4.5
Silver	0.927	0.525	8.4	-0.1 0.2 0.3 0.5	2.3 2.0 1.8 1.7 1.7
Soybean Meal	0.431	0.331	19.3	-0.3 0.0 -0.3 0.5	2.7 2.0 3.0 0.8 4.7
Soybean Oil	1.024	0.994	6.4	0.4 0.5 0.6 0.7	2.0 2.2 2.4 2.6 2.9
Soybeans	0.978	1.272	10.8	-0.3 -0.2 -0.2 -0.3	2.2 2.3 2.5 2.8 3.1
Sugar	0.748	0.625	21.8	0.3 0.6 0.8 1.0	2.0 1.8 1.9 2.1 2.6
Wheat	0.621	0.311	17.5	0.1 0.1 0.1 0.2	0.6 0.6 0.7 0.9 1.9

Notes: Minimax loss represents the worst case loss of a portfolio hedged with the minimax hedge ratio. Coskewness values correspond to sample versions of the following population moments: $E((X-\mu_X)^3)/\sigma_X^3, E((X-\mu_X)^2(Y-\mu_Y))/(\sigma_X^2\sigma_Y), E((X-\mu_X)(Y-\mu_Y)^2)/(\sigma_X\sigma_Y^2), E((Y-\mu_Y)^3)/\sigma_Y^3$. Excess cokurtosis values correspond to sample versions of the following population moments: $E((X-\mu_X)^4)/\sigma_X^4-3, E((X-\mu_X)^3(Y-\mu_Y))/(\sigma_X^3\sigma_Y)-3\rho_{XY}, E((X-\mu_X)^2(Y-\mu_Y)^2)/(\sigma_X^2\sigma_Y^2)-1-2\rho_{XY}^2, E((X-\mu_X)(Y-\mu_Y)^3)/(\sigma_X\sigma_Y^3)-3\rho_{XY}, E((Y-\mu_Y)^4)/\sigma_Y^4-3$. Here μ_X, μ_Y are the means and σ_X, σ_Y are the standard deviations of the futures and spot returns, respectively. ρ_{XY} denotes the correlation between futures and spot returns. The slight discrepancy between skewness and kurtosis values in this table and table 4 is caused by different estimators. This table uses consistent estimators of co-moments which however are not unbiased in a normal model. Table 4 employs so-called G_1, G_2 estimators which are unbiased in a normal model (cf. Joanes and Gill, 1998). Similar estimators for normalized co-moments are not readily available.

Table 6: In-sample hedge ratios and their *ex ante* performance.

commodity		γ							
OLS HR		-1	-3	1		5		∞	
				HARA	POLY	HARA	POLY	HARA	POLY
ALUMINUM	OHR	0.1826	0.1866	0.1973	0.1967	0.1904	0.1903	0.1889	0.1889
0.1828	OLS HP	0.0348	0.0363	0.0418	0.0415	0.0381	0.0381	0.0374	0.0374
	OHR HP	0.0348	0.0363	0.0421	0.0417	0.0382	0.0381	0.0374	0.0374
COCOA	OHR	0.8463	0.8563	0.8732	0.8713	0.8635	0.8633	0.8609	0.8608
0.8530	OLS HP	3.0529	3.1315	3.3875	3.3689	3.2177	3.2158	3.1828	3.1821
	OHR HP	3.0531	3.1315	3.3894	3.3705	3.2182	3.2162	3.1831	3.1824
COFFEE	OHR	0.853	0.8527	0.8484	0.8485	0.8516	0.8516	0.8521	0.8521
0.8553	OLS HP	5.3606	5.2869	5.2057	5.222	5.2456	5.2477	5.2595	5.2604
	OHR HP	5.3607	5.287	5.206	5.2224	5.2457	5.2478	5.2596	5.2605
COPPER	OHR	0.8991	0.8948	0.8878	0.8881	0.8918	0.8918	0.8929	0.8929
0.8978	OLS HP	3.9998	3.8743	3.8181	3.8388	3.8222	3.8256	3.8371	3.8385
	OHR HP	3.9999	3.8744	3.8186	3.8392	3.8224	3.8258	3.8372	3.8386
CORN	OHR	1.0184	1.0306	1.0668	1.052	1.0419	1.0403	1.0374	1.0368
1.0329	OLS HP	2.5082	2.5881	3.2604	3.007	2.7459	2.7244	2.6744	2.667
	OHR HP	2.5087	2.5882	3.2642	3.008	2.7461	2.7246	2.6744	2.667
COTTON	OHR	0.8604	0.8533	0.7674	0.8176	0.8401	0.8424	0.8463	0.847
0.8623	OLS HP	2.2765	2.2187	1.8207	2.0259	2.1441	2.1539	2.1769	2.1801
	OHR HP	2.2765	2.2189	1.8494	2.032	2.1456	2.1551	2.1777	2.1808
FROZEN PORK	OHR	0.9337	0.9518	0.9544	0.9494	0.9585	0.9573	0.9568	0.9563
0.9477	OLS HP	6.7604	6.7039	6.6696	6.7109	6.6832	6.6885	6.6892	6.6914
	OHR HP	6.7619	6.704	6.67	6.711	6.6841	6.6892	6.6898	6.6919
GOLD	OHR	0.9598	0.9742	1.001	0.9977	0.9851	0.9847	0.9811	0.9809
0.9662	OLS HP	1.6244	1.6579	1.7733	1.7676	1.6964	1.6959	1.6808	1.6806
	OHR HP	1.6245	1.658	1.7755	1.7694	1.697	1.6965	1.6812	1.681
HEATING OIL	OHR	0.8197	0.8074	0.8023	0.8165	0.803	0.8057	0.8043	0.8054
0.8246	OLS HP	4.1521	3.7077	4.4804	4.7656	3.7341	3.8307	3.6915	3.734
	OHR HP	4.1522	3.7094	4.4836	4.7661	3.7368	3.8328	3.6939	3.7361
LEAN HOGS	OHR	0.9556	0.9423	0.9302	0.9323	0.9352	0.9356	0.9375	0.9377
0.9555	OLS HP	2.6371	2.5399	2.5633	2.5928	2.5165	2.5217	2.5202	2.5223
	OHR HP	2.6371	2.5403	2.5652	2.5943	2.5177	2.5228	2.5211	2.5232

Continued on next page.

Table 6: Continued from previous page.

commodity		γ							
OLS HR		-1	-3	1		5		∞	
				HARA	POLY	HARA	POLY	HARA	POLY
LIGHT CRUDE	OHR	1.0762	1.0505	1.0223	1.0391	1.0363	1.039 1	1.0411	1.0422
1.0781	OLS HP	6.3277	5.7828	6.6023	6.9565	5.8014	5.9019	5.7558	5.7999
	OHR HP	6.3277	5.7868	6.6199	6.9653	5.8106	5.9101	5.763	5.8067
LUMBER	OHR	0.6778	0.6726	0.6524	0.6542	0.6662	0.6664	0.6689	0.6689
0.6827	OLS HP	2.2788	2.2512	2.1974	2.208	2.2306	2.2319	2.2383	2.2388
	OHR HP	2.2789	2.2517	2.2023	2.2123	2.232	2.2332	2.2393	2.2398
OATS	OHR	0.7671	0.7922	0.8746	0.8433	0.8185	0.8146	0.8078	0.8063
0.7775	OLS HP	2.6935	2.9648	4.0755	3.7347	3.3014	3.2611	3.1604	3.1455
	OHR HP	2.694	2.9658	4.1356	3.7604	3.3103	3.2683	3.1651	3.1497
PLATINUM	OHR	0.9215	0.921	0.9203	0.9203	0.9207	0.9207	0.9208	0.9208
0.9214	OLS HP	3.0384	3.0122	3.0285	3.0259	3.007	3.0071	3.0073	3.0074
	OHR HP	3.0384	3.0122	3.0285	3.0259	3.007	3.0071	3.0073	3.0074
SILVER	OHR	0.9076	0.9419	0.9947	0.9668	0.9651	0.9601	0.9567	0.9547
0.9265	OLS HP	2.9066	2.9765	3.7111	3.5862	3.167	3.1621	3.0811	3.0804
	OHR HP	2.9079	2.9773	3.73	3.5929	3.1721	3.166	3.0842	3.0831
SOYBEAN MEAL	OHR	0.4322	0.436	0.449	0.4478	0.4402	0.4401	0.4385	0.4385
0.4314	OLS HP	0.522	0.5309	0.5988	0.5941	0.5492	0.549	0.5412	0.5411
	OHR HP	0.522	0.531	0.5997	0.5949	0.5494	0.5492	0.5413	0.5413
SOYB. OIL	OHR	1.0182	1.0272	1.0459	1.0442	1.0345	1.0343	1.0318	1.0317
1.0245	OLS HP	3.4276	3.4665	3.6222	3.6086	3.5155	3.5142	3.4952	3.4947
	OHR HP	3.4277	3.4665	3.6238	3.6099	3.5158	3.5145	3.4953	3.4949
SOYBEANS	OHR	0.9732	0.9775	0.9868	0.986	0.9811	0.981	0.9797	0.9797
0.9785	OLS HP	2.3901	2.3724	2.4063	2.4079	2.3749	2.3753	2.3722	2.3724
	OHR HP	2.3901	2.3724	2.4065	2.4081	2.3749	2.3754	2.3722	2.3724
SUGAR	OHR	0.7416	0.7573	0.7798	0.7765	0.7677	0.7672	0.764	0.7638
0.7485	OLS HP	4.8832	4.9641	5.1873	5.1688	5.0432	5.041	5.012	5.0112
	OHR HP	4.8836	4.9648	5.1961	5.1759	5.0465	5.0441	5.0141	5.0132
WHEAT	OHR	0.6088	0.6178	0.6297	0.6296	0.624	0.6238	0.6218	0.6218
0.6213	OLS HP	0.8838	0.8977	1.0113	0.9908	0.927	0.9256	0.9139	0.9135
	OHR HP	0.8842	0.8977	1.0115	0.991	0.927	0.9256	0.9139	0.9135

Notes: The OLS hedge is computed as $h_{\mathrm{OLS}} = Cov(X,Y)/Var(X)$. The optimal hedge for each utility function is given by $\hat{h}(1)$ from Theorem 2.4. OLS HP shows the welfare gain from using the OLS hedge ratio, $g(1, h_{\mathrm{OLS}})$. The OHR HP gives the welfare gain from using the optimal hedge ratio, $g(1, \hat{h}(1))$, see Theorem 2.4. For definitions of HARA utility and its polynomial approximation (POLY), see equations (2.8, 2.9) and Appendix B.

Table 7: Moments of hedged commodity returns; in-sample results.

commodity	OLS	utility							
		-1	-3	1		5		∞	
				HARA	POLY	HARA	POLY	HARA	POLY
ALUMINUM									
mean (% p.a.)	-0.12	-0.12	-0.13	-0.17	-0.16	-0.14	-0.14	-0.14	-0.14
std (% p.a.)	33.42	33.42	33.42	33.43	33.43	33.42	33.42	33.42	33.42
skew	0.44	0.44	0.44	0.45	0.45	0.45	0.45	0.44	0.44
kurt	4.72	4.72	4.73	4.75	4.75	4.74	4.74	4.73	4.73
min (% p.c.m.)	-8.62	-8.62	-8.63	-8.64	-8.64	-8.63	-8.63	-8.63	-8.63
COCOA									
mean (% p.a.)	8.43	8.37	8.45	8.59	8.57	8.51	8.51	8.49	8.49
std (% p.a.)	30.86	30.87	30.86	30.92	30.91	30.88	30.88	30.87	30.87
skew	-0.59	-0.59	-0.59	-0.58	-0.58	-0.59	-0.59	-0.59	-0.59
kurt	8.00	7.86	8.06	8.34	8.31	8.19	8.19	8.15	8.15
min (% p.c.m.)	-12.30	-12.29	-12.31	-12.34	-12.34	-12.32	-12.32	-12.32	-12.32
COFFEE									
mean (% p.a.)	8.63	8.62	8.62	8.60	8.60	8.61	8.61	8.61	8.61
std (% p.a.)	66.69	66.69	66.69	66.70	66.70	66.69	66.69	66.69	66.69
skew	0.68	0.68	0.69	0.70	0.70	0.69	0.69	0.69	0.69
kurt	4.29	4.30	4.30	4.31	4.31	4.30	4.30	4.30	4.30
min (% p.c.m.)	-22.66	-22.66	-22.66	-22.67	-22.67	-22.66	-22.66	-22.66	-22.66
COPPER									
mean (% p.a.)	6.44	6.44	6.45	6.46	6.46	6.45	6.45	6.45	6.45
std (% p.a.)	66.98	66.98	66.98	66.99	66.99	66.99	66.99	66.99	66.99
skew	0.07	0.07	0.07	0.07	0.07	0.07	0.07	0.07	0.07
kurt	-0.35	-0.36	-0.35	-0.33	-0.33	-0.34	-0.34	-0.34	-0.34
min (% p.c.m.)	-12.00	-11.99	-12.02	-12.05	-12.05	-12.03	-12.03	-12.03	-12.03
CORN									
mean (% p.a.)	10.38	10.27	10.37	10.66	10.54	10.46	10.44	10.42	10.42
std (% p.a.)	41.47	41.48	41.47	41.55	41.49	41.47	41.47	41.47	41.47
skew	-1.09	-1.09	-1.09	-1.08	-1.09	-1.09	-1.09	-1.09	-1.09
kurt	4.96	4.92	4.95	4.99	4.99	4.97	4.97	4.96	4.96
min (% p.c.m.)	-17.56	-17.49	-17.55	-17.71	-17.64	-17.60	-17.59	-17.58	-17.57
COTTON									
mean (% p.a.)	4.60	4.59	4.57	4.32	4.47	4.53	4.54	4.55	4.55
std (% p.a.)	64.50	64.50	64.50	65.01	64.61	64.52	64.52	64.51	64.51
skew	-6.45	-6.45	-6.42	-5.98	-6.26	-6.37	-6.38	-6.39	-6.40
kurt	80.00	79.91	79.56	73.34	77.41	78.84	78.97	79.19	79.23
min (% p.c.m.)	-66.53	-66.51	-66.44	-65.64	-66.11	-66.32	-66.34	-66.38	-66.38
FROZEN PORK									
mean (% p.a.)	26.67	26.54	26.71	26.74	26.69	26.78	26.77	26.76	26.75
std (% p.a.)	155.67	155.68	155.67	155.67	155.67	155.68	155.67	155.67	155.67
skew	0.47	0.48	0.47	0.46	0.47	0.46	0.46	0.46	0.46
kurt	2.89	2.92	2.89	2.88	2.89	2.87	2.87	2.87	2.88
min (% p.c.m.)	-36.79	-36.75	-36.80	-36.81	-36.79	-36.82	-36.82	-36.81	-36.81

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Table 7: Continued from previous page.

commodity	OLS	utility							
		-1	-3	1		5		∞	
				HARA	POLY	HARA	POLY	HARA	POLY
GOLD									
mean (% p.a.)	7.46	7.44	7.49	7.58	7.57	7.53	7.53	7.51	7.51
std (% p.a.)	18.13	18.14	18.14	18.27	18.25	18.17	18.17	18.16	18.16
skew	1.15	1.16	1.13	1.10	1.10	1.12	1.12	1.12	1.12
kurt	10.50	10.47	10.52	10.53	10.53	10.54	10.54	10.53	10.53
min (% p.c.m.)	-7.55	-7.54	-7.56	-7.60	-7.59	-7.58	-7.58	-7.57	-7.57
HEATING OIL									
mean (% p.a.)	-4.66	-4.58	-4.39	-4.31	-4.53	-4.32	-4.36	-4.34	-4.36
std (% p.a.)	87.70	87.70	87.73	87.74	87.71	87.74	87.73	87.74	87.73
skew	0.13	0.15	0.18	0.19	0.16	0.19	0.19	0.19	0.19
kurt	5.07	5.09	5.13	5.15	5.10	5.14	5.14	5.14	5.14
min (% p.c.m.)	-32.51	-32.54	-32.62	-32.65	-32.56	-32.65	-32.63	-32.64	-32.63
LEAN HOGS									
mean (% p.a.)	0.39	0.39	0.43	0.47	0.47	0.46	0.46	0.45	0.45
std (% p.a.)	85.58	85.58	85.58	85.60	85.60	85.59	85.59	85.59	85.59
skew	0.14	0.14	0.15	0.15	0.15	0.15	0.15	0.15	0.15
kurt	1.43	1.43	1.48	1.52	1.52	1.51	1.51	1.50	1.50
min (% p.c.m.)	-27.00	-27.00	-27.10	-27.20	-27.18	-27.16	-27.16	-27.14	-27.14
LIGHT CRUDE									
mean (% p.a.)	-2.29	-2.27	-1.98	-1.66	-1.85	-1.82	-1.85	-1.87	-1.89
std (% p.a.)	55.08	55.08	55.17	55.45	55.26	55.28	55.26	55.24	55.23
skew	1.15	1.15	1.12	1.09	1.11	1.10	1.11	1.11	1.11
kurt	5.67	5.63	5.25	4.92	5.11	5.07	5.11	5.13	5.15
min (% p.c.m.)	-12.58	-12.57	-12.45	-12.33	-12.40	-12.39	-12.40	-12.41	-12.42
LUMBER									
mean (% p.a.)	13.26	13.24	13.21	13.12	13.13	13.19	13.19	13.20	13.20
std (% p.a.)	113.57	113.57	113.58	113.62	113.61	113.59	113.58	113.58	113.58
skew	0.93	0.94	0.94	0.95	0.95	0.94	0.94	0.94	0.94
kurt	1.44	1.45	1.45	1.48	1.48	1.46	1.46	1.46	1.46
min (% p.c.m.)	-16.19	-16.23	-16.28	-16.45	-16.44	-16.33	-16.33	-16.31	-16.31
OATS									
mean (% p.a.)	11.71	11.61	11.85	12.64	12.34	12.10	12.06	12.00	11.98
std (% p.a.)	63.09	63.10	63.11	63.90	63.47	63.24	63.21	63.17	63.17
skew	-0.08	-0.05	-0.11	-0.23	-0.19	-0.15	-0.15	-0.13	-0.13
kurt	2.09	2.13	2.04	1.89	1.93	1.97	1.98	2.00	2.00
min (% p.c.m.)	-23.76	-23.71	-23.83	-24.25	-24.09	-23.96	-23.94	-23.91	-23.90
PLATINUM									
mean (% p.a.)	6.29	6.29	6.29	6.29	6.29	6.29	6.29	6.29	6.29
std (% p.a.)	26.79	26.79	26.79	26.79	26.79	26.79	26.79	26.79	26.79
skew	0.97	0.97	0.97	0.97	0.97	0.97	0.97	0.97	0.97
kurt	4.83	4.83	4.84	4.84	4.84	4.84	4.84	4.84	4.84
min (% p.c.m.)	-6.63	-6.63	-6.63	-6.63	-6.63	-6.63	-6.63	-6.63	-6.63

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Table 7: Continued from previous page.

commodity	OLS	utility							
		-1	-3	1		5		∞	
				HARA	POLY	HARA	POLY	HARA	POLY
SILVER									
mean (% p.a.)	11.66	11.41	11.87	12.58	12.20	12.18	12.11	12.07	12.04
std (% p.a.)	27.29	27.34	27.32	27.97	27.53	27.51	27.46	27.42	27.41
skew	1.05	1.01	1.08	1.21	1.14	1.14	1.13	1.12	1.11
kurt	6.24	5.84	6.54	7.34	6.97	6.94	6.86	6.81	6.77
min (% p.c.m.)	-8.88	-8.86	-8.90	-8.96	-8.93	-8.93	-8.92	-8.92	-8.92
SOYBEAN MEAL									
mean (% p.a.)	2.41	2.41	2.40	2.37	2.38	2.39	2.39	2.40	2.40
std (% p.a.)	73.70	73.70	73.70	73.71	73.71	73.70	73.70	73.70	73.70
skew	1.14	1.14	1.15	1.17	1.16	1.15	1.15	1.15	1.15
kurt	9.33	9.34	9.39	9.56	9.55	9.44	9.44	9.42	9.42
min (% p.c.m.)	-19.94	-19.95	-19.97	-20.05	-20.04	-20.00	-20.00	-19.99	-19.98
SOYBEAN OIL									
mean (% p.a.)	9.02	8.98	9.03	9.14	9.13	9.07	9.07	9.06	9.06
std (% p.a.)	34.59	34.59	34.59	34.64	34.63	34.60	34.60	34.60	34.60
skew	0.40	0.41	0.40	0.37	0.37	0.38	0.39	0.39	0.39
kurt	1.19	1.24	1.18	1.04	1.06	1.12	1.13	1.14	1.14
min (% p.c.m.)	-6.50	-6.48	-6.51	-6.57	-6.57	-6.54	-6.53	-6.53	-6.53
SOYBEANS									
mean (% p.a.)	6.30	6.27	6.29	6.34	6.34	6.31	6.31	6.30	6.30
std (% p.a.)	24.19	24.19	24.19	24.20	24.20	24.19	24.19	24.19	24.19
skew	-1.51	-1.51	-1.51	-1.49	-1.49	-1.50	-1.50	-1.50	-1.50
kurt	5.55	5.58	5.55	5.49	5.50	5.53	5.53	5.54	5.54
min (% p.c.m.)	-10.95	-10.95	-10.95	-10.94	-10.94	-10.95	-10.95	-10.95	-10.95
SUGAR									
mean (% p.a.)	14.75	14.69	14.84	15.06	15.03	14.94	14.94	14.91	14.90
std (% p.a.)	70.69	70.70	70.70	70.84	70.81	70.74	70.74	70.73	70.72
skew	0.00	0.01	0.00	-0.02	-0.02	-0.01	-0.01	-0.01	-0.01
kurt	2.89	2.87	2.91	2.98	2.97	2.94	2.94	2.93	2.93
min (% p.c.m.)	-24.22	-24.08	-24.39	-24.84	-24.77	-24.60	-24.59	-24.53	-24.52
WHEAT									
mean (% p.a.)	8.34	8.23	8.31	8.41	8.41	8.36	8.36	8.34	8.34
std (% p.a.)	52.97	52.98	52.97	52.97	52.97	52.97	52.97	52.97	52.97
skew	0.06	0.07	0.06	0.06	0.06	0.06	0.06	0.06	0.06
kurt	2.19	2.22	2.20	2.17	2.17	2.18	2.18	2.19	2.19
min (% p.c.m.)	-18.14	-18.12	-18.13	-18.16	-18.16	-18.15	-18.15	-18.14	-18.14

Notes: The values in parentheses below the series labels denote the number of in-sample and out-of-sample observations respectively. The entries for each asset show in rows the first four sample moments of realized returns on (un)hedged portfolios, followed by a row specifying the lowest monthly return of the (un)hedged position. Columns correspond to no hedge, OLS hedge and utility-based hedge ratios for different utility functions. For definitions of HARA utility and its polynomial approximation (POLY), see equations (2.8, 2.9) and Appendix B.

Table 8: Out-of-sample hedging performance.

commodity		OLS	utility							
			-1	-3	1		5		∞	
					HARA	POLY	HARA	POLY	HARA	POLY
ALUMINUM	OLS HP		-3.13	-3.13	-3.13	-3.13	-3.13	-3.13	-3.13	-3.13
(40, 21)	OHR HP		-3.15	-3.12	-3.05	-3.06	-3.09	-3.09	-3.10	-3.10
	mean HR	0.18	0.18	0.18	0.17	0.17	0.18	0.18	0.18	0.18
	std HR	0.03	0.03	0.03	0.04	0.04	0.04	0.04	0.04	0.04
COCOA	OLS HP		8.74	8.67	8.59	8.59	8.63	8.63	8.64	8.64
(165, 84)	OHR HP		8.70	8.69	8.66	8.67	8.68	8.68	8.68	8.68
	mean HR	0.85	0.84	0.86	0.88	0.88	0.87	0.87	0.86	0.86
	std HR	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01
COFFEE	OLS HP		13.56	13.50	13.54	13.54	13.49	13.49	13.49	13.49
(204, 103)	OHR HP		13.57	13.45	13.40	13.41	13.40	13.41	13.42	13.42
	mean HR	0.85	0.85	0.85	0.84	0.84	0.84	0.84	0.84	0.84
	std HR	0.01	0.01	0.01	0.02	0.02	0.02	0.02	0.02	0.02
COPPER	OLS HP		-26.66	-26.87	-27.14	-27.12	-27.00	-26.99	-26.95	-26.95
(86, 44)	OHR HP		-26.28	-27.28	-28.66	-27.90	-27.91	-27.78	-27.69	-27.64
	mean HR	0.93	0.92	0.94	0.97	0.94	0.95	0.94	0.95	0.94
	std HR	0.03	0.02	0.05	0.09	0.05	0.06	0.06	0.06	0.05
CORN	OLS HP		17.83	17.87	17.98	17.98	17.91	17.90	17.89	17.89
(204, 103)	OHR HP		17.69	17.89	18.43	18.30	18.07	18.06	18.00	18.00
	mean HR	1.00	0.99	1.00	1.04	1.03	1.01	1.01	1.01	1.01
	std HR	0.01	0.01	0.01	0.02	0.01	0.01	0.01	0.01	0.01
COTTON	OLS HP		19.85	19.83	19.83	19.83	19.82	19.82	19.82	19.82
(196, 99)	OHR HP		19.85	19.42	17.50	18.32	18.97	19.02	19.16	19.17
	mean HR	0.83	0.83	0.81	0.71	0.76	0.79	0.79	0.80	0.80
	std HR	0.02	0.02	0.02	0.02	0.03	0.02	0.02	0.02	0.02
FROZEN PORK	OLS HP		-5.84	-6.11	-6.11	-6.08	-6.20	-6.19	-6.18	-6.18
(204, 103)	OHR HP		-5.50	-6.33	-6.71	-6.24	-6.71	-6.60	-6.60	-6.55
	mean HR	0.96	0.93	0.97	1.00	0.97	0.99	0.99	0.98	0.98
	std HR	0.02	0.02	0.02	0.04	0.03	0.03	0.03	0.03	0.03
GOLD	OLS HP		3.63	3.62	3.61	3.61	3.62	3.62	3.62	3.62
(204, 103)	OHR HP		3.61	3.65	3.71	3.71	3.67	3.67	3.66	3.66
	mean HR	0.97	0.96	0.98	1.01	1.01	0.99	0.99	0.99	0.99
	std HR	0.00	0.00	0.00	0.01	0.01	0.01	0.01	0.01	0.01
HEATING OIL	OLS HP		-11.60	-11.75	-11.93	-11.90	-11.84	-11.84	-11.81	-11.81
(180, 91)	OHR HP		-11.55	-11.39	-11.39	-11.66	-11.36	-11.40	-11.37	-11.38
	mean HR	0.83	0.82	0.81	0.81	0.82	0.80	0.81	0.80	0.81
	std HR	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01
LEAN HOGS	OLS HP		12.28	12.17	12.17	12.19	12.14	12.14	12.15	12.15
(139, 70)	OHR HP		12.22	12.08	12.20	12.25	12.07	12.08	12.06	12.07
	mean HR	0.91	0.91	0.90	0.90	0.91	0.90	0.90	0.90	0.90
	std HR	0.04	0.04	0.03	0.03	0.03	0.03	0.03	0.03	0.03

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Table 8: Continued from previous page.

commodity		OLS	utility							
			-1	-3	1		5		∞	
					HARA	POLY	HARA	POLY	HARA	POLY
LIGHT CRUDE	OLS HP		-13.01	-13.17	-13.27	-13.25	-13.24	-13.23	-13.22	-13.21
(171, 86)	OHR HP		-13.01	-12.89	-12.67	-12.76	-12.80	-12.82	-12.83	-12.84
	mean HR	1.06	1.06	1.04	1.02	1.03	1.03	1.04	1.04	1.04
	std HR	0.02	0.02	0.01	0.01	0.01	0.01	0.01	0.01	0.01
LUMBER	OLS HP		4.42	4.38	4.39	4.41	4.37	4.37	4.37	4.37
(76, 39)	OHR HP		4.48	4.35	4.38	4.36	4.31	4.31	4.32	4.32
	mean HR	0.62	0.61	0.60	0.56	0.57	0.59	0.59	0.59	0.59
	std HR	0.04	0.04	0.04	0.05	0.05	0.04	0.04	0.04	0.04
OATS	OLS HP		6.64	6.62	6.62	6.62	6.62	6.62	6.62	6.62
(204, 103)	OHR HP		6.59	6.76	7.09	7.03	6.91	6.89	6.86	6.85
	mean HR	0.79	0.78	0.82	0.94	0.89	0.86	0.85	0.84	0.84
	std HR	0.02	0.02	0.02	0.04	0.03	0.03	0.03	0.03	0.02
PLATINUM	OLS HP		-9.22	-9.23	-9.23	-9.23	-9.23	-9.23	-9.23	-9.23
(204, 103)	OHR HP		-9.20	-9.28	-9.42	-9.40	-9.34	-9.34	-9.32	-9.32
	mean HR	0.93	0.92	0.93	0.95	0.94	0.94	0.94	0.94	0.94
	std HR	0.00	0.00	0.01	0.01	0.01	0.01	0.01	0.01	0.01
SILVER	OLS HP		14.11	14.12	14.14	14.14	14.12	14.12	14.12	14.12
(170, 86)	OHR HP		13.91	14.30	14.89	14.63	14.56	14.51	14.46	14.45
	mean HR	0.94	0.92	0.96	1.00	0.98	0.98	0.97	0.97	0.97
	std HR	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01
SOYBEAN MEAL	OLS HP		-0.88	-0.89	-0.86	-0.86	-0.88	-0.88	-0.89	-0.89
(129, 65)	OHR HP		-0.87	-1.21	-1.39	-1.18	-1.35	-1.32	-1.31	-1.30
	mean HR	0.25	0.25	0.26	0.27	0.26	0.26	0.26	0.26	0.26
	std HR	0.08	0.08	0.07	0.07	0.07	0.07	0.07	0.07	0.07
SOYBEAN OIL	OLS HP		12.15	12.09	12.06	12.06	12.07	12.07	12.08	12.08
(204, 103)	OHR HP		12.12	12.11	12.18	12.18	12.12	12.13	12.12	12.12
	mean HR	0.99	0.99	1.00	1.02	1.02	1.01	1.00	1.00	1.00
	std HR	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02
SOYBEANS	OLS HP		3.18	3.23	3.41	3.40	3.29	3.29	3.27	3.27
(204, 103)	OHR HP		3.20	3.23	3.35	3.36	3.27	3.27	3.25	3.25
	mean HR	0.96	0.95	0.97	1.00	0.99	0.98	0.98	0.98	0.97
	std HR	0.00	0.00	0.00	0.01	0.01	0.01	0.01	0.01	0.01
SUGAR	OLS HP		2.20	2.17	2.15	2.15	2.16	2.16	2.17	2.17
(204, 103)	OHR HP		2.21	2.16	2.08	2.10	2.12	2.13	2.14	2.14
	mean HR	0.76	0.75	0.77	0.80	0.80	0.79	0.79	0.78	0.78
	std HR	0.01	0.01	0.01	0.02	0.01	0.01	0.01	0.01	0.01
WHEAT	OLS HP		14.24	14.24	14.24	14.24	14.24	14.24	14.24	14.24
(204, 103)	OHR HP		14.10	14.24	14.46	14.45	14.34	14.34	14.31	14.31
	mean HR	0.64	0.63	0.64	0.65	0.65	0.64	0.64	0.64	0.64
	std HR	0.01	0.02	0.02	0.01	0.01	0.01	0.01	0.01	0.01

Notes: The values in parentheses below the series labels denote the number of in-sample and out-of-sample observations respectively. Below that, means and standard deviations of the OLS hedge ratios are also presented in the first column. The entries for each asset in the remaining columns give first the hedging potentials of the OLS and of the utility-based hedges respectively, followed by the mean and standard deviations of the hedge ratios for the OLS and utility-based hedges. For definitions of HARA utility and its polynomial approximation (POLY), see equations (2.8, 2.9) and Appendix B.

Table 9: Moments of hedged commodity return, out-of-sample results.

commodity	no hedge	OLS	utility							
			-1	-3	1		5		∞	
					HARA	POLY	HARA	POLY	HARA	POLY
ALUMINUM										
mean (% p.a.)	6.13	2.95	2.93	2.96	3.04	3.04	2.99	2.99	2.98	2.98
std (% p.a.)	39.99	38.46	38.45	38.49	38.61	38.61	38.53	38.53	38.52	38.52
skew	-0.88	-1.06	-1.06	-1.05	-1.05	-1.05	-1.05	-1.05	-1.05	-1.05
kurt	0.67	0.93	0.93	0.92	0.90	0.90	0.91	0.91	0.92	0.92
min (% p.c.m.)	-8.44	-8.67	-8.67	-8.67	-8.67	-8.67	-8.67	-8.67	-8.67	-8.67
COCOA										
mean (% p.a.)	2.81	7.09	7.05	7.11	7.17	7.16	7.14	7.14	7.13	7.13
std (% p.a.)	106.41	22.25	22.44	22.19	22.25	22.23	22.15	22.16	22.15	22.16
skew	0.74	-0.36	-0.31	-0.38	-0.40	-0.40	-0.40	-0.40	-0.40	-0.40
kurt	1.22	3.25	3.21	3.26	3.18	3.19	3.26	3.25	3.26	3.26
min (% p.c.m.)	-22.85	-6.96	-6.97	-6.95	-6.93	-6.93	-6.94	-6.94	-6.95	-6.95
COFFEE										
mean (% p.a.)	2.59	9.61	9.62	9.57	9.49	9.50	9.54	9.54	9.55	9.55
std (% p.a.)	143.60	68.63	68.62	68.78	69.07	69.04	68.90	68.90	68.86	68.86
skew	0.22	-0.47	-0.47	-0.46	-0.45	-0.45	-0.46	-0.46	-0.46	-0.46
kurt	0.01	2.38	2.38	2.34	2.28	2.28	2.31	2.31	2.32	2.32
min (% p.c.m.)	-32.81	-22.65	-22.65	-22.65	-22.66	-22.66	-22.66	-22.66	-22.66	-22.66
COPPER										
mean (% p.a.)	19.22	-13.27	-12.88	-13.68	-14.78	-14.05	-14.18	-14.05	-14.00	-13.95
std (% p.a.)	132.69	59.94	59.94	59.92	60.04	59.97	59.95	59.94	59.93	59.93
skew	0.68	-0.02	-0.02	-0.03	-0.04	-0.02	-0.03	-0.03	-0.03	-0.03
kurt	0.66	0.24	0.29	0.19	0.04	0.17	0.12	0.15	0.15	0.16
min (% p.c.m.)	-20.00	-11.93	-11.95	-11.92	-11.87	-11.88	-11.90	-11.90	-11.90	-11.90
CORN										
mean (% p.a.)	-5.11	9.59	9.46	9.62	10.03	9.91	9.76	9.74	9.70	9.70
std (% p.a.)	97.37	42.98	43.16	43.00	42.62	42.76	42.86	42.88	42.91	42.92
skew	-0.45	-1.54	-1.52	-1.55	-1.61	-1.59	-1.57	-1.57	-1.56	-1.56
kurt	0.05	6.26	6.11	6.27	6.64	6.53	6.40	6.39	6.35	6.35
min (% p.c.m.)	-22.66	-17.36	-17.33	-17.38	-17.47	-17.46	-17.41	-17.41	-17.40	-17.40
COTTON										
mean (% p.a.)	-2.83	13.50	13.49	13.10	11.30	12.05	12.66	12.71	12.85	12.86
std (% p.a.)	102.49	43.68	43.70	43.92	47.23	45.29	44.37	44.30	44.15	44.13
skew	0.45	0.36	0.36	0.38	0.49	0.45	0.41	0.41	0.40	0.40
kurt	0.04	1.02	1.02	0.96	0.54	0.75	0.88	0.89	0.92	0.92
min (% p.c.m.)	-18.64	-7.51	-7.51	-7.40	-7.46	-7.23	-7.31	-7.32	-7.35	-7.35
FROZEN PORK										
mean (% p.a.)	25.26	11.71	12.09	11.47	11.07	11.53	11.18	11.27	11.27	11.31
std (% p.a.)	218.20	171.46	171.73	171.39	171.28	171.43	171.31	171.34	171.33	171.34
skew	0.16	0.07	0.07	0.07	0.08	0.07	0.07	0.07	0.07	0.07
kurt	0.61	1.53	1.53	1.53	1.52	1.52	1.52	1.52	1.52	1.52
min (% p.c.m.)	-41.30	-36.85	-36.79	-36.87	-36.92	-36.88	-36.91	-36.90	-36.90	-36.90

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Table 9: Continued from previous page.

commodity	no hedge	OLS	utility							
			-1	-3	1		5		∞	
					HARA	POLY	HARA	POLY	HARA	POLY
GOLD										
mean (% p.a.)	1.37	4.11	4.09	4.13	4.21	4.21	4.17	4.17	4.15	4.15
std (% p.a.)	47.13	8.13	8.21	8.07	8.08	8.07	8.03	8.03	8.03	8.04
skew	1.03	1.09	1.24	0.90	0.35	0.46	0.66	0.68	0.75	0.76
kurt	2.99	4.27	5.07	3.38	1.32	1.67	2.34	2.42	2.70	2.73
min (% p.c.m.)	-8.82	-1.21	-1.13	-1.32	-1.66	-1.59	-1.46	-1.45	-1.40	-1.40
HEATING OIL										
mean (% p.a.)	20.26	3.80	3.85	4.16	4.32	4.04	4.28	4.23	4.24	4.22
std (% p.a.)	142.73	94.05	94.04	94.02	94.03	94.04	94.03	94.03	94.02	94.02
skew	0.28	-0.20	-0.20	-0.18	-0.17	-0.19	-0.17	-0.17	-0.17	-0.17
kurt	0.71	1.99	1.96	1.86	1.82	1.91	1.82	1.84	1.83	1.84
min (% p.c.m.)	-31.18	-24.24	-24.12	-23.47	-23.08	-23.65	-23.20	-23.29	-23.28	-23.32
LEAN HOGS										
mean (% p.a.)	7.49	14.94	14.87	14.86	14.98	15.01	14.88	14.89	14.87	14.87
std (% p.a.)	151.07	104.79	104.86	104.97	104.99	104.94	105.01	105.00	105.00	104.99
skew	0.40	-0.16	-0.16	-0.15	-0.15	-0.15	-0.15	-0.15	-0.15	-0.15
kurt	1.65	1.77	1.77	1.79	1.79	1.79	1.80	1.80	1.79	1.79
min (% p.c.m.)	-34.38	-27.16	-27.18	-27.21	-27.18	-27.16	-27.21	-27.21	-27.22	-27.22
LIGHT CRUDE										
mean (% p.a.)	21.39	0.61	0.62	0.91	1.26	1.13	1.08	1.06	1.02	1.01
std (% p.a.)	146.35	53.60	53.61	54.01	54.59	54.35	54.29	54.25	54.19	54.18
skew	0.25	0.33	0.33	0.31	0.29	0.30	0.30	0.30	0.30	0.30
kurt	0.55	-0.05	-0.05	-0.01	0.04	0.02	0.01	0.01	0.00	0.00
min (% p.c.m.)	-0.31	-0.10	-0.10	-0.11	-0.11	-0.11	-0.11	-0.11	-0.11	-0.11
LUMBER										
mean (% p.a.)	18.28	19.50	19.57	19.47	19.51	19.47	19.45	19.45	19.45	19.45
std (% p.a.)	161.97	135.62	135.74	135.65	135.82	135.84	135.66	135.67	135.65	135.65
skew	0.43	0.62	0.62	0.62	0.63	0.63	0.63	0.63	0.63	0.63
kurt	-0.58	-0.76	-0.76	-0.76	-0.73	-0.74	-0.75	-0.75	-0.75	-0.75
min (% p.c.m.)	-0.19	-0.12	-0.12	-0.12	-0.12	-0.12	-0.12	-0.12	-0.12	-0.12
OATS										
mean (% p.a.)	-0.66	4.14	4.08	4.32	4.91	4.73	4.55	4.52	4.46	4.45
std (% p.a.)	96.27	69.23	68.87	69.88	74.04	72.13	71.16	70.89	70.62	70.51
skew	0.10	-0.58	-0.58	-0.58	-0.57	-0.58	-0.58	-0.58	-0.58	-0.58
kurt	0.99	2.73	2.75	2.68	2.33	2.51	2.57	2.60	2.62	2.63
min (% p.c.m.)	-0.20	-0.24	-0.24	-0.24	-0.24	-0.24	-0.24	-0.24	-0.24	-0.24
PLATINUM										
mean (% p.a.)	10.16	-0.53	-0.51	-0.59	-0.73	-0.70	-0.65	-0.64	-0.62	-0.62
std (% p.a.)	63.44	21.88	21.90	21.84	21.80	21.82	21.82	21.82	21.83	21.83
skew	-0.20	-0.04	-0.03	-0.07	-0.12	-0.11	-0.09	-0.09	-0.08	-0.08
kurt	-0.05	2.47	2.45	2.50	2.57	2.55	2.53	2.53	2.52	2.52
min (% p.c.m.)	-0.15	-0.06	-0.06	-0.06	-0.07	-0.07	-0.06	-0.06	-0.06	-0.06

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Table 9: Continued from previous page.

commodity	no hedge	OLS	utility							
			-1	-3	1		5		∞	
					HARA	POLY	HARA	POLY	HARA	POLY
SILVER										
mean (% p.a.)	-0.11	12.26	12.05	12.45	13.05	12.78	12.71	12.67	12.62	12.60
std (% p.a.)	69.71	24.97	24.67	25.30	26.65	25.93	25.84	25.73	25.63	25.59
skew	0.01	2.04	2.07	1.98	1.78	1.88	1.89	1.91	1.92	1.93
kurt	0.85	10.30	10.36	10.13	9.28	9.72	9.80	9.86	9.93	9.95
min (% p.c.m.)	-0.19	-0.05	-0.05	-0.05	-0.06	-0.06	-0.06	-0.05	-0.05	-0.05
SOYBEAN MEAL										
mean (% p.a.)	10.10	7.80	7.81	7.46	7.25	7.45	7.32	7.35	7.36	7.37
std (% p.a.)	94.68	74.30	74.23	74.16	73.84	73.85	74.07	74.07	74.11	74.11
skew	0.23	0.61	0.61	0.59	0.56	0.58	0.57	0.58	0.58	0.58
kurt	3.02	2.11	2.10	2.16	2.18	2.15	2.18	2.18	2.18	2.18
min (% p.c.m.)	-0.21	-0.14	-0.14	-0.13	-0.14	-0.13	-0.14	-0.13	-0.13	-0.13
SOYBEAN OIL										
mean (% p.a.)	4.82	12.61	12.58	12.62	12.71	12.71	12.65	12.65	12.64	12.64
std (% p.a.)	109.24	37.04	37.24	36.96	36.36	36.47	36.73	36.74	36.82	36.82
skew	0.54	0.96	0.98	0.95	0.87	0.88	0.92	0.92	0.93	0.93
kurt	1.46	2.94	3.00	2.91	2.66	2.71	2.82	2.83	2.86	2.86
min (% p.c.m.)	-0.23	-0.06	-0.06	-0.06	-0.06	-0.06	-0.06	-0.06	-0.06	-0.06
SOYBEANS										
mean (% p.a.)	-0.55	-0.63	-0.61	-0.64	-0.69	-0.67	-0.66	-0.66	-0.65	-0.65
std (% p.a.)	93.79	29.48	29.50	29.48	29.56	29.51	29.50	29.49	29.49	29.49
skew	-0.88	-1.67	-1.67	-1.67	-1.66	-1.66	-1.67	-1.67	-1.67	-1.67
kurt	2.62	4.91	4.92	4.90	4.78	4.81	4.86	4.86	4.88	4.88
min (% p.c.m.)	-0.33	-0.11	-0.11	-0.11	-0.11	-0.11	-0.11	-0.11	-0.11	-0.11
SUGAR										
mean (% p.a.)	-0.02	-0.18	-0.16	-0.18	-0.21	-0.19	-0.20	-0.19	-0.19	-0.19
std (% p.a.)	93.40	54.88	54.81	55.01	55.49	55.39	55.20	55.19	55.13	55.13
skew	0.27	-0.42	-0.40	-0.46	-0.55	-0.53	-0.50	-0.50	-0.48	-0.48
kurt	-0.23	2.31	2.26	2.39	2.56	2.53	2.47	2.46	2.44	2.44
min (% p.c.m.)	-0.16	-0.17	-0.17	-0.17	-0.18	-0.17	-0.17	-0.17	-0.17	-0.17
WHEAT										
mean (% p.a.)	-1.30	12.03	11.89	12.04	12.25	12.24	12.14	12.14	12.10	12.10
std (% p.a.)	74.38	57.37	57.29	57.37	57.49	57.49	57.43	57.42	57.40	57.40
skew	0.20	-0.37	-0.36	-0.37	-0.39	-0.38	-0.38	-0.38	-0.37	-0.37
kurt	0.60	1.99	1.99	1.99	1.98	1.98	1.99	1.99	1.99	1.99
min (% p.c.m.)	-0.17	-0.18	-0.18	-0.18	-0.18	-0.18	-0.18	-0.18	-0.18	-0.18

Notes: The values in parentheses below the series labels denote the number of in-sample and out-of-sample observations respectively. The entries for each asset show in rows the first four sample moments of realized returns on (un)hedged portfolios, followed by a row specifying the lowest monthly return of the (un)hedged position. Columns correspond to no hedge, OLS hedge and utility-based hedge ratios for different utility functions. For definitions of HARA utility and its polynomial approximation (POLY), see equations (2.8, 2.9) and Appendix B.

Table 10: Out-of-sample hedging performance for cotton, $\lambda = 10$.

	No hedge	OLS	utility							
			-1	-3	1		5		∞	
					HARA	POLY	HARA	POLY	HARA	POLY
OLS HP			4.4615	4.4077	4.4956	4.5088	4.6169	4.6442	4.6942	4.7241
OHR HP			4.4608	3.5488	2.4636	3.1528	-1.2095	2.7604	-5.2704	2.5831
mean HR		0.8306	0.8303	0.5598	0.3187	0.4503	-0.1872	0.3549	-0.5499	0.3151
std HR		0.0156	0.0152	0.0599	0.0855	0.0684	0.1045	0.0683	0.1006	0.0658
mean (% p.a.)	-2.83	13.50	13.49	8.27	3.62	6.16	-6.23	4.30	-13.33	3.53
std (% p.a.)	102.49	43.68	43.70	53.77	70.34	60.91	114.65	68.26	149.75	71.63
skew	0.45	0.36	0.36	0.56	0.54	0.56	0.42	0.55	0.37	0.54
kurt	0.04	1.02	1.02	0.10	-0.18	-0.09	-0.05	-0.15	0.08	-0.16
min (% p.c.m.)	-18.64	-7.51	-7.51	-8.11	-10.33	-8.69	-19.69	-9.84	-27.95	-10.53

Notes: The entries for each asset show in rows the hedging potential of the OLS and optimal hedge ratios, the mean and standard deviations of the hedge ratios, and then the first four sample moments of realized returns on (un)hedged portfolios, followed by a row specifying the lowest monthly return of the (un)hedged position. Columns correspond to no hedge, OLS hedge and utility-based hedge ratios for different utility functions. For definitions of HARA utility and its polynomial approximation (POLY), see equations (2.8, 2.9) and Appendix B.

Figure 1: Out-of-sample hedge ratios for cotton, $\lambda = 1$.

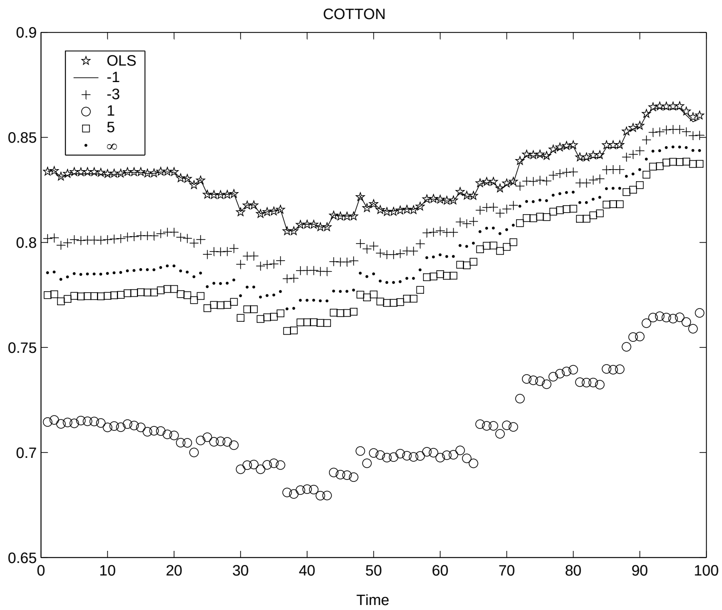

Figure 2: Out-of-sample hedge ratios for gold, $\lambda = 1$.