Multi-Asset Noisy Rational Expectations Equilibrium with Contingent Claims*

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Abstract

We study a noisy rational expectations equilibrium in a multi-asset economy populated by informed and uninformed investors, and noise traders. The assets can include state contingent claims such as Arrow-Debreu securities, assets with only positive payoffs, options or other derivative securities. The probabilities of states depend on a shock, which is observed only by the informed investor. We provide conditions under which the informed investor's asset demands contain information about the shock. These conditions imply that adding derivative securities in some widely studied economies with one risky asset does not reveal any additional information about the shock. We also show that introducing volatility derivatives in incomplete markets makes these markets effectively complete, that is, allows investors to achieve Pareto optimal asset allocations. We find asset prices in closed form in (effectively) complete markets.

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1. Introduction

Informed investors in financial markets often use their private information to extract gains from trading in a multitude of financial assets, such as stocks, corporate bonds, and derivative securities with payoffs contingent on payoffs of other assets. This information can be partially learned by uninformed investors from asset prices and used in their own trading. Learning from asset prices in realistic multi-asset economies is complicated by the joint effects of the non-normality of asset payoff distributions, complex payoff structures of derivative securities, and the interdependence of asset prices arising because each price may contain information about the payoffs of other assets. Therefore, despite the fact that the asymmetry of information across investors is a prominent feature of financial markets, its impact on equilibrium asset prices in multi-asset economies remains relatively unexplored. In this paper, we develop a new tractable approach for studying asset prices in multi-asset noisy rational expectations equilibrium (REE) economies with realistic asset payoffs and asymmetric information.

We consider a multi-asset economy with two dates and a finite but arbitrary number of random states. The probabilities of states are exponential functions of an aggregate shock normalized to sum up to one, and the shock has a certain conjugate prior distribution. The assets are state contingent claims such as Arrow-Debreu securities, assets with only positive payoffs, and derivative securities. The economy is populated by two types of price-taking rational investors, an informed and an uninformed, with constant absolute risk aversion (CARA) preferences over terminal wealth. There are also noise traders with exogenous jointly normally distributed asset demands. The investors trade assets at the initial date, and asset prices are determined from the market clearing conditions. The informed investor observes the shock, whereas the uninformed investor uses asset prices to extract information about the shock, which is obfuscated by the noise traders.

We begin the analysis by studying markets where the number of states equals the number of assets, henceforth complete markets. Our unique setup allows us to extend the methods of risk-neutral valuation and portfolio choice to complete markets with asymmetric information. By employing the risk-neutral valuation, we obtain closed-form asset prices, investors' portfolios of assets, and intuitive comparative statics. Asset prices are given by expected discounted payoffs with respect to risk-neutral probabilities.

We then also solve for the equilibrium in a subset of realistic incomplete-market economies, where the number of states exceeds the number of assets. These economies include economies with one risky asset studied in the previous literature [e.g., Grossman and Stiglitz (1980), Breon-Drish (2015)], the latter economies with added derivative securities, and economies with risky corporate debt and equity, among others. The asset prices in these economies solve a system of non-linear equations. We prove the existence and uniqueness of equilibrium in these incomplete-market economies using a novel method which relies only on well-known results of basic calculus. The proofs in the related literature only apply to economies with a single risky asset.

First, we apply our setup to answer a fundamental question: which assets help mitigate the information asymmetry in the economy? We show that the information about the aggregate shock is impounded into asset prices only via the informed investor's demand for those assets that make up a replicating portfolio for the sensitivities of state probabilities to the shock. These sensitivities are defined as coefficients multiplying the aggregate shock in the expressions for probabilities, and hence, determine which states are more likely given the shock. The intuition is that investing in a replicating portfolio for shock sensitivities allows the informed investor to transfer more wealth to more likely states. The amount of wealth invested in this portfolio depends on the aggregate shock. The information about the shock then becomes incorporated into asset prices via the market clearing conditions. The replicating portfolio exists in all complete markets and the subset of incomplete markets, described above. We call the existence of this portfolio the informational spanning condition. This condition makes the investor's portfolio separable in the shock and prices, which simplifies the learning from prices and the derivation of equilibrium.

Assets that do not belong to the replicating portfolio for shock sensitivities of probabilities are informationally irrelevant. The name stems from our result that when noise trader demands are uncorrelated across markets, adding such assets to the economy does not reveal any additional information about the shock. The intuition is that the informed investor's demands for these assets do not depend on the shock, as discussed above. However, when noise trader demands are correlated across markets, the prices of these securities carry information about the noises in other markets, which facilitates the extraction of information about the shock from the prices. Guided by the latter results, we show that adding options or other contingent claims to the economies with one risky asset in Grossman and Stiglitz (1980) and Breon-Drish (2015) does not reveal any additional information about the underlying asset, provided that noises across markets are uncorrelated. In another application of the informational irrelevance of assets, we show that the information jointly revealed by the prices of risky corporate debt and equity does not depend on the face value of debt.

The multi-asset nature of our setup also allows us to unravel the important economic role of volatility derivatives. These derivatives are widely traded in financial markets, and adding them to our model makes it both realistic and tractable. The study of these derivatives when the payoff of the underlying asset is not normally distributed is unique to our model. Specifically, we study a quadratic security with a payoff equal to the squared payoff of the underlying, which we interpret as a volatility derivative. We show that introducing this security in an incomplete market with one risky underlying asset makes this market effectively complete, that is, allows informed and uninformed investors to achieve Pareto optimal asset allocations. Moreover, the asset prices in the resulting effectively complete market are available in closed form. We also generalize our results and show that a market is effectively complete if and only if there exists a portfolio of assets that replicates the squares of shock sensitivities of probabilities. Hence, while the replication of shock probabilities allows investors to extract all available information, the replication of the squares allows them to share risks effectively.

We build on the role of volatility derivatives discussed above to develop a new method for studying the effects of market completeness and asymmetric information on asset prices. We consider an incomplete market with a single risky asset and then make it effectively complete by adding a volatility derivative. To facilitate the comparison of incomplete and effectively complete markets, we exclude noise traders from trading the derivative; even then we show that the equilibrium is not fully revealing because the derivative is informationally irrelevant (in this particular case). For several broad classes of asset payoff distributions we find the price of the underlying asset in the effectively complete market in closed-form. As an application of our methodology, we find the price of an asset with a skew-normal payoff distribution, which generalizes the normal distribution to allow for skewness. We show that skewness has a large effect on prices. Finally, we provide examples showing that the price of the underlying asset in effectively complete markets has similar properties and magnitude as in the incomplete economy with a single risky asset (which we compute numerically in most cases). Therefore, market incompleteness appears to have a small effect on prices.

Related Literature. Our paper is related to a large literature on noisy REE models, which was pioneered by Grossman (1976), Grossman and Stiglitz (1980) and Hellwig (1980). These early works typically consider economies with CARA investors and one risky asset with normally distributed payoffs. Admati (1985) extends these models to the case of multiple securities and Wang (1993) develops a dynamic model. Pálvölgyi and Venter (2014) prove the uniqueness of equilibrium in Grossman and Stiglitz (1980) in the class of continuous prices and find multiple equilibria with discontinuous prices. Diamond and

Verrecchia (1981), Vives (2008), García and Urošević (2013) and Kurlat and Veldkamp (2013) discuss further extensions and applications of CARA-normal models. In contrast to this literature, we allow for more general payoff distributions and multiple assets.

Breon-Drish (2015) studies an economy with CARA investors and one risky asset but without normality. The latter work derives prices in terms of inverse functions and proves the existence and uniqueness of equilibrium when the asset payoff conditional on the informed investor's signal has a distribution from the exponential family. Relative to the latter paper, we study a muti-asset economy. We show that market completeness offers significant additional tractability and yields closed-form prices. We also provide new economic implications such as conditions for the informational irrelevance of assets and uncover the important new role of quadratic derivatives in a wide class of markets with asymmetric information.

Malamud (2015) studies an REE with Arrow-Debreu securities in a continuous-space complete-market economy and shows that the equilibria with non-CARA preferences are fully revealing. Albagli, Hellwig, and Tsyvinski (2013) consider a noisy REE model where the information is dispersed across all investors, preferences are general, and investors face position limits. Our model differs from their setup in that our investors are either informed or uninformed and their trades are unconstrained. Bernardo and Judd (2000) solve models with general distributions and preferences numerically and demonstrate that the REE in Grossman and Stiglitz (1980) is not robust to parametric assumptions. Other related models that do not rely on the normality of asset payoffs include Barlevy and Veronesi (2000), Peress (2004), and Yuan (2005) among others.

Our model is also related to works on the informational role of derivatives. Brennan and Cao (1996) consider a CARA-normal model with one risky underlying asset and a quadratic derivative written on it. They show that the derivative effectively completes the market and does not reveal any information which is not already in the price of the underlying asset. We show that the latter result extends to economies without payoff normality. Moreover, the quadratic derivative in our model plays a unique role by allowing us to obtain closed-form prices whereas in their work introducing this derivative does not affect the price of the underlying asset. Other related models in this literature include Back (1993), Biais and Hillion (1994), Vanden (2008), and Huang (2014) among others.

2. Model

2.1. Securities Markets and Information Structure

We consider an economy with two dates $t = 0$ and $t = T$, and N states $\omega_1,\ldots,\omega_N$ at the terminal date, where $N \geq 2$. The probabilities $\pi_n(\varepsilon)$ of states ω_n are normalized exponential functions of the *aggregate shock* $\varepsilon \in \mathbb{R}$ adopted from logit models in econometrics:

$$\pi_n(\varepsilon) = \frac{e^{a_n + b_n \varepsilon}}{\sum_{j=1}^{N} e^{a_j + b_j \varepsilon}}, \quad n = 1, \ldots, N, \tag{1}$$

where a_n and b_n are arbitrary parameters. We label parameter b_n as the *shock sensitivity* of probability $\pi_n(\varepsilon)$.

There are $M \geq 2$ assets traded in the economy: a riskless bond in perfectly elastic supply paying \$1 at date T and $M-1$ state-contingent risky assets in zero net supply with terminal payoff $C_m(\omega)$ in state ω, where $m = 1, \ldots, M-1$. The risky assets may include Arrow-Debreu securities, assets with only positive payoffs, options or other derivatives. All assets are non-redundant in the sense that each asset's terminal payoff cannot be replicated by trading in other assets. We denote the vector of $M-1$ risky asset payoffs in state ω by $C(\omega) = \big(C_1(\omega), \ldots, C_{M-1}(\omega)\big)^\top$, and use C_m and C as shorthand notation for $C_m(\omega)$ and $C(\omega)$. The bond price is set to $p_0 = e^{-rT}$, where r is an exogenous risk-free interest rate. We denote the vector of observed date-0 prices of the risky assets by $p = (p_1, \ldots, p_{M-1})^\top$. These prices are determined in equilibrium, defined below.

The economy is populated by three representative investors, informed and uninformed investors, labeled I and U, and noise traders. Each investor stands for a group of a continuum of identical investors that act as price-takers. Investors I and U have CARA preferences over terminal wealth with risk aversions γ_I and γ_U, respectively. Noise traders submit exogenous normally distributed demands $\nu = (\nu_1, \ldots, \nu_{M-1})^\top \sim \mathcal{N}(0, \Sigma_\nu)$ for risky assets, where $\Sigma_\nu \in \mathbb{R}^{(M-1)\times(M-1)}$ is a symmetric positive-definite matrix.

Both investors I and U know all asset payoffs in all states, $C_m(\omega_n)$. Before the markets open, investor I observes shock ε. Investor U observes only asset prices p at date $t = 0$ and knows how the equilibrium prices, given by some function $P(\varepsilon, \nu) \in \mathbb{R}^{M-1}$, depend on shock ε and noises ν. Investor U has the following conjugate prior probability density function (PDF) over ε:

$$\varphi_\varepsilon(x) = \frac{\Big(\sum_{j=1}^{N} e^{a_j + b_j x}\Big) e^{-0.5(x-\mu_0)^2/\sigma_0^2}}{\int_{-\infty}^{\infty} \Big(\sum_{j=1}^{N} e^{a_j + b_j x}\Big) e^{-0.5(x-\mu_0)^2/\sigma_0^2} dx}. \tag{2}$$

In Remark 1 below, we provide examples of economies where PDF $\varphi_\varepsilon(x)$ arises endogenously. After observing prices p, the uninformed investor updates the prior PDF (2) conditioning on the information that shock ε and noise ν satisfy equation $P(\varepsilon, \nu) = p$.

For fixed parameters a_n and b_n, we choose μ_0 and σ_0^2 such that ε has any desired mean μ_ε and variance σ_ε^2. We refer to the distribution of ε as *generalized normal* $\widehat{\mathcal{N}}(\mu_\varepsilon, \sigma_\varepsilon^2)$ with mean $\mathbb{E}[\varepsilon] = \mu_\varepsilon$ and variance $\text{var}[\varepsilon] = \sigma_\varepsilon^2$, and note that it can be rewritten as a weighted average of PDFs of normally distributed random variables. The relationship between (μ_0, σ_0^2) and $(\mu_\varepsilon, \sigma_\varepsilon^2)$ is given by Equations (A.1) and (A.2) in the Appendix.[1]

2.2. Investors' Optimization and Definition of Equilibrium

Each investor $i = I, U$ is endowed with initial wealth $W_{i,0}$, and allocates it to buy α_i units of the riskless asset and $\theta_{i,m}$ units of the risky asset m. By $\theta_i = (\theta_{i,1}, \ldots, \theta_{i,M-1})^\top$ we denote the vector of units of risky assets purchased by investor i. Investors I and U maximize their expected utilities over terminal wealth:

$$\max_{\theta_I} \mathbb{E}\Big[-e^{-\gamma_I W_{I,T}}\Big|\varepsilon, p\Big], \tag{3}$$

$$\max_{\theta_U} \mathbb{E}\Big[-e^{-\gamma_U W_{U,T}}\Big|P(\varepsilon, \nu) = p\Big], \tag{4}$$

respectively, subject to their self-financing budget constraints

$$W_{i,T} = W_{i,0}e^{rT} + (C - e^{rT}p)^\top \theta_i, \quad i = I, U. \tag{5}$$

Definition of Equilibrium. *A competitive noisy rational expectations equilibrium is a vector of risky asset prices* $P(\varepsilon, \nu)$ *and investor portfolios* $\theta_I^*(p; \varepsilon)$ *and* $\theta_U^*(p)$ *that solve optimization problems* (3) *and* (4) *subject to self-financing budget constraints* (5), *taking asset prices as given, and satisfy the market clearing condition:*

$$\theta_I^*(P(\varepsilon, \nu); \varepsilon) + \theta_U^*(P(\varepsilon, \nu)) + \nu = 0. \tag{6}$$

Remark 1 (Particular Cases). Our model incorporates the economies in Grossman and Stiglitz (1980) and Breon-Drish (2015) for particular choices of distribution parameters a_n and b_n in Equation (1) in the limit $N \to \infty$. In particular, following Breon-Drish (2015), consider a single risky asset with payoff C_1 that has general PDF $\varphi_C(x)$. Investor I receives a noisy signal $\varepsilon = C_1 + u$, where $u \sim \mathcal{N}(\mu_0, \sigma_0^2)$, while investor U observes the price of the

[1] Probabilities (1) and distribution (2) allow for considerable tractability due to the fact the unconditional expectation $\mathbb{E}[\pi_n(\varepsilon)]$ and posterior probabilities of states conditional on information in prices can be obtained in closed form.

risky asset and updates the prior distribution $\varphi_C(x)$ in a Bayesian fashion. The Grossman and Stiglitz (1980) economy is a special case when $C_1 \sim \mathcal{N}(\mu_C, \sigma_C^2)$. Lemma A.2 in the Appendix demonstrates that the latter economy is a limiting case of our N-state economy with asset payoffs $C_1(\omega_n) = \underline{C}_N + (\overline{C}_N - \underline{C}_N)(n-1)/(N-1)$ and distribution parameters

$$a_n = -\frac{C_1(\omega_n)^2}{2\sigma_0^2} - \frac{\mu_0 C_1(\omega_n)}{\sigma_0^2} + \ln\left[\varphi_C(C_1(\omega_n))\right], \quad b_n = \frac{C_1(\omega_n)}{\sigma_0^2}, \tag{7}$$

when $N \to \infty$ and $\underline{C}_N$ and $\overline{C}_N$ converge to the lower and upper limits of payoff C_1. Finally, we note that our model can be easily extended to the case of multiple shocks and probabilities $\pi_n(\varepsilon) = e^{a_n + b_n^\top \varepsilon} / \left(\sum_{j=1}^N e^{a_j + b_j^\top \varepsilon}\right)$.[2]

3. Characterization of Equilibrium

In this section, we first consider our baseline economy where the number of assets equals the number of states, $M = N$, and find the equilibrium in closed form. Then, we consider a more general economy with $M \leq N$ assets and derive the equilibrium in terms of inverse functions. We use the latter economy to establish conditions for the effective completeness of financial markets, and to study the impact of market incompleteness on asset prices.

3.1. Economy with $M = N$ Securities

We start with an economy where the number of assets equals the number of states, that is, $M = N$, which we label as a *complete-market economy*.[3] Our methodological contribution is to show that market completeness substantially simplifies the derivation of equilibrium, so that asset prices and investors' portfolios are available in closed form. In Section IA.1 of the Internet Appendix, we extend our methodology to general state probabilities $\pi_n(\varepsilon)$, shock distribution $\varphi_\varepsilon(x)$, and noise trader demand distribution $\varphi_\nu(x)$, and apply it to study closed-form asset prices when distribution $\varphi_\nu(x)$ is a mixture of normal distributions.

[2] Lemma IA.5 in the Internet Appendix shows that such an extension includes as a special case an economy with two risky assets where investor I receives a signal on each asset payoff $\varepsilon_1 = C_1 + u_1$ and $\varepsilon_2 = C_2 + u_2$, where u_i are independent and normally distributed. Lemma IA.6 in the Internet Appendix further shows that when there is one risky asset with payoff C_1 and two signals $\varepsilon_1 = C_1 + u_1$ and $\varepsilon_2 = C_1 + u_2$, these signals can be combined into one signal ε, and we revert to our baseline model.

[3] The realizations of shock ε can be interpreted as a continuum of states of the economy in addition to states ω_n. However, N non-redundant assets still suffice to replicate any contingent claim in our economy because the payoffs of such claims do not vary across ε for a fixed state ω_n. In other words, ε-states can be clumped together so that only ω_n states matter for replication. Furthermore, noise trader demands do not contribute to market incompleteness because they affect date-0 prices but do not affect future payoffs.

Due to market completeness, we look for equilibrium prices p in the following form:

$$p = \Big[\pi_1^{\text{RN}} C(\omega_1) + \pi_2^{\text{RN}} C(\omega_2) + \ldots + \pi_N^{\text{RN}} C(\omega_N)\Big] e^{-rT}, \tag{8}$$

where π_n^{RN} is the *risk-neutral* probability of state ω_n. The risk-neutral probabilities exist and are unique in equilibrium because investors I and U are unconstrained and can eliminate any arising arbitrage opportunities [e.g., Duffie (2001, p. 4)].

Informed and uninformed investors agree on risk-neutral probabilities because these probabilities are uniquely determined from Equation (8) as functions of prices p. However, investors assign different real probabilities to states ω_n. In particular, investor U's real *posterior probabilities* are given by expected probabilities $\pi_n(\varepsilon)$ conditional on the information in prices: $\pi_n^U(p) = \mathbb{E}\Big[\pi_n(\varepsilon)|P(\varepsilon,\nu) = p\Big]$. The expression for $\pi_n^U(p)$ can be obtained by rewriting investor U's expected utility (4) as follows:

$$\begin{aligned}\mathbb{E}\Big[-e^{-\gamma_U W_{U,T}}|P(\varepsilon,\nu) = p\Big] &= -\textstyle\sum_{n=1}^N \Big(\mathbb{E}\Big[\pi_n(\varepsilon)|P(\varepsilon,\nu) = p\Big] e^{-\gamma_U W_{U,T,n}}\Big)\\ &= -\textstyle\sum_{n=1}^N \pi_n^U(p) e^{-\gamma_U W_{U,T,n}}.\end{aligned} \tag{9}$$

Taking probabilities $\pi_n(\varepsilon)$ and $\pi_n^U(p)$ as given, investors' optimizations can be solved using the methods of complete-market portfolio choice. The informed and uninformed investors have different state price densities (SPDs), which are given by discounted ratios of risk-neutral and their real probabilities [e.g., Duffie (2001, p. 11)]: $\pi_n^{\text{RN}} e^{-rT}/\pi_n(\varepsilon)$ and $\pi_n^{\text{RN}} e^{-rT}/\pi_n^U(p)$, respectively. The first order conditions (FOCs) of investors equate their marginal utilities and SPDs [e.g., Duffie (2001, p. 5)] and are given by:

$$\gamma_I e^{-\gamma_I W_{I,T,n}} = \ell_I \frac{\pi_n^{\text{RN}} e^{-rT}}{\pi_n(\varepsilon)}, \quad \gamma_U e^{-\gamma_U W_{U,T,n}} = \ell_U \frac{\pi_n^{\text{RN}} e^{-rT}}{\pi_n^U(p)}, \tag{10}$$

where ℓ_i are Lagrange multipliers for investors' budget constraints. From Equations (10), we find optimal wealths $W_{i,T}$. Then, we use these wealths to recover optimal portfolios from budget constraints (5). Lemma 1 below reports investor I's portfolio in closed form.

Lemma 1 (Investor I's optimal portfolio). *Investor I's optimal portfolio is given by:*

$$\theta_I^*(p;\varepsilon) = \frac{\lambda\varepsilon}{\gamma_I} - \frac{\Omega^{-1}(\widetilde{v}(p) - \widetilde{a})}{\gamma_I}, \tag{11}$$

where $\Omega \in \mathbb{R}^{(N-1)\times(N-1)}$ *is a matrix of excess payoffs with elements* $\Omega_{n,k} = C_k(\omega_n) - C_k(\omega_N)$*,* $\lambda = \Omega^{-1}(b_1 - b_N, \ldots, b_{N-1} - b_N)^\top \in \mathbb{R}^{N-1}$*,* $\widetilde{a} = (a_1 - a_N, \ldots, a_{N-1} - a_N)^\top \in \mathbb{R}^{N-1}$*, and* $\widetilde{v}(p) \in \mathbb{R}^{N-1}$ *is the vector of log ratios of risk-neutral probabilities, given by:*

$$\widetilde{v}(p) = \left(\ln\left(\frac{\pi_1^{\text{RN}}}{\pi_N^{\text{RN}}}\right), \ldots, \ln\left(\frac{\pi_{N-1}^{\text{RN}}}{\pi_N^{\text{RN}}}\right)\right)^\top. \tag{12}$$

Equation (11) decomposes portfolio $\theta_I^*(p;\varepsilon)$ into two terms, which we label as *information-sensitive* and *information-insensitive* demands, respectively. The information-sensitive demand is linear in shock ε, and the portfolio is separable in ε and prices p, as in the related literature [e.g., Grossman and Stiglitz (1980), Breon-Drish (2015)]. The information-insensitive demand depends on risk-neutral probabilities, which can be found analytically as functions of prices p by solving the risk-neutral pricing equations (8). Hence, portfolio (11) is in closed form in terms of shock ε and price p, which are observable by investor I.

Portfolio (11) can be rewritten as $\theta_I^*(p;\varepsilon) = \Omega^{-1}\left(\widetilde{b}\varepsilon - (\widetilde{v}(p) - \widetilde{a})\right)/\gamma_I$, where $\widetilde{a}, \widetilde{b} \in \mathbb{R}^{N-1}$ are vectors with elements $a_n - a_N$ and $b_n - b_N$, respectively. Portfolio $\theta_I^*(p;\varepsilon)$ resembles investor I's portfolio in Grossman and Stiglitz (1980), which in our notation is given by $(\varepsilon - pe^{rT})/(\gamma_I \sigma_C^2)$, where σ_C is the payoff volatility. In particular, $\theta_I^*(p;\varepsilon)$ retains the separability in shock ε and prices p, and matrix Ω plays a similar role as variance σ_C^2.

The separability of portfolio (11) in shock ε and prices p is due to CARA utility of investor I. Taking logs on both sides of investor I's FOC in (10), we find that the wealth of investor I is separable in ε and p and given by

$$W_{I,T,n} = \frac{\ln(\pi_n(\varepsilon))}{\gamma_I} - \left(\frac{\ln(\pi_n^{\mathrm{RN}})}{\gamma_I} + const\right). \tag{13}$$

Decomposition (13) demonstrates that investor I has incentives to allocate more wealth to more likely states, and less wealth to states with higher risk-neutral probabilities.[4] The information-sensitive and information-insensitive components of portfolio (11) replicate the two components of wealth $W_{I,T,n}$ in (13).[5]

Vector λ in portfolio (11) has the following interpretation. The definition of λ in Lemma 1 implies that λ satisfies equations $b_n = \lambda_0 + C(\omega_n)^\top \lambda$, where λ_0 is a constant. Therefore, λ can be interpreted as a replicating portfolio for the realisations of shock sensitivities b_n in states ω_n, up to a constant λ_0. Hence, $\lambda\varepsilon$ replicates $b\varepsilon$. Investing in this portfolio allows investor I to replicate (up to a constant) the dependence on ε of term $\ln(\pi_n(\varepsilon))/\gamma_I$ in decomposition (13), and hence, have more wealth in more likely states.

Substituting $\theta_I^*(p;\varepsilon)$ and $\theta_U^*(p)$ into the market clearing condition (6), we find that

$$\frac{\lambda\varepsilon}{\gamma_I} + \nu + H(p) = 0, \tag{14}$$

[4] The incentive to allocate less wealth to states with higher risk-neutral probabilities arises due to a price effect because $\pi_n^{\mathrm{RN}} e^{-rT}$ is the value of \$1 in state ω_n.

[5] We note that separability breaks down for general utility $U(W_{I,T})$. The first order condition $U'(W_{I,T,n}) = \ell_I \pi_n^{\mathrm{RN}} e^{-rT}/\pi_n(\varepsilon)$ implies that wealth $W_{I,T,n}$ cannot be split into two additive components that depend on ε and p, respectively, for non-CARA utilities.

where $H(p)$ is a function of asset prices p, discussed below, which is given by:

$$H(p) = \theta_U^*(p) - \frac{1}{\gamma_I}\Omega^{-1}\Big(\widetilde{v}(p) - \widetilde{a}\Big), \tag{15}$$

and $\widetilde{v}(p)$, $\widetilde{a}$ are as in Lemma 1.

Investor U uses Equation (14) to infer information about shock ε. Substituting prices p into this equation, investor U learns *sufficient statistic* $\lambda\varepsilon/\gamma_I + \nu$, which is equal to $-H(p)$. Therefore, function $H(p)$ reveals information in prices, and hence, we label it *the informational content of prices*. Next, investor U finds posterior probabilities of states given by the conditional expectation $\pi_n^U(p) = \mathbb{E}\Big[\pi_n(\varepsilon)|\lambda\varepsilon/\gamma_I + \nu = -H(p)\Big]$. The derivation of probabilities $\pi_n^U(p)$ is simplified by the structures of probabilities $\pi_n(\varepsilon)$ in (1) and the prior distribution $\varphi_\varepsilon(x)$ of shock ε in (2). It is also simplified by the fact that the sufficient statistic does not depend on asset prices, which is due to the separability of investor I's portfolio (11) in shock ε and prices p. Lemma 2 reports the posterior probabilities.

Lemma 2 (Posterior probabilities of states). *The posterior probabilities of states ω_n conditional on observing the sufficient statistic $\lambda\varepsilon/\gamma_I + \nu$ are given by:*

$$\pi_n^U(p) = \frac{1}{G(p)} \exp\left\{ a_n + \frac{1}{2}\frac{b_n^2 + 2b_n\Big(\mu_0/\sigma_0^2 - \lambda^\top\Sigma_\nu^{-1}H(p)/\gamma_I\Big)}{\lambda^\top\Sigma_\nu^{-1}\lambda/\gamma_I^2 + 1/\sigma_0^2} \right\}, \tag{16}$$

where function $H(p)$ is given by Equation (15)*, and $G(p)$ is a normalizing function.*

We then find the equilibrium as follows. From the FOC (10) for investor U we find investor U's optimal portfolio $\theta_U^*(p)$ in terms of probabilities (16), similar to finding investor I's portfolio (11). We observe, that probabilities (16) themselves depend on portfolio $\theta_U^*(p)$ via the informational content of prices $H(p)$. Hence, portfolio $\theta_U^*(p)$ solves a fixed-point problem [Equation (A.15) in the Appendix], which we do not show here for brevity. The fixed-point problem turns out to be linear, and hence, we find portfolio $\theta_U^*(p)$ in closed form. Then, we find the equilibrium risk-neutral probabilities and asset prices from the market clearing condition (14). In this paper, we focus on equilibria in which asset prices are continuous functions of the sufficient statistic $\lambda\varepsilon/\gamma_I + \nu$, and the sufficient statistic is the only information revealed by prices.[6] Proposition 1 reports the equilibrium.

Proposition 1 (Equilibrium with $M = N$ assets).

i) *There exists unique equilibrium in which prices only reveal $\lambda\varepsilon/\gamma_I + \nu$. In this equilibrium,*

[6] Pálvölgyi and Venter (2015) show that our complete market equilibrium is unique among all equilibria with continuous prices. However, Pálvölgyi and Venter (2014) demonstrate the existence of multiple discontinuous equilibria in a Grossman and Stiglitz (1980) economy, which may also exist in our model. Finding such equilibria in multi-asset economies is a challenging task and is beyond the scope of our work.

the vector of risky asset prices $P(\varepsilon,\nu)$ and risk-neutral probabilities π_n^{RN} are given by:

$$P(\varepsilon,\nu) = \Big[\pi_1^{\text{RN}} C(\omega_1) + \pi_2^{\text{RN}} C(\omega_2) + \ldots + \pi_N^{\text{RN}} C(\omega_N)\Big] e^{-rT}, \tag{17}$$

$$\pi_n^{\text{RN}} = \frac{e^{v_n}}{\sum_{j=1}^N e^{v_j}}, \tag{18}$$

where probability parameters v_n are given in closed form by:

$$v_n = a_n + \frac{1}{2}\frac{\gamma_I}{\gamma_I+\gamma_U}\frac{b_n^2 + 2(\mu_0/\sigma_0^2)b_n}{\lambda^\top\Sigma_\nu^{-1}\lambda/\gamma_I^2 + 1/\sigma_0^2} + \frac{\gamma_I\gamma_U}{\gamma_I+\gamma_U} C(\omega_n)^\top \Big(E+Q\Big)\Big(\frac{\lambda\varepsilon}{\gamma_I} + \nu\Big), \tag{19}$$

$$Q = \frac{\lambda\lambda^\top\Sigma_\nu^{-1}}{\gamma_U\gamma_I\Big(\lambda^\top\Sigma_\nu^{-1}\lambda/\gamma_I^2 + 1/\sigma_0^2\Big)}, \tag{20}$$

where E is an identity matrix, $Q, E \in \mathbb{R}^{(N-1)\times(N-1)}$, and $\lambda = \Omega^{-1}(b_1 - b_N, \ldots, b_{N-1} - b_N)^\top$.

ii) *Portfolio $\theta_I^*(p;\varepsilon)$ is given by equation* (11) *and portfolio $\theta_U^*(p)$ is given by*

$$\theta_U^*(p) = \Big(E+Q\Big)^{-1}\left(\frac{Q\Omega^{-1}(\widetilde{v}(p) - \widetilde{a})}{\gamma_I} - \frac{\Omega^{-1}(\widetilde{v}(p)-\widehat{a})}{\gamma_U} + \frac{\mu_0/(\gamma_U\sigma_0^2)\lambda}{\lambda^\top\Sigma_\nu^{-1}\lambda/\gamma_I^2 + 1/\sigma_0^2}\right), \tag{21}$$

where $\widetilde{v}(p) \in \mathbb{R}^{N-1}$ is given by (12) *and has elements $v_n - v_N$ in equilibrium, and $\widetilde{a}, \widehat{a} \in \mathbb{R}^{N-1}$ have elements $a_n - a_N$ and $a_n - a_N + 0.5(b_n^2 - b_N^2)/(\lambda^\top\Sigma_\nu^{-1}\lambda/\gamma_I^2 + 1/\sigma_0^2)$, respectively.*

Proposition 1 extends the no-arbitrage valuation approach to economies with asymmetric information and provides asset prices in closed form in terms of expected discounted payoffs under risk-neutral probabilities, familiar from the asset-pricing literature. The equilibrium prices are non-linear functions of shock ε and noise ν, in contrast to CARA-normal noisy REE models [e.g., Grossman and Stiglitz (1980); Admati (1985), among others]. The tractability of equilibrium prices and portfolios allows us to study various comparative statics, which we report in Proposition 2 below.

Proposition 2 (Comparative statics). *The comparative statics for price $P_m(\varepsilon,\nu)$ of risky asset m with respect to shock ε and noisy demands ν are as follows:*

$$\frac{\partial P_m(\varepsilon,\nu)}{\partial\varepsilon} = \frac{\gamma_U}{\gamma_U+\gamma_I}\left(1 + \frac{\lambda^\top\Sigma_\nu^{-1}\lambda/(\gamma_U\gamma_I)}{\lambda^\top\Sigma_\nu^{-1}\lambda/\gamma_I^2 + 1/\sigma_0^2}\right)\text{cov}^{\text{RN}}(b, C_m)e^{-rT}, \tag{22}$$

$$\frac{\partial P_m(\varepsilon,\nu)}{\partial\nu_l} = \frac{\gamma_U\gamma_I}{\gamma_U+\gamma_I}\left(\text{cov}^{\text{RN}}(C_l, C_m) + \frac{\lambda^\top\Sigma_\nu^{-1}e_l/(\gamma_U\gamma_I)}{\lambda^\top\Sigma_\nu^{-1}\lambda/\gamma_I^2 + 1/\sigma_0^2}\text{cov}^{\text{RN}}(b, C_m)\right)e^{-rT}. \tag{23}$$

The comparative statics for investors' portfolios with respect to prices p are as follows:

$$\frac{\partial\theta_I^*(p;\varepsilon)}{\partial p} = -\frac{1}{\gamma_I}\Big(\text{var}^{\text{RN}}[C]\Big)^{-1}e^{rT}, \tag{24}$$

$$\frac{\partial\theta_U^*(p)}{\partial p} = -\left(\frac{1}{\gamma_U}E - \frac{\gamma_U+\gamma_I}{\gamma_U\gamma_I}\frac{\lambda\lambda^\top\Sigma_\nu^{-1}}{\gamma_U\gamma_I\Big(\lambda^\top\Sigma_\nu^{-1}\lambda/\gamma_I^2 + 1/\sigma_0^2\Big) + \lambda^\top\Sigma_\nu^{-1}\lambda}\right)\Big(\text{var}^{\text{RN}}[C]\Big)^{-1}e^{rT}, \tag{25}$$

where $\mathrm{cov}^{\mathrm{RN}}(\cdot,\cdot)$ *is covariance and* $\mathrm{var}^{\mathrm{RN}}[\cdot]$ *is the variance-covariance matrix under the risk-neutral probability measure. Furthermore, informed investor's demand for risky asset* m *is a downward-sloping function of that asset's price* p_m*, holding other prices fixed.*

The intuition for the effect of shock ε on asset prices is as follows. Positive shock ε increases the probabilities (both real and risk-neutral) of states with higher shock sensitivities b_n. Therefore, the prices of assets that pay more in states with higher b_n increase, and the opposite happens for a negative shock, which gives rise to the covariance term in (22). The asymmetry of information increases the sensitivity of asset prices to shock ε as captured by the second term in brackets in (22). Intuitively, the uninformed investor interprets high prices of assets that positively covary with b as the information that shock ε is positive, and hence, further increases the demand for such assets, which generates an amplification.

The effect of noise traders' demand on asset prices can be decomposed into substitution and information effects, which correspond to the first and second terms in the brackets in (23), respectively. The first term demonstrates that positive demand shock ν_l to asset l exerts positive pressure on the price of asset m if the payoffs of assets m and l have positive covariance. This is because following positive demand shock ν_l and the resulting increase in the price of asset l investors partially substitute asset l with asset m which positively covaries with the former. As a result, the price of asset m increases.

The second term in (23) arises due to the difficulty of disentangling the effects of noise ν_l and shock ε. Consider, for example, asset m with payoffs that positively covary with shock sensitivities b. If noisy demand ν_l increases the price of asset m, such increases may partially be attributed to a positive shock ε, as outlined in the intuition for Equation (22) above. Hence, noisy demand ν_l affects the posterior distribution of ε, and through this distribution affects the prices of other assets. Thus, learning from prices generates contagions by spreading demand shocks across assets.

Equations (24) and (25) show the sensitivities of investors' asset demands to prices. The demand (24) of the informed investor is determined by the inverse risk-neutral variance-covariance matrix. Hence, the informed investor's demand for risky asset m is a downward sloping function of that asset's own price p_m because the elements on the main diagonal of a positive-definite matrix $(\mathrm{var}^{\mathrm{RN}}[C])^{-1}$ are all positive (see proof of Proposition 2).

The sensitivity of portfolio $\theta^*_U(p)$ to prices p is determined by the two terms in brackets in Equation (25), which capture the substitution and information effects, respectively. The demand of investor U for asset m can be an upward sloping function of p_m because the product of the matrices in (25), in general, is not a positive-definite matrix. Intuitively, high asset prices can be interpreted as positive information about shock ε, in which case the demand for asset m may increase despite high price p_m. Admati (1985) finds a similar result in a multi-asset CARA-normal model. Our analysis extends the finding of Admati (1985) to economies without normality and reveals the important roles of the replicating portfolio λ and the risk-neutral variance of asset payoffs in generating these effects.

3.2. General Economy with $M \leq N$ Securities

In this section, we study an economy with M securities, where $M \leq N$, which subsumes complete and incomplete market economies as special cases. For tractability, we focus on a subset of economies which satisfy the following condition.

Informational spanning condition: *There exists a replicating portfolio for shock sensitivities b. That is, there exist constant λ_0 and vector $\lambda = (\lambda_1, \ldots, \lambda_{M-1})^\top$ such that*

$$b_n = \lambda_0 + C(\omega_n)^\top \lambda. \tag{26}$$

We provide several plausible economies that satisfy Condition (26). A first example is the complete-market economy, where $M = N$, and hence, there always exist constant λ_0 and vector λ satisfying Equation (26). A second example is an incomplete-market economy with only one risky asset with payoff $C_1 = b/\lambda_1$, where λ_1 is a constant. Lemma A.2 in the Appendix shows that single risky asset economies in Grossman and Stiglitz (1980) and Breon-Drish (2015) satisfy condition $C_1 = b/\lambda_1$. A third example is an economy with an asset paying $C_1 = b/\lambda_1$, and call options on C_1 with payoffs $C_2 = \max(C_1 - K_2, 0)$,..., $C_{M-1} = \max(C_1 - K_{M-1}, 0)$, in which case $\lambda_0 = 0$, $\lambda = (\lambda_1, 0, \ldots, 0)^\top$. A fourth example is an economy with a firm that has cash flow b and issues risky debt and equity with payoffs $\min(b, K)$ and $\max(b - K, 0)$, where K is the face value of debt. It is easy to observe that in this last example $\lambda_0 = 0$ and $\lambda = (1, 1)^\top$ because $b = \min(b, K) + \max(b - K, 0)$.

We begin with the derivation of informed investor's portfolio in Lemma 3 below.

Lemma 3 (Investor I's optimal portfolio). *Suppose that Condition* (26) *is satisfied. Then, the informed investor's demand is a linear function of ε, given by*

$$\theta_I^*(p; \varepsilon) = \frac{\lambda \varepsilon}{\gamma_I} - \frac{\widehat{\theta}_I^*(p)}{\gamma_I}, \tag{27}$$

where vector λ is such that Condition (26) *is satisfied, and $\widehat{\theta}_I^*(p)$ is a function of p.*

Investor I's portfolio (11) in the complete market economy is a special case of portfolio (27) when $M = N$. The difference between these portfolios is that portfolio (11) is in closed form whereas component $\widehat{\theta}_I^*(p)$ of portfolio (27), in general, is not.

Portfolio (27) is separable in shock ε and price p and linear in ε. Breon-Drish (2015) shows that in an economy with one risky asset these properties of investor I's portfolio arise when the PDF of asset payoff C conditional on ε belongs to the *exponential family* of distributions, that is, has the form $\varphi_{C|\varepsilon}(x) = H(\varepsilon)\exp(A(x) + B(x)\varepsilon)$ [see Casella and Berger (2002, p. 114)], with a linear function $B(x) = \lambda x$. From Equation (1) for real probabilities, we observe that a random variable with realizations b_n and probabilities $\pi_n(\varepsilon)$ has a distribution from that family with $B(x) = x$. Condition (26) then implies that a linear combination of asset payoffs $C(\omega_n)^\top\lambda$ has the same distribution as b_n, which then yields the separability of portfolio (27) following the same steps as in Breon-Drish (2015). We remark, however, that the conditional payoff distribution of an individual asset m, $\text{Prob}(C_m|\varepsilon)$, need not be in the exponential family.[7]

The intuition for Condition (26) is as follows. Although investor I observes shock ε, the realization of state ω remains uncertain. Investor I hedges the exposure to this uncertainty by constructing a portfolio that replicates a state-dependent random variable with realizations b_n. Investing in this portfolio helps investor I allocate more wealth to more likely states because the log-likelihoods of states are given by $\ln(\pi_n(\varepsilon)) = a_n + b_n\varepsilon + const$. Condition (26) guarantees that shock exposures $b\varepsilon$ can be perfectly replicated by a portfolio with $\lambda_m\varepsilon$ units of each risky asset $m = 1, \ldots, M-1$, giving rise to the information-sensitive demand $\lambda\varepsilon/\gamma_I$ in Equation (27). Our intuition also explains why the exponential family distributions in the literature are only tractable with a linear function $B(x)$: this linearity implies the existence of the replicating portfolio for $b\varepsilon$.

CARA utility plays important role in the separability of portfolio (27) in ε and p and the intuition for Condition (26). Using duality theory, Lemma IA.3 in the Internet Appendix shows the existence of a risk-neutral measure $\pi_{I,n}^{\text{RN}}$ such that the optimal wealth of investor I satisfies the FOC $\gamma_I e^{-\gamma_I W_{I,T,n}} = \ell_I \pi_{I,n}^{\text{RN}} e^{-rT}/\pi_n(\varepsilon)$. This FOC implies that $W_{I,T,n} = \ln(\pi_n(\varepsilon))/\gamma_I - \ln(\pi_n^{\text{RN}})/\gamma_I + const$. Then, similar to the complete-market case, we conclude that CARA utility is essential for separability and that portfolio $\lambda\varepsilon$ replicates the exposure of log-likelihoods $\ln(\pi_n(\varepsilon))$ to shock ε.

The above intuition captures important aspects of the informed investor's portfolio choice even when Condition (26) is violated. In Lemma IA.4 in the Internet Appendix we consider an economy where Condition (26) is violated and approximate investor I's expected utility (3) by a quadratic function of portfolio weights. We show that the first-

[7] Consider an economy where $b = (1,2,3,4)^\top$, $C_1 = (1,2,3,4)^\top$ and $C_2 = (1,0,0,0)^\top$. Then, Condition (26) is satisfied with $\lambda = (1,0)^\top$. However, probabilities $\text{Prob}(C_2(\omega) = 1|\varepsilon) = \pi_1(\varepsilon)$ and $\text{Prob}(C_2(\omega) = 0|\varepsilon) = 1 - \pi_1(\varepsilon)$ cannot be rewritten as $\text{Prob}(C_2(\omega) = x|\varepsilon) = H(\varepsilon)\exp(A(x) + B(x)\varepsilon)$. We further note that the distribution and the moments of payoff $C_2(\omega)$ depend on ε despite $\lambda_2 = 0$.

order term of the optimal portfolio has the same structure as portfolio (27), where vector λ is such that $\lambda_0 + C^\top\lambda$ is a projection of the shock sensitivities b on the linear space of all portfolios. Hence, investor I replicates shock exposures $b\varepsilon$ as closely as possible.

Similar to the complete-market economy, the market clearing condition now is given by $\lambda\varepsilon/\gamma_I + \nu + \widehat{H}(p) = 0$, where $\widehat{H}(p) \equiv \theta_U^*(p) - \widehat{\theta}_I^*(p)/\gamma_I$ is the informational content of prices. Investor U finds the posterior probabilities following the same steps as in the complete-market case. Given these probabilities, we derive the portfolios, and then the asset prices from the market clearing condition. Proposition 3 reports the equilibrium.

Proposition 3 (Equilibrium with $M \leq N$ assets).

i) *Let the informational spanning condition* (26) *be satisfied. Then, there exists unique price vector $P(\varepsilon, \nu)$ that is a continuous, differentiable, and invertible on its range function of sufficient statistic $\lambda\varepsilon/\gamma_I + \nu$. Price vector $P(\varepsilon, \nu)$ is the unique solution of equation*

$$\frac{f_I^{-1}\Big(e^{rT}P(\varepsilon,\nu)\Big)}{\gamma_I} + \frac{f_U^{-1}\Big(e^{rT}P(\varepsilon,\nu)\Big)}{\gamma_U} = \Big(E+Q\Big)\left(\frac{\lambda\varepsilon}{\gamma_I}+\nu\right) + \frac{\mu_0/(\gamma_U\sigma_0^2)\lambda}{\lambda^\top\Sigma_\nu^{-1}\lambda/\gamma_I^2 + 1/\sigma_0^2}, \quad (28)$$

where E is the identity matrix, Q is a matrix given by Equation (20), $E, Q \in \mathbb{R}^{(M-1)\times(M-1)}$, *and functions $f_I, f_U : \mathbb{R}^{M-1} \to \mathbb{R}^{M-1}$ are invertible on their ranges and given by*

$$f_I(x) = \frac{\sum_{j=1}^N C(\omega_j)\exp\{a_j + C(\omega_j)^\top x\}}{\sum_{j=1}^N \exp\{a_j + C(\omega_j)^\top x\}}, \quad (29)$$

$$f_U(x) = \frac{\sum_{j=1}^N C(\omega_j)\exp\{a_j + \frac{1}{2}\frac{b_j^2}{\lambda^\top\Sigma_\nu^{-1}\lambda/\gamma_I^2+1/\sigma_0^2} + C(\omega_j)^\top x\}}{\sum_{j=1}^N \exp\{a_j + \frac{1}{2}\frac{b_j^2}{\lambda^\top\Sigma_\nu^{-1}\lambda/\gamma_I^2+1/\sigma_0^2} + C(\omega_j)^\top x\}}. \quad (30)$$

ii) *The informed and uninformed investors' optimal portfolios are given by*

$$\theta_I^*(p;\varepsilon) = \frac{\lambda\varepsilon}{\gamma_I} - \frac{f_I^{-1}\Big(e^{rT}p\Big)}{\gamma_I}, \quad (31)$$

$$\theta_U^*(p) = \Big(E+Q\Big)^{-1}\left(\frac{Qf_I^{-1}\Big(e^{rT}p\Big)}{\gamma_I} - \frac{f_U^{-1}\Big(e^{rT}p\Big)}{\gamma_U} + \frac{\mu_0/(\gamma_U\sigma_0^2)\lambda}{\lambda^\top\Sigma_\nu^{-1}\lambda/\gamma_I^2+1/\sigma_0^2}\right). \quad (32)$$

Portfolios (31) and (32) have similar structure as the complete-market portfolios. Furthermore, both in incomplete and complete markets prices are non-linear functions of the sufficient statistic $\lambda\varepsilon/\gamma_I + \nu$. However, in incomplete markets, in general, the prices are not available in closed form and satisfy non-linear equations (28). Solving and analyzing these equations, in general, is a challenging task when the number of unknowns is large. The intractability of the incomplete-market equilibrium is in stark contrast with our complete-market equilibrium where prices are available in closed form.

Another contribution of Proposition 3 is the proof of the existence of equilibrium. This proof is complicated by market incompleteness and the multiplicity of risky assets. As a result, proofs in the related models with a single risky asset [e.g., Pálvölgyi and Venter (2014), Breon-Drish (2015)], which are based on the intermediate value theorem, do not apply in our model. Our proof in the Appendix relies on well-known results of basic calculus such as Weierstrass's compactness theorem ([e.g., Rudin (1976, Theorem 2.42)]) and an implicit function theorem ([e.g., Rudin (1976, Theorem 9.28)]). Although the latter theorem guarantees only local existence and uniqueness, using Weierstrass's theorem and the fact that functions $f_I(x)$ and $f_U(x)$ in (29) and (30) have globally invertible Jacobians we show that the solution can be extended globally.

4. Economic Applications

In this section, we provide several applications of our theory. In Section 4.1 we study the informativeness of derivative prices and introduce a new concept of informationally irrelevant securities. In the same section, we explore the informational content of corporate debt and equity. Then, in Section 4.2, we study volatility derivatives and show that they make financial markets effectively complete. In Section 4.3 we derive new closed-form complete-market prices, compare them with their less tractable incomplete-market counterparts, and use them to study the effect of payoff skewness on asset prices.

4.1. Information Revelation in Asset Markets

In this section, we study how the information about the aggregate shock revealed by the prices depends on the type of traded assets. We start with the analysis of the informed investor's portfolio in Equation (27). This equation implies that the informed investor's demand for asset m depends on shock ε if and only if $\lambda_m \neq 0$, where λ_m is the m^{th} component of vector λ. Otherwise, if $\lambda_m = 0$, the informed investor's demand for asset m does not contain any information about ε, and hence, asset m is *informationally irrelevant.*

The informational irrelevance of derivatives is a generic property of economies with one underlying asset with payoff $C_1 = b/\lambda_1$ and derivative securities written on this asset. In such economies, the informational spanning condition (26) is satisfied with the replicating portfolio weights $\lambda = (\lambda_1, 0, \dots, 0)^\top$. As demonstrated in Lemma A.2 in the Appendix, condition $C_1 = b/\lambda_1$ is satisfied in the single risky asset economies of Grossman and Stiglitz (1980) and Breon-Drish (2015). Consequently, adding any non-redundant securities to these economies does not reveal more information about ε (provided that noises ν are uncorrelated across assets, as elaborated below).

As shown in Section 3, the assets with $\lambda_m \neq 0$ are used by investor I for replicating shock exposures $b\varepsilon$ to achieve higher utility given the realization of ε. From Equations (31) and (32) we observe that assets with $\lambda_m = 0$ are still held by investors for reasons unrelated to information, such as trading with each other and against noise traders. Moreover, the conditional payoff distributions $\text{Prob}(C_m|\varepsilon)$ and the prices of such assets still depend on shock ε through real probabilities $\pi_n(\varepsilon)$ and risk-neutral probabilities of states.

The informativeness of prices also depends on the correlations of noises. Consider an example with two risky assets and perfectly correlated noises $\nu_1 = \nu_2$. Taking the difference of the market clearing conditions (14) for the two markets we find that $(\lambda_1 - \lambda_2)\varepsilon/\gamma_I + (\nu_1 - \nu_2) + (1, -1)^\top H(p) = 0$. Shock ε can then be perfectly learned from prices if $\lambda_1 \neq \lambda_2$ and is given by $\varepsilon = \gamma_I(-1, 1)^\top H(p)/(\lambda_1 - \lambda_2)$. The effects of the informed investor's demand and noise correlations on the learning from prices by investor U are captured by vector λ and matrix Σ_ν in Equation (16) for investor U's posterior probabilities, respectively.

To illustrate informational irrelevance, we consider an economy with a risky asset with payoff $C_1 = b/\lambda_1$, as in Grossman and Stiglitz (1980) and Breon-Drish (2015). Then, we add $M - 2$ derivative securities written on that asset and assume that all noises ν_m are i.i.d. In this economy, $\lambda = (\lambda_1, 0, \dots, 0)^\top$. Prices then reveal sufficient statistic $\lambda\varepsilon/\gamma_I + \nu = (\lambda_1\varepsilon/\gamma_I + \nu_1, \nu_2, \dots, \nu_{M-1})^\top$. However, because noises ν_m are i.i.d., only the first element $\lambda_1\varepsilon/\gamma_I + \nu_1$ of the sufficient statistic provides information about ε. Hence, the number of assets does not affect the posterior distribution of ε and posterior probabilities $\pi_n^U(p)$. The derivative securities are thus informationally irrelevant.

The empirical literature on price discovery in financial markets finds that the prices and trading volumes of derivative securities reveal information about the payoffs of the underlying asset [e.g., Easley, O'Hara, and Srinivas (1998); Chakravarty, Gulen, and Mayhew (2004); Pan and Poteshman (2006)]. This evidence is consistent with our model when the derivatives help replicate shock sensitivities b or noise trader demands are correlated.

We now apply the results of this section to corporate finance, and study the informational content of risky debt and equity. Consider a firm with cash flow b_n in state ω_n. The real probabilities of states $\pi_n(\varepsilon)$ and PDF $\varphi_\varepsilon(x)$ of shock ε are given by equations (1) and (2), respectively. Suppose, investors trade the firm's debt and equity with payoffs $\min(b, K)$ and $\max(b - K, 0)$, respectively, where $K > 0$ is the face value of debt. The informational spanning condition is satisfied for debt and equity with the replicating portfolio weights $\lambda = (1,1)^\top$ because $b = \min(b, K) + \max(b - K, 0)$. The prices of debt and equity can be obtained by solving Equation (28). Because weights $\lambda = (1,1)^\top$ do not depend K, the sufficient statistic, the posterior distribution of ε, and posterior probabilities of states also do not depend on K. Hence, the face value of debt K is irrelevant for the amount of information jointly revealed by the prices of debt and equity.

Our irrelevance result is not a mere consequence of the Modigliani and Miller theorem because of a market imperfection in the form informational asymmetry. This result emerges because investor I buys the same number of units of debt and equity to replicate sensitivities b, which is similar to buying directly a security with payoff b. Such portfolio composition of investor I is difficult to anticipate without our theory because, in general, the uninformed investor may want to purchase less informationally sensitive securities such as debt $\min(b, K)$ when K is small. Then, the informed investor would hold more equity than debt in equilibrium.

4.2. Effective Completeness and Volatility Derivatives

We now apply our model to study the economic role of volatility derivatives. We show that adding certain volatility derivatives to incomplete markets makes these markets *effectively complete*, that is, allows investors to achieve Pareto-optimal allocations. In a setting where investors have different probabilities of states Pareto optimality requires the ratio of probability-weighted marginal utilities of investors (i.e., marginal rate of substitution) to be the same across all states [e.g., Amershi (1985); Brennan and Cao (1996)]:

$$\frac{\pi_n(\varepsilon)\exp\{-\gamma_I W_{I,T,n}\}}{\pi_n^U(p)\exp\{-\gamma_U W_{U,T,n}\}} = \ell, \tag{33}$$

where ℓ is constant across states ω. Proposition 4 below provides necessary and sufficient conditions for effective completeness.

Proposition 4 (Conditions for Effective Completeness).

i) *Suppose, the market is incomplete, that is, $M < N$, and the informational spanning condition* (26) *is satisfied. Then, the market is effectively complete if and only if there exists a replicating portfolio for squared sensitivities b_n^2, that is, there exist constant $\widehat{\lambda}_0$ and vector $\widehat{\lambda} \in \mathbb{R}^{M-1}$ such that $b_n^2 = \widehat{\lambda}_0 + C(\omega_n)^\top \widehat{\lambda}$, for all $n = 1, \ldots, N$.*

ii) *If the market is effectively complete, the prices of risky assets are given by Equation* (17), *as in a complete market, with the only difference that vectors λ, ν, and matrices Ω, E and Q are of lower dimensions: $\lambda, \nu \in \mathbb{R}^{M-1}$, $\Omega \in \mathbb{R}^{(N-1)\times(M-1)}$, and $E, Q \in \mathbb{R}^{(M-1)\times(M-1)}$.*

We demonstrate the implications of Proposition 4 in a simple economy with one asset that has payoff $C_1 = b/\lambda_1$. The proposition implies that the market can be effectively completed by introducing a derivative security with a quadratic payoff such as C_1^2 or $(C_1 - \mathbb{E}^{\text{RN}}[C_1])^2$. The last security is similar to a simple variance swap (SVIX) introduced in Martin (2013). The prices of this security and the underlying asset are then given in closed form by $\text{var}^{\text{RN}}[C_1]e^{-rT}$ and $\mathbb{E}^{\text{RN}}[C_1]e^{-rT}$, respectively. Therefore, we interpret assets with quadratic payoffs as volatility derivatives. Volatility derivatives are widely traded in financial markets, and hence, adding them to our model makes it more realistic.

Next, we explain the importance of volatility derivatives for achieving effective completeness. Taking logs on both sides of Equation (33), after simple algebra, we find that $\gamma_U W_{U,T,n} - \gamma_I W_{I,T,n} - \ln(\ell) = \ln\left(\pi_n^U(p)/\pi_n(\varepsilon)\right)$. The expression $\gamma_U W_{U,T,n} - \gamma_I W_{I,T,n} - \ln(\ell)$ can be interpreted as the payoff of a portfolio that invests in bonds and $\gamma_U \theta_U^* - \gamma_I \theta_I^*$ units of the risky assets. Therefore, effective completeness is feasible only if the log-ratio of investors' probabilities $\ln\left(\pi_n^U(p)/\pi_n(\varepsilon)\right)$ is replicated by a portfolio of tradable assets. Moreover, the log-probability $\ln(\pi_n^U(p))$ is a quadratic function of b_n (see equations 16 and A.33), and hence, Pareto allocations of wealth are quadratic functions of b_n. Consequently, the existence of a replicating portfolio for b_n^2 is essential for achieving Pareto optimality.

Previous literature has noted that introducing contracts and securities with quadratic payoffs can lead to Pareto-optimality in CARA-normal models of syndicates with belief heterogeneity [e.g., Wilson (1968)] and REE models of asset markets [e.g., Brennan and Cao (1996)]. We show that the role of quadratic derivatives in achieving Pareto optimality extends to models with more realistic and more general payoff distributions. Moreover, adding these securities to incomplete-market economies yields asset prices in closed form, which are in general different from their incomplete-market counterparts. This is in contrast to the CARA-normal case where adding a volatility derivative does not affect the price of the underlying asset (see Lemma 4 below). We further explore these closed-form prices in the next section.

4.3. Effective Completeness, Asset Prices, and Payoff Skewness

In this section, for a broad class of payoff distributions, we derive new analytic expressions for asset prices in effectively complete markets with traded volatility derivatives. We study the properties of these prices and compare them to prices in incomplete markets. We then apply our methodology to study the effect of payoff skewness on asset prices. The fact that volatility derivatives are actively traded in financial markets makes our effectively complete markets an economically realistic and tractable alternative to incomplete markets for studying the effects of asymmetric information on asset prices.

Our benchmark is the economy with one risky asset with payoff C_1 which has a continuous unconditional PDF $\varphi_C(x)$. The informed investor observes signal $\varepsilon = C_1 + u$, where $u \sim \mathcal{N}(\mu_0, \sigma_0^2)$. The uninformed investor learns about ε from prices, and noise traders submit exogenous demand ν_1. This economy is a special limiting case of our economy in Section 2 with distribution parameters a_n and b_n given by Equations (7) when $N \to \infty$, as noted in Remark 1 and demonstrated in Lemma A.2 in the Appendix. From Equations (7) we observe that the informational spanning condition is satisfied because $b_n = C_1(\omega_n)/\sigma_0^2$.

Following Section 4.2, we make the economy effectively complete by adding a volatility derivative with payoff C_1^2. To facilitate the comparison of the economies with and without the volatility derivative, we assume that the noise traders do not trade in the derivative. As a result, the prices in the incomplete and effectively complete markets depend on the same scalar sufficient statistic $s = \varepsilon/(\gamma_I \sigma_0^2) + \nu_1$. The equilibrium is not fully revealing because the derivative is informationally irrelevant[8] (as defined in Section 4.1), that is, $b = C_1/\sigma_0^2 + 0 \cdot C_1^2$. Proposition 5 below reports asset prices in the two economies.

Proposition 5 (Equilibrium Prices). *The prices of the asset with payoff C_1 that has PDF $\varphi_C(x)$ in the effectively complete and incomplete market economies are given by*

$$P_{com}(s) = P_C\left(\gamma_I \sigma_0^2 s - \mu_0, \sigma_{com}\right) e^{-rT}, \quad P_{inc}(s) = P_C\left(\gamma_I \sigma_0^2 \widehat{s}(s) - \mu_0, \sigma_0\right) e^{-rT}, \tag{34}$$

respectively, where the pricing function $P_C(\mu, \sigma)$ is given by equation

$$P_C(\mu, \sigma) = \frac{\mathbb{E}\left[C_1 \exp\left(-\frac{(C_1-\mu)^2}{2\sigma^2}\right)\right]}{\mathbb{E}\left[\exp\left(-\frac{(C_1-\mu)^2}{2\sigma^2}\right)\right]}, \tag{35}$$

$\mathbb{E}[\cdot]$ is the expectation with respect to PDF $\varphi_C(x)$, $s = \varepsilon/(\gamma_I\sigma_0^2) + \nu_1$, $\widehat{s}(s)$ in the equation for $P_{inc}(s)$ is an increasing implicit function of s satisfying equation

$$P_C\left(\gamma_I \sigma_0^2 \widehat{s}(s) - \mu_0, \sigma_0\right) = P_C\left(\gamma_I \sigma_0^2 s - \mu_0 + \gamma_U \sigma_{inc}^2 (s - \widehat{s}(s)), \sigma_{inc}\right), \tag{36}$$

and the volatility parameters σ_{com} and σ_{inc} are given by:

$$\sigma_{com} = \frac{\sigma_0}{\sqrt{1 - \frac{\gamma_I}{\gamma_I + \gamma_U} \frac{1}{1 + 1/\left(\gamma_I^2 \sigma_\nu^2 \sigma_0^2\right)}}}, \quad \sigma_{inc} = \frac{\sigma_0}{\sqrt{1 - \frac{1}{1 + 1/\left(\gamma_I^2 \sigma_\nu^2 \sigma_0^2\right)}}}. \tag{37}$$

Moreover, both prices $P_{com}(s)$ and $P_{inc}(s)$ are increasing functions of s.

[8]The volatility derivative is made informationally irrelevant only to facilitate the comparison between complete and incomplete markets in this application. However, in general, the volatility derivative does not need to be informationally irrelevant as, for example, when $b = C_1/\sigma_0^2 + C_1^2$.

Equations (34) show that the price of an asset with payoff C_1 admits similar representations in terms of pricing function (35) both in effectively complete and incomplete markets. Price $P_{com}(s)$ is given in closed form, that is, finding this price does not require solving any equations and it is in terms of only exogenous parameters specified in Section 2. Price $P_{inc}(s)$ in the incomplete market is less tractable because it is given in terms of an implicit function $\widehat{s}(s)$, which solves non-linear Equation (36). The latter equation is a consistency condition, which makes informed and uninformed investors agree on the incomplete-market price, as shown in the proof of Proposition 5 in the Appendix.

Next, we provide analytic pricing functions (35) for the following payoff distributions:

$$\varphi_C(x) = \sum_{l=1}^{L} w_l \frac{1}{\sqrt{2\pi}\widehat{\sigma}_{C,l}} \exp\left(-\frac{(x-\widehat{\mu}_{C,l})^2}{2\widehat{\sigma}_{C,l}^2}\right), \ x \in \mathbb{R}, \quad \text{(Mixture of Normals)}, \tag{38}$$

$$\varphi_C(x) = \frac{1}{\Lambda} x^{k-1} \exp\left(-\frac{x^2}{2\widehat{\sigma}_C^2} - \delta x\right), \ x \geq 0, \quad \text{(Generalized Gamma)}, \tag{39}$$

$$\varphi_C(x) = \frac{2}{\sqrt{2\pi}\widehat{\sigma}_C} \exp\left(-\frac{(x-\widehat{\mu}_C)^2}{2\widehat{\sigma}_C^2}\right) \Phi\left(\alpha \frac{x-\widehat{\mu}_C}{\widehat{\sigma}_C}\right), \ x \in \mathbb{R}, \quad \text{(Skew-normal)}, \tag{40}$$

respectively, where $w_1 + \ldots + w_L = 1$, $w_l \geq 0$, Λ is a normalizing constant, $\widehat{\sigma}_{C,l} > 0$ and $\widehat{\sigma}_C > 0$ are scale parameters, $\widehat{\mu}_{C,l}$, δ and $\widehat{\mu}_C$ are shift parameters, and $k \geq 1$ is an integer power, α is a skewness parameter, and $\Phi(x)$ is the standard normal cumulative distribution function (CDF). We note that: mixture of normals (38) is widely employed for non-parametric estimation of general PDFs [Greene (2008, p. 416)]; generalized gamma (39) has positive support and incorporates exponential ($k = 1$, $\widehat{\sigma}_C \to \infty$), gamma ($\widehat{\sigma}_C \to \infty$), Rayleigh ($k = 2$), and truncated normal ($k = 1$) distributions; and the skew-normal distribution (40) extends the normal distribution to allow for skewness [Azzalini (1985)]. Proposition 6 reports pricing functions (35) for distributions (38)–(40).

Proposition 6 (Pricing Functions in Analytic Form).

i) *When $\varphi_C(x)$ is a mixture of normals* (38), *the pricing function $P_C(\mu, \sigma)$ is given by:*

$$P_C(\mu,\sigma) = \frac{\displaystyle\sum_{l=1}^{L} w_l \frac{\mu\widehat{\sigma}_{C,l}^2 + \widehat{\mu}_{C,l}\sigma^2}{\left(\widehat{\sigma}_{C,l}^2 + \sigma^2\right)^{3/2}} \exp\left(-\frac{1}{2}\frac{(\mu-\widehat{\mu}_{C,l})^2}{\widehat{\sigma}_{C,l}^2+\sigma^2}\right)}{\displaystyle\sum_{l=1}^{L} w_l \frac{1}{\sqrt{\widehat{\sigma}_{C,l}^2+\sigma^2}} \exp\left(-\frac{1}{2}\frac{(\mu-\widehat{\mu}_{C,l})^2}{\widehat{\sigma}_{C,l}^2+\sigma^2}\right)}. \tag{41}$$

ii) *When $\varphi_C(x)$ is a generalized gamma PDF* (39) *with an integer power k the pricing* *function* $P_C(\mu,\sigma;k)$ *is given in terms of* $P_C(\mu,\sigma;k-1)$ *by the following recursive formula:*

$$P_C(\mu,\sigma;k)=\begin{cases}\dfrac{\widehat{\sigma}_C^2\sigma^2}{\widehat{\sigma}_C^2+\sigma^2}\left(\dfrac{\mu}{\sigma^2}-\delta+\dfrac{k-1}{P_C(\mu,\sigma;k-1)}\right), & \textit{if } k>1,\\[2ex] \dfrac{\widehat{\sigma}_C^2\sigma^2}{\widehat{\sigma}_C^2+\sigma^2}\left(\dfrac{\mu}{\sigma^2}-\delta\right)+\dfrac{\widehat{\sigma}_C\sigma}{\sqrt{\widehat{\sigma}_C^2+\sigma^2}}\widehat{\Phi}\left(\dfrac{\widehat{\sigma}_C\sigma}{\sqrt{\widehat{\sigma}_C^2+\sigma^2}}\left(\dfrac{\mu}{\sigma^2}-\delta\right)\right), & \textit{if } k=1,\end{cases} \tag{42}$$

where $\widehat{\Phi}(x)=\exp(-0.5x^2)/\left(\sqrt{2\pi}\Phi(x)\right)$*, and* $\Phi(x)$ *is the standard normal CDF.*
iii) *When* $\varphi_C(x)$ *is a skew-normal PDF* (40) *the pricing function* $P_C(\mu,\sigma)$ *is given by:*

$$P_C(\mu,\sigma)=\frac{\mu\widehat{\sigma}_C^2+\widehat{\mu}_C\sigma^2}{\widehat{\sigma}_C^2+\sigma^2}+\frac{\widehat{\sigma}_C^2\sigma^2}{\widehat{\sigma}_C^2+\sigma^2}\frac{\operatorname{sgn}(\alpha)}{\sqrt{\dfrac{\widehat{\sigma}_C^2}{\alpha^2}+\dfrac{\widehat{\sigma}_C^2\sigma^2}{\widehat{\sigma}_C^2+\sigma^2}}}\widehat{\Phi}\left(\frac{\widehat{\sigma}_C^2}{\widehat{\sigma}_C^2+\sigma^2}\frac{\operatorname{sgn}(\alpha)(\mu-\widehat{\mu}_C)}{\sqrt{\dfrac{\widehat{\sigma}_C^2}{\alpha^2}+\dfrac{\widehat{\sigma}_C^2\sigma^2}{\widehat{\sigma}_C^2+\sigma^2}}}\right), \tag{43}$$

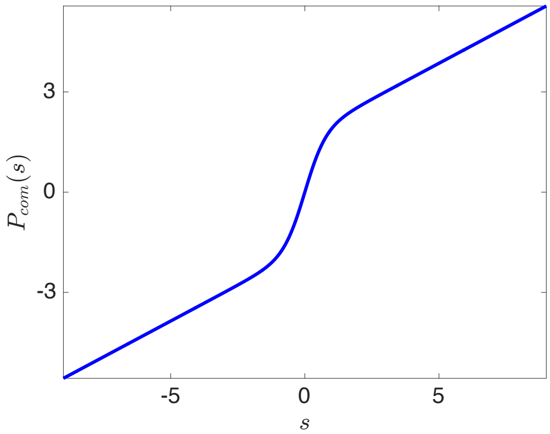
where $\widehat{\Phi}(x)=\exp(-0.5x^2)/\left(\sqrt{2\pi}\Phi(x)\right)$, $\Phi(x)$ *is the standard normal CDF, and* $\operatorname{sgn}(x)=x/|x|$ *when* $x\neq 0$ *and* $\operatorname{sgn}(0)=0$.

The pricing function (41) for the mixture of normals is derived in terms of elementary functions, and pricing functions (42) and (43) for the generalized gamma and the skew-normal distribution are in terms of the inverse Mill's ratio $\widehat{\Phi}(x)=\exp(-0.5x^2)/\left(\sqrt{2\pi}\Phi(x)\right)$, widely employed in statistics. The pricing functions (41)–(43) then give rise to analytic complete-market price $P_{com}(s)$ in (34) of the asset with payoff C_1, which is in terms of basic and widely used functions. Pricing functions also facilitate the computation of incomplete-market price $P_{inc}(s)$ in (34), which involves solving non-linear Equation (36) numerically.

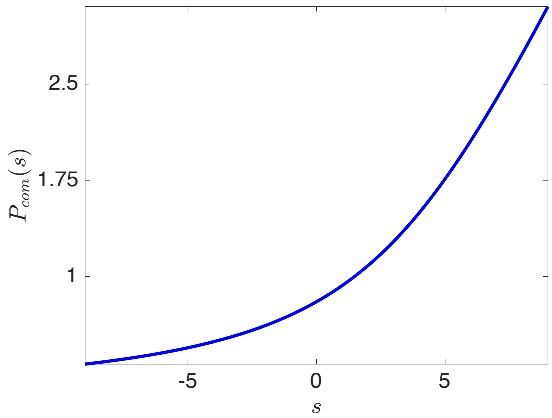
Panels (a), (b), and (c) of Figure 1 plot complete-market prices $P_{com}(s)$ when asset payoff C is drawn from distributions (38)–(40), respectively. Panels (a) and (b) show that non-normality of payoff C_1 makes asset prices non-linear functions of the sufficient statistic $s=\varepsilon/(\gamma_I\sigma_0^2)+\nu_1$. In particular, panel (a.i) demonstrates that even a small change in s can lead to large price changes [see also Breon-Drish (2010)]. Therefore, more general distributions give rise to effects that are not captured by CARA-normal models, where prices are linear functions of s.

Panel(c) of Figure 1 plots price $P_{com}(s)$ for the standard normal distribution with zero skewness and the skew-normal distribution for which the distribution parameters are chosen in such a way that payoff C_1 has mean $\mu_C=0$ and variance $\sigma_C^2=1$, as for the standard normal, but has skewness of 0.75.[9] Equation (43) decomposes the pricing function

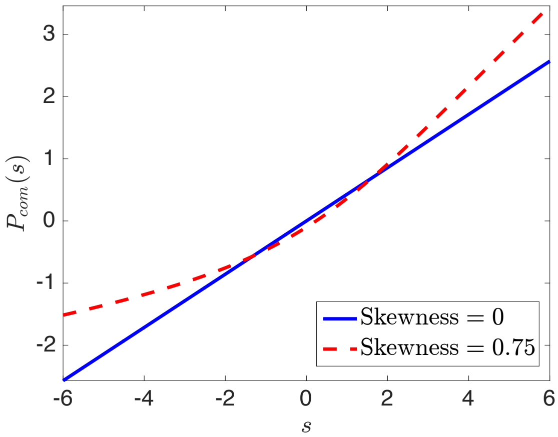
[9]The mean, variance, and skewness of a skew-normal random variable are given by $\mu_C=\widehat{\mu}_C+\widehat{\sigma}_C\sqrt{2/\pi}\widehat{\alpha}$, $\sigma_C^2=\widehat{\sigma}_C^2\left(1-2\widehat{\alpha}^2/\pi\right)$, $\kappa=0.5(4-\pi)\left(\widehat{\alpha}\sqrt{2/\pi}\right)^3/\left(1-\widehat{\alpha}^2 2/\pi\right)^{3/2}$, respectively, where $\widehat{\alpha}=\alpha/\sqrt{1+\alpha^2}$ [e.g., Azzalini (1985)]. We calibrate the parameters so that $\mu_C=0$, $\sigma_C=1$, and $\kappa=0.75$.



(a) Mixture of normals



(b) Generalized Gamma distribution



(c) Skew-normal distribution

Figure 1: Asset prices in effectively complete markets.
Panels (a), (b), and (c) show the effectively complete market price $P_{com}(s)$ when payoff PDF $\varphi_C(x)$ is a mixture of normals (38), a generalized gamma (39), and a skew-normal (40), respectively. Panel (c) shows price $P_{com}(s)$ for the case of zero skewness (solid blue line) and positive skewness 0.75 (dashed red line). For panel (a) $\widehat{\sigma}_{C,1} = \widehat{\sigma}_{C,2} = 1$, $\widehat{\mu}_{C,1} = -\widehat{\mu}_{C,2} = 3$, $w_1 = w_2 = 0.5$; for panel (b) $\widehat{\sigma}_C = 1$, $\delta = 2$, $k = 3$; for panel (c) $\mu_C = 0$, $\sigma_C^2 = 1$. The remaining parameters are: $\mu_0 = 0$, $\sigma_0 = 1$, $\sigma_\nu = 1$, $\gamma_I = \gamma_U = 1$, $r = 0$, $T = 1$.

$P_C(\mu, \sigma)$ for the skew-normal distribution (40) into two terms, where the first term is the pricing function for a normally distributed payoff and the second term isolates the effect of skewness and shows that skewness is priced. We now evaluate the effect of the second term in (43). The comparison of prices on panel (c) reveals that skewness introduces non-linearity and convexity in asset prices and gives rise to higher prices that substantially deviate from prices in the case of normally distributed payoffs when $|s|$ is large. Positive skewness makes large negative payoffs less likely. As a result, the uninformed investor attributes large negative prices to shock u and noise ν_1 rather than payoff C_1, which gives rise to higher prices with convexity, relative to the case with zero skewness.

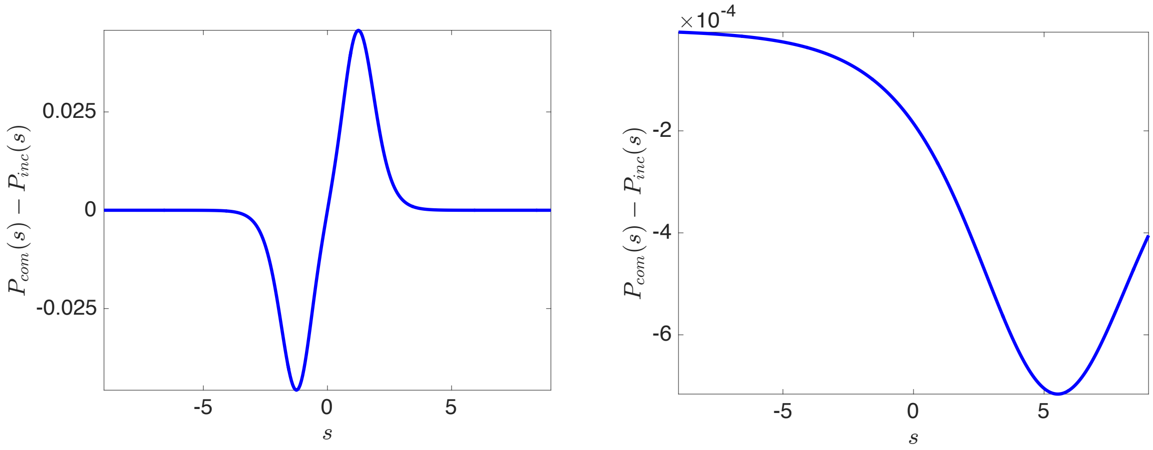
Comparison with Incomplete-market Prices. Finally, we compare the complete and incomplete market prices $P_{com}(s)$ and $P_{inc}(s)$. Panels (a) and (b) of Figure 2 plot the difference $P_{com}(s) - P_{inc}(s)$ for the mixture of normals (38) and generalized gamma (39) PDFs. The two prices are different despite the fact that both prices reveal the same sufficient statistic s, and hence, investor U has the same information in both markets. The difference in prices arises because the quadratic derivative is not traded in the incomplete market, and so, as shown in Proposition 4, a Pareto-optimal allocation is not feasible (in contrast to the effectively complete market). However, panels (a) and (b) reveal that the price difference is small and converges to zero for large $|s|$. Similar results hold for the skew-normal distribution (40), but are not reported for brevity.

Although we do not have a formal proof that prices $P_{com}(s)$ and $P_{inc}(s)$ are close, as measured by $\max_s |P_{com}(s) - P_{inc}(s)|$, we provide the economic intuition. First, these prices are close because they have very similar structure (34) and are increasing functions of s as shown in Proposition 5. Second, we argue that $\max_s |P_{com}(s) - P_{inc}(s)|$ is close to zero for large $|s|$. To demonstrate the latter result, we need the following Lemma 4 showing that $P_{com}(s) = P_{inc}(s)$ in CARA-normal economies.

Lemma 4. *In the case of normally distributed asset payoff $C_1 \sim \mathcal{N}(\mu_C, \sigma_C^2)$ the price of this asset is the same in incomplete and complete markets, is linear in s, and is given by:*

$$P(s) = \frac{\left(1 + \frac{\gamma_U + \gamma_I}{\gamma_U \gamma_I} \frac{1}{\gamma_I \sigma_\nu^2 \sigma_0^2}\right) s + \frac{\mu_0}{\gamma_U \sigma_0^2} + \frac{\gamma_U + \gamma_I}{\gamma_U \gamma_I}\left(1 + \frac{1}{\gamma_I^2 \sigma_\nu^2 \sigma_0^2}\right)\left(\frac{\mu_C}{\sigma_C^2} - \frac{\mu_0}{\sigma_0^2}\right)}{\frac{\gamma_U + \gamma_I}{\gamma_U \gamma_I}\left(1 + \frac{1}{\gamma_I^2 \sigma_\nu^2 \sigma_0^2}\right)\left(\frac{1}{\sigma_0^2} + \frac{1}{\sigma_C^2}\right) - \frac{1}{\gamma_U \sigma_0^2}} e^{-rT}. \quad (44)$$

Lemma 4 sheds light on the closeness of prices $P_{com}(s)$ and $P_{inc}(s)$ for large values of s even when C_1 is not normally distributed. From expression (38) for mixture of normals PDF we observe that this distribution is approximately normal for large values of $|x|$. Therefore, large values of the sufficient statistic s signal that payoff C_1 is high and approximately normal, and hence the difference $P_{com}(s) - P_{inc}(s)$ is small, consistent with Lemma 4. Similarly, the generalized gamma distribution (39) is approximately normal for large $x > 0$ because it is dominated by the exponential term $\exp(-x^2/(2\widehat{\sigma}_C^2) - \delta x)$. Hence, $P_{com}(s) - P_{inc}(s)$ is small for large $s > 0$ by Lemma 4. The price difference is also small for



(a) Mixture of normals (b) Generalized Gamma distribution

Figure 2: Differences of effectively complete and incomplete market prices
Panels (a) and (b) show price differences $P_{com}(s) - P_{inc}(s)$ for PDFs (38) and (39), respectively. For panel (a) $\widehat{\sigma}_{C,1} = \widehat{\sigma}_{C,2} = 1$, $\widehat{\mu}_{C,1} = -\widehat{\mu}_{C,2} = 3$, $w_1 = w_2 = 0.5$; for panel (b) $\widehat{\sigma}_C = 1$, $\delta = 2$, $k = 3$, the remaining parameters are: $\mu_0 = 0$, $\sigma_0 = 1$, $\sigma_\nu = 1$, $\gamma_I = \gamma_U = 1$, $r = 0$, $T = 1$.

large $s < 0$ because both prices converge to their common lower bound at zero. Intuitively, large $s < 0$ is more likely when payoff C_1 is low, which leads to low prices.

5. Conclusion

We develop a tractable REE model with multiple risky assets with realistic payoff structures, such as assets with strictly positive payoffs and derivative securities. We solve the model in closed form when the market is complete and in terms of inverse functions in several plausible incomplete-market settings. Our results yield necessary and sufficient conditions under which the informed demands for derivative securities reveal information about the underlying asset. Moreover, we show that adding volatility derivatives to incomplete markets makes these markets effectively complete, that is, allows informed and uninformed investors to achieve Pareto optimal allocations. Furthermore, effectively completing the market allows obtaining the price of the underlying asset in closed form. We provide several numerical examples where the properties and the magnitude of prices in incomplete and complete markets are similar.

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Appendix: Proofs

Lemma A.1 (Prior mean and prior variance of ε and prior probabilities). *Let ε have PDF (2). Then, its mean μ_ε and variance σ_ε^2 in terms of (μ_0, σ_0^2) are given by:*

$$\mu_\varepsilon = \frac{\sum_{j=1}^N \exp\left(a_j + \dfrac{\mu_j^2}{2\sigma_0^2}\right)\mu_j}{\sum_{j=1}^N \exp\left(a_j + \dfrac{\mu_j^2}{2\sigma_0^2}\right)}, \tag{A.1}$$

$$\sigma_\varepsilon^2 = \sigma_0^2 + \left[\frac{\sum_{j=1}^N \exp\left(a_j + \dfrac{\mu_j^2}{2\sigma_0^2}\right)\mu_j^2}{\sum_{j=1}^N \exp\left(a_j + \dfrac{\mu_j^2}{2\sigma_0^2}\right)} - \frac{\left(\sum_{j=1}^N \exp\left(a_j + \dfrac{\mu_j^2}{2\sigma_0^2}\right)\mu_j\right)^2}{\left(\sum_{j=1}^N \exp\left(a_j + \dfrac{\mu_j^2}{2\sigma_0^2}\right)\right)^2}\right], \tag{A.2}$$

where $\mu_j = b_j\sigma_0^2 + \mu_0$.

Proof of Lemma A.1. We compute $\mu_\varepsilon = \mathbb{E}[\varepsilon]$ and $\sigma_\varepsilon^2 = \text{var}[\varepsilon]$ with PDF $\varphi_\varepsilon(x)$ given by (2), and after straightforward integration, we obtain Equations (A.1) and (A.2). ■

Lemma A.2 (Special Cases). *Let payoff C_1 have general unconditional continuous PDF $\varphi_C(x)$. Suppose, the informed investor receives signal $\varepsilon = C_1 + u$, where $u \sim \mathcal{N}(\mu_0, \sigma_0^2)$. Then, the latter economy is a limiting case (when $N \to \infty$) of our N-state economy with $C_1(\omega_n) = \underline{C}_N + (\overline{C}_N - \underline{C}_N)(n-1)/(N-1)$ and distribution parameters*

$$a_n = -\frac{C_1(\omega_n)^2}{2\sigma_0^2} - \frac{\mu_0 C_1(\omega_n)}{\sigma_0^2} + \ln\left[\varphi_C(C_1(\omega_n))\right], \quad b_n = \frac{C_1(\omega_n)}{\sigma_0^2}, \tag{A.3}$$

when $N \to \infty$ and $\underline{C}_N$ and $\overline{C}_N$ converge to lower and upper limits of payoff C_1.

Proof of Lemma A.2. Consider a discretized economy with one risky asset with payoff $C_1(\omega_n) = \underline{C}_N + (\overline{C}_N - \underline{C}_N)(n-1)/(N-1)$, where $n = 1, \ldots, N$, and let the informed investor receive signal $\varepsilon = C_1(\omega) + u$, where $u \sim \mathcal{N}(\mu_0, \sigma_0^2)$. The unconditional probabilities of $C_1(\omega_n)$ are given by $\text{Prob}(C_1(\omega_n)) = \varphi_C(C_1(\omega_n))/\left(\sum_{n=1}^N \varphi_C(C_1(\omega_n))\right)$. The original continuous-space economy is a limiting case of the latter economy because as $N \to \infty$, the distributions $\text{Prob}(C_1(\omega) \le x)$ and $\text{Prob}(C_1(\omega) \le x|\varepsilon)$ converge pointwise to the respective continuous-space distributions. We show that the latter discretized economy is a special case of ours when parameters a_n and b_n are given by Equations (A.3) by verifying that in our economy $\text{Prob}(C_1(\omega_n)) = \varphi_C(C_1(\omega_n))/\left(\sum_{n=1}^N \varphi_C(C_1(\omega_n))\right)$ and $\varepsilon = C_1(\omega) + u$.

Consider the unconditional probability $\text{Prob}(C_1(\omega_n))$ in the model of Section 2:

$$\text{Prob}(C_1(\omega_n)) = \int_{-\infty}^{\infty} \pi_n(x)\varphi_\varepsilon(x)dx = \frac{1}{\Lambda}\int_{-\infty}^{\infty} e^{a_n+b_nx-0.5(x-\mu_0)^2/\sigma_0^2}dx = \frac{1}{\widetilde{\Lambda}}e^{a_n+0.5(\mu_0+b_n\sigma_0^2)^2/\sigma_0^2},$$

where Λ and $\widetilde{\Lambda}$ are constants. Substituting a_n and b_n from Equations (A.3) into the above equation, after some algebra, we verify that $\text{Prob}(C_1(\omega_n)) = \varphi_C(C_1(\omega_n))/\sum_{n=1}^N \varphi_C(C_1(\omega_n))$.

Next, we verify that $\varepsilon = C_1(\omega) + u$, where $u \sim \mathcal{N}(\mu_0, \sigma_0^2)$. Substituting a_n and b_n from (A.3) into Equation (2) for PDF $\varphi_\varepsilon(x)$, after some algebra, we obtain:

$$\varphi_\varepsilon(x) = \frac{\sum_{n=1}^N e^{-0.5(x-C_1(\omega_n)-\mu_0)^2/\sigma_0^2}\varphi_C(C_1(\omega_n))}{\int_{-\infty}^{\infty}\left(\sum_{n=1}^N e^{-0.5(x-C_1(\omega_n)-\mu_0)^2/\sigma_0^2}\varphi_C(C_1(\omega_n))\right)dx}.$$

The above PDF is the convolution of the unconditional distributions of $C_1(\omega)$ and a normal distribution $\mathcal{N}(\mu_0, \sigma_0^2)$, and hence is the PDF of $\varepsilon = C_1(\omega) + u$, where $u \sim \mathcal{N}(\mu_0, \sigma_0^2)$. ■

Proof of Lemma 1. Taking log on both sides of investor I's FOC (10), and substituting wealth $W_{I,T,n}$ from the budget constraint (5), we obtain:

$$(\theta_I^*)^\top(C(\omega_n) - e^{rT}p) = \frac{1}{\gamma_I}\Big(\ln\big(\pi_n(\varepsilon)\big) - \ln\big(\pi_n^{\text{RN}}\big)\Big) + \textit{const}, \quad n = 1, \dots, N, \tag{A.4}$$

where *const* is a constant. Subtracting Equation (A.4) for $n = N$ from the other equations in (A.4), we obtain a system of $N-1$ equations with $N-1$ unknown components of θ_I^*:

$$(\theta_I^*)^\top(C(\omega_n) - C(\omega_N)) = \frac{1}{\gamma_I}\left(\ln\left(\frac{\pi_n(\varepsilon)}{\pi_N(\varepsilon)}\right) - \ln\left(\frac{\pi_n^{\text{RN}}}{\pi_N^{\text{RN}}}\right)\right), \quad n = 1, \dots, N-1, \tag{A.5}$$

Solving the system of equations (A.5), we obtain investor I's optimal portfolio

$$\theta_I^*(p;\varepsilon) = \frac{\Omega^{-1}}{\gamma_I}\left\{\left(\ln\left(\frac{\pi_1(\varepsilon)}{\pi_N(\varepsilon)}\right), \dots, \ln\left(\frac{\pi_{N-1}(\varepsilon)}{\pi_N(\varepsilon)}\right)\right) - \left(\ln\left(\frac{\pi_1^{\text{RN}}}{\pi_N^{\text{RN}}}\right), \dots, \ln\left(\frac{\pi_{N-1}^{\text{RN}}}{\pi_N^{\text{RN}}}\right)\right)\right\}^\top. \tag{A.6}$$

Finally, substituting $\pi_n(\varepsilon)$ from (1) into the above equation, we obtain portfolio (11). ■

Proof of Lemma 2. Let $s \equiv \lambda\varepsilon/\gamma_I + \nu$ denote the sufficient statistic. From Bayes rule, the PDF of ε conditional on s is given by:

$$\varphi_{\varepsilon|s}(x|y) = \frac{\varphi_{s|\varepsilon}(y|x)\varphi_\varepsilon(x)}{\int_{-\infty}^{\infty}\varphi_{s|\varepsilon}(y|x)\varphi_\varepsilon(x)dx}. \tag{A.7}$$

Because $\nu \sim \mathcal{N}(0, \Sigma_\nu)$, $s = \lambda\varepsilon/\gamma_I + \nu$ conditional on ε has a multivariate normal distribution $\mathcal{N}(\lambda\varepsilon/\gamma_I, \Sigma_\nu)$. Hence, substituting $\varphi_{s|\varepsilon}(y|x)$ into Equation (A.7), we have

$$\varphi_{\varepsilon|s}(x|y) = \frac{\exp\Big\{-0.5\big(y - \lambda x/\gamma_I\big)^\top\Sigma_\nu^{-1}\big(y - \lambda x/\gamma_I\big)\Big\}\varphi_\varepsilon(x)}{G_1(y)}, \tag{A.8}$$

where $G_1(y)$ normalizes the density. Next, to find probability $\pi_n^U(p)$, from the market clearing condition (14), we note that in equilibrium $s = -H(p)$. We focus on equilibrium where prices only reveal the sufficient statistic s. Hence, $\pi_n^U(p) = \mathbb{E}\Big[\pi_n(\varepsilon)|P(\varepsilon,\nu) = p\Big] = \mathbb{E}[\pi_n(\varepsilon)|s = -H(p)]$. Calculating the last conditional expectation, we obtain:

$$\begin{aligned}\pi_n^U(p) &= \mathbb{E}[\pi_n(\varepsilon)|s = -H(p)] \\ &= \int_{-\infty}^{\infty} \frac{e^{a_n+b_n x}}{\sum_{j=1}^N e^{a_j+b_j x}} \varphi_{\varepsilon|s}\Big(x|-H(p)\Big)dx = \frac{1}{G_1(y)}\int_{-\infty}^{\infty} e^{d_n(x)}dx,\end{aligned} \tag{A.9}$$

where $d_n(x)$ is a quadratic function of x given by:

$$\begin{aligned}d_n(x) &= a_n + b_n x - 0.5\Big(\lambda x/\gamma_I + H(p)\Big)^\top \Sigma_\nu^{-1}\Big(\lambda x/\gamma_I + H(p)\Big) - 0.5(x-\mu_0)^2/\sigma_0^2 \\ &= -\frac{\lambda^\top\Sigma_\nu^{-1}\lambda/\gamma_I^2 + 1/\sigma_0^2}{2}\left(x - \frac{\mu_0/\sigma_0^2 + b_n - \lambda^\top\Sigma_\nu^{-1}H(p)/\gamma_I}{\lambda^\top\Sigma_\nu^{-1}\lambda/\gamma_I^2 + 1/\sigma_0^2}\right)^2 \\ &\quad + a_n + \frac{1}{2}\frac{b_n^2 + 2b_n\Big(\mu_0/\sigma_0^2 - \lambda^\top\Sigma_\nu^{-1}H(p)/\gamma_I\Big)}{\lambda^\top\Sigma_\nu^{-1}\lambda/\gamma_I^2 + 1/\sigma_0^2} + g(p),\end{aligned} \tag{A.10}$$

where $g(p)$ is a normalizing function. Substituting $d_n(x)$ from Equation (A.10) into integral (A.9) and integrating, we obtain Equation (16) for $\pi_n^U(p)$. ■

Proof of Proposition 1. First, we find the optimal portfolios of investors, and then recover the equilibrium prices from the market clearing condition. The portfolio of investor I is given by Equation (11) in Lemma 1. To find investor U's portfolio $\theta_U^*(p)$, we follow similar steps as in Lemma 1: 1) take the log of both sides of investor U's FOC (10); 2) subtract the N^{th} equation from the rest; 3) solve the $N-1$ equations for the $N-1$ positions of U's portfolio. This gives $\theta_U^*(p)$ in terms of investor U's probabilities $\pi_n^U(p)$

$$\theta_U^*(p) = \frac{1}{\gamma_U}\Omega^{-1}\left\{\left(\ln\left(\frac{\pi_1^U(p)}{\pi_N^U(p)}\right), \dots, \ln\left(\frac{\pi_{N-1}^U(p)}{\pi_N^U(p)}\right)\right)^\top - \widetilde{v}(p)\right\}, \tag{A.11}$$

where $\widetilde{v}$ is given by Equation (12). Substituting $\pi_n^U(p)$ from (16) into (A.11) we obtain:

$$\theta_U^*(p) = \frac{1}{\gamma_U}\Omega^{-1}\left\{\widetilde{a} + \frac{1}{2}\frac{\widetilde{b}^{(2)} - 2\widetilde{b}\Big(\lambda^\top\Sigma_\nu^{-1}H(p)/\gamma_I - \mu_0/\sigma_0^2\Big)}{\lambda^\top\Sigma_\nu^{-1}\lambda/\gamma_I^2 + 1/\sigma_0^2} - \widetilde{v}(p)\right\}, \tag{A.12}$$

where vectors $\widetilde{a}, \widetilde{b}, \widetilde{b}^{(2)} \in \mathbb{R}^{N-1}$ are given by:

$$\widetilde{a} = (a_1 - a_N, \dots, a_{N-1} - a_N), \quad \widetilde{b} = (b_1 - b_N, \dots, b_{N-1} - b_N), \tag{A.13}$$

$$\widetilde{b}^{(2)} = (b_1^2 - b_N^2, \dots, b_{N-1}^2 - b_N^2). \tag{A.14}$$

Substituting $\lambda = \Omega^{-1}\widetilde{b}$ and $H(p) = \theta_U^* - \Omega^{-1}\Big(\widetilde{v} - \widetilde{a}\Big)/\gamma_I$ from (15) into (A.12), we obtain:

$$\begin{aligned}\theta_U^*(p) &= \frac{\Omega^{-1}\Big(\widehat{a} - \widetilde{v}(p)\Big)}{\gamma_U} + \frac{\mu_0/(\gamma_U\sigma_0^2)\lambda}{\lambda^\top\Sigma_\nu^{-1}\lambda/\gamma_I^2 + 1/\sigma_0^2} - QH(p), \\ &= \frac{Q\Omega^{-1}\Big(\widetilde{v}(p) - \widetilde{a}\Big)}{\gamma_I} - \frac{\Omega^{-1}(\widetilde{v}(p) - \widehat{a})}{\gamma_U} + \frac{\mu_0/(\gamma_U\sigma_0^2)\lambda}{\lambda^\top\Sigma_\nu^{-1}\lambda/\gamma_I^2 + 1/\sigma_0^2} - Q\theta_U^*(p),\end{aligned} \tag{A.15}$$

where $\widehat{a}$ and matrix Q are are given by:

$$\widehat{a} = \widetilde{a} + \frac{1}{2}\frac{\widetilde{b}^{(2)}}{(\lambda^\top\Sigma_\nu^{-1}\lambda/\gamma_I^2 + 1/\sigma_0^2)}, \quad Q = \frac{\lambda\lambda^\top\Sigma_\nu^{-1}}{\gamma_U\gamma_I\Big(\lambda^\top\Sigma_\nu^{-1}\lambda/\gamma_I^2 + 1/\sigma_0^2\Big)}. \tag{A.16}$$

Solving linear Equation (A.15) for portfolio $\theta_U^*(p)$, we obtain portfolio $\theta_U^*(p)$ in (21).

Next, we find the equilibrium prices. Substituting optimal portfolios $\theta_I^*(p;\varepsilon)$ and $\theta_U^*(p)$ from Equations (11) and (21) into the market clearing condition $\theta_I^*(p;\varepsilon) + \theta_U^*(p) + \nu = 0$, after rearranging terms, we obtain the following equation for vector $\widetilde{v}(p)$:

$$\begin{aligned}\Big(E + Q\Big)^{-1}\left(\frac{Q\Omega^{-1}(\widetilde{v}(p) - \widetilde{a})}{\gamma_I} - \frac{\Omega^{-1}(\widetilde{v}(p) - \widehat{a})}{\gamma_U} + \frac{\mu_0/(\gamma_U\sigma_0^2)\lambda}{(\lambda^\top\Sigma_\nu^{-1}\lambda/\gamma_I^2 + 1/\sigma_0^2)}\right) \\ - \frac{\Omega^{-1}\Big(\widetilde{v}(p) - \widetilde{a}\Big)}{\gamma_I} + \frac{\lambda\varepsilon}{\gamma_I} + \nu = 0.\end{aligned} \tag{A.17}$$

The above equation can be further simplified by noting that

$$\begin{aligned}(E + Q)^{-1}\frac{1}{\gamma_I}Q\Omega^{-1}(\widetilde{v}(p) - \widetilde{a}) &= (E + Q)^{-1}(E + Q - E)\frac{1}{\gamma_I}\Omega^{-1}(\widetilde{v}(p) - \widetilde{a}) \\ &= \frac{1}{\gamma_I}\Omega^{-1}(\widetilde{v}(p) - \widetilde{a}) - (E + Q)^{-1}\frac{1}{\gamma_I}\Omega^{-1}(\widetilde{v}(p) - \widetilde{a}).\end{aligned}$$

Substituting the latter expression into Equation (A.17), canceling like terms, substituting $\widehat{a}$ from Equation (A.16) into Equation (A.17), and solving it for $\widetilde{v}(p) - \widetilde{a}$ we obtain

$$\widetilde{v}(p) = \widetilde{a} + \frac{1}{2}\frac{\gamma_I}{\gamma_I + \gamma_U}\frac{\widetilde{b}^{(2)} + 2(\mu_0/\sigma_0^2)\tilde{b}}{\lambda^\top\Sigma_\nu^{-1}\lambda/\gamma_I^2 + 1/\sigma_0^2} + \frac{\gamma_I\gamma_U}{\gamma_I + \gamma_U}\Omega\Big(E + Q\Big)\Big(\frac{\lambda\varepsilon}{\gamma_I} + \nu\Big), \tag{A.18}$$

where $\tilde{a}$, $\tilde{b}$, and $\tilde{b}^{(2)}$ are given by (A.13) and (A.14), or element-wise for $n = 1, \dots, N-1$:

$$\widetilde{v}_n = a_n - a_N + \frac{1}{2}\frac{\gamma_I}{\gamma_I + \gamma_U}\frac{(b_n^2 - b_N^2) + 2(\mu_0/\sigma_0^2)(b_n - b_N)}{\lambda^\top\Sigma_\nu^{-1}\lambda/\gamma_I^2 + 1/\sigma_0^2} + \frac{\gamma_I\gamma_U}{\gamma_I + \gamma_U}(C(\omega_n) - C(\omega_N))^\top\Big(E + Q\Big)\Big(\frac{\lambda\varepsilon}{\gamma_I} + \nu\Big).$$

Let v_n be given by Equation (19). Then, vector $\widetilde{v}(p) \in \mathbb{R}^{N-1}$ has elements $v_n - v_N$. From the definition of vector $\widetilde{v}$ in Equation (12), we find that $\pi_n^{\text{RN}} = e^{v_n - v_N}/\Big(\sum_{j=1}^N e^{v_j - v_N}\Big)$ for $n = 1, \dots, N$. Canceling e^{-v_N}, we obtain probabilities (18).

Finally, we show that $P(\varepsilon, \nu)$ is an invertible on its range function of $\lambda\varepsilon/\gamma_I + \nu$, and hence, observing prices reveals unique $\lambda\varepsilon/\gamma_I + \nu$. First, because the risk-neutral probabilities are unique, there is a one-to-one mapping between these probabilities and prices. From Equation (12) we observe that there is an one-to-one mapping between $\tilde{v}$ and the vector of risk-neutral probabilities. Finally, from Equation (A.18) there is an one-to-one mapping between $\tilde{v}$ and $\lambda\varepsilon/\gamma_I + \nu$, which, by transitivity, completes the proof. ■

Proof of Proposition 2. We start with the comparative statics with respect to ε. Differentiating risk-neutral probability π_n^{RN} given by (18) with respect to ε we obtain:

$$\frac{\partial \pi_n^{\mathrm{RN}}}{\partial \varepsilon} = \pi_n^{\mathrm{RN}} \frac{\partial v_n}{\partial \varepsilon} - \frac{\pi_n^{\mathrm{RN}}}{\sum_{k=1}^N e^{v_k}} \sum_{k=1}^N \frac{\partial v_k}{\partial \varepsilon} e^{v_k} = \pi_n^{\mathrm{RN}} \frac{\partial v_n}{\partial \varepsilon} - \pi_n^{\mathrm{RN}} \mathbb{E}^{\mathrm{RN}} \left[\frac{\partial v(\omega)}{\partial \varepsilon} \right], \tag{A.19}$$

where $v(\omega)$ now denotes a random variable that takes value v_n in state ω_n. Next, differentiating price (17) with respect to ε, and using Equation (A.19), we obtain:

$$\begin{aligned} \frac{\partial P_m(\varepsilon, \nu)}{\partial \varepsilon} &= \mathbb{E}^{\mathrm{RN}} \left[\frac{\partial v(\omega)}{\partial \varepsilon} C_m(\omega) \right] - \mathbb{E}^{\mathrm{RN}} \left[\frac{\partial v(\omega)}{\partial \varepsilon} \right] \mathbb{E}^{\mathrm{RN}} \big[C_m(\omega) \big] \\ &= \mathrm{cov}^{\mathrm{RN}} \left(\frac{\partial v(\omega)}{\partial \varepsilon}, C_m(\omega) \right). \end{aligned} \tag{A.20}$$

Differentiating Equation (19) for v_n and substituting Q from Equation (20), we obtain:

$$\frac{\partial v_n}{\partial \varepsilon} = \frac{\gamma_U (b_n - \lambda_0)}{\gamma_U + \gamma_I} \left(1 + \frac{\lambda^\top \Sigma_\nu^{-1} \lambda / (\gamma_U \gamma_I)}{\lambda^\top \Sigma_\nu^{-1} \lambda / \gamma_I^2 + 1/\sigma_0^2} \right), \tag{A.21}$$

where we use $b_n = \lambda_0 + C(\omega_n)^\top \lambda$. Then, using (A.21) we compute the covariance in (A.20):

$$\frac{\partial P_m(\varepsilon, \nu)}{\partial \varepsilon} = \frac{\gamma_U}{\gamma_U + \gamma_I} \left(1 + \frac{\lambda^\top \Sigma_\nu^{-1} \lambda / (\gamma_U \gamma_I)}{\lambda^\top \Sigma_\nu^{-1} \lambda / \gamma_I^2 + 1/\sigma_0^2} \right) \mathrm{cov}^{\mathrm{RN}}(b, C_m) e^{-rT}. \tag{A.22}$$

Next, we find $\partial P_m(\varepsilon, \nu)/\partial \nu_l$. Following the same steps as above, we find:

$$\frac{\partial P_m(\varepsilon, \nu)}{\partial \nu_l} = \mathrm{cov}^{\mathrm{RN}} \left(\frac{\partial v(\omega)}{\partial \nu_l}, C_m(\omega) \right), \tag{A.23}$$

where $l = 1, \ldots, M-1$. Computing $\partial v_n/\nu_l$, similarly to Equation (A.21), and denoting by $e_l \in \mathbb{R}^{N-1}$ a vector with l^{th} element equal to 1 and other elements equal to 0, we obtain:

$$\begin{aligned} \frac{\partial v_n}{\partial \nu_l} &= \frac{\gamma_I \gamma_U}{\gamma_U + \gamma_I} C(\omega_n)^\top \big(E + Q \big) e_l = \frac{\gamma_I \gamma_U}{\gamma_U + \gamma_I} \left(C(\omega_n)^\top e_l + \frac{C(\omega_n)^\top \lambda \lambda^\top \Sigma_\nu^{-1} e_l / (\gamma_U \gamma_I)}{\lambda^\top \Sigma_\nu^{-1} \lambda / \gamma_I^2 + 1/\sigma_0^2} \right), \\ &= \frac{\gamma_I \gamma_U}{\gamma_U + \gamma_I} \left(C_n(\omega_l) + \frac{(b_n - \lambda_0) \lambda^\top \Sigma_\nu^{-1} e_l / (\gamma_U \gamma_I)}{\lambda^\top \Sigma_\nu^{-1} \lambda / \gamma_I^2 + 1/\sigma_0^2} \right), \end{aligned} \tag{A.24}$$

where the last line uses the fact that by the definition of vectors C and λ, $C(\omega_n)^\top e_l = C_n(\omega_l)$ and $C(\omega_n)^\top \lambda = b_n - \lambda_0$. Substituting (A.24) into (A.23), we obtain Equation (23).

Finally, we find the derivatives of portfolios θ_i^* with respect to prices p. Substituting $\pi_N^{\text{RN}} = 1 - \pi_1^{\text{RN}} - \ldots - \pi_{N-1}^{\text{RN}}$ into the risk-neutral valuation Equation (8), we obtain:

$$p_m = \Big[\pi_1^{\text{RN}}\Big(C_m(\omega_1) - C_m(\omega_N)\Big) + \ldots + \pi_{N-1}^{\text{RN}}\Big(C_m(\omega_{N-1}) - C_m(\omega_N)\Big) + C_m(\omega_N)\Big]e^{-rT}, \tag{A.25}$$

where $m = 1, \ldots, N-1$. From the definition of vector $\widetilde{v}$ in (12) we observe that $\pi_n^{\text{RN}} = e^{\widetilde{v}_n}/(1 + \sum_{n=1}^{N-1} e^{\widetilde{v}_n})$. First, we need to compute $\partial\widetilde{v}/\partial p$. To do this, we find the Jacobian $J_p = \partial p/\partial\widetilde{v}$ and then by the inverse function theorem we have $\partial\widetilde{v}/\partial p = J_p^{-1}$. Let J_π be the Jacobian of vector $(\pi_1^{\text{RN}}, \ldots, \pi_{N-1}^{\text{RN}})^\top$, that is, a matrix with (n,k) element given by $\partial\pi_n^{\text{RN}}/\partial\widetilde{v}_k$. Differentiating Equation (A.25) we find that $J_p = \Omega^\top J_\pi e^{-rT}$, and hence

$$J_p \Omega\, e^{rT} = \Omega^\top J_\pi \Omega. \tag{A.26}$$

To find J_π we first calculate $\partial\pi_n^{\text{RN}}/\partial\widetilde{v}_k$, where π_n^{RN} is given by Equation (18):

$$\frac{\partial\pi_n^{\text{RN}}}{\partial\widetilde{v}_k} = \frac{\partial\pi_n^{\text{RN}}}{\partial v_k} = \begin{cases} -\pi_n^{\text{RN}}\pi_k^{\text{RN}}, & \text{if } n \neq k, \\ \pi_n^{\text{RN}} - (\pi_n^{\text{RN}})^2, & \text{if } n = k. \end{cases} \tag{A.27}$$

for $n, k = 1, \ldots, N-1$, where the first equality follows from $\widetilde{v}_k = v_k - v_N$. From Equation (A.27) we find $J_\pi = \text{diag}\{\pi_1^{\text{RN}}, \ldots, \pi_{N-1}^{\text{RN}}\} - (\pi_1^{\text{RN}}, \ldots, \pi_{N-1}^{\text{RN}})^\top(\pi_1^{\text{RN}}, \ldots, \pi_{N-1}^{\text{RN}})$, where $\text{diag}\{\ldots\}$ is a diagonal matrix. Substituting J_π into Equation (A.26) we obtain:

$$J_p\Omega\, e^{rT} = \Omega^\top\Big(\text{diag}\{\pi_1^{\text{RN}}, \ldots, \pi_{N-1}^{\text{RN}}\} - (\pi_1^{\text{RN}}, \ldots, \pi_{N-1}^{\text{RN}})^\top(\pi_1^{\text{RN}}, \ldots, \pi_{N-1}^{\text{RN}})\Big)\Omega. \tag{A.28}$$

Recalling that Ω is a matrix with rows $(C(\omega_n) - C(\omega_N))^\top$, and denoting $\widetilde{C}_n = \Big(C_n(\omega_1) - C_n(\omega_N), \ldots, C_n(\omega_{N-1}) - C_n(\omega_N)\Big)^\top$, we find that the (n,k) element of matrix $J_p\Omega e^{rT}$ is:

$$\begin{aligned}
\{J_p\Omega\, e^{rT}\}_{n,k} &= \widetilde{C}_n^\top \text{diag}\{\pi_1^{\text{RN}}, \ldots, \pi_{N-1}^{\text{RN}}\}\widetilde{C}_k - \widetilde{C}_n^\top(\pi_1^{\text{RN}}, \ldots, \pi_{N-1}^{\text{RN}})^\top(\pi_1^{\text{RN}}, \ldots, \pi_{N-1}^{\text{RN}})\widetilde{C}_k \\
&= \textstyle\sum_{i=1}^N \Big(C_n(\omega_i) - C_n(\omega_N)\Big)\Big(C_k(\omega_i) - C_k(\omega_N)\Big)\pi_i^{\text{RN}} \\
&\quad - \Big(\textstyle\sum_{i=1}^N \Big(C_n(\omega_i) - C_n(\omega_N)\Big)\pi_i^{\text{RN}}\Big)\Big(\textstyle\sum_{i=1}^N \Big(C_k(\omega_i) - C_k(\omega_N)\Big)\pi_i^{\text{RN}}\Big) \\
&= \text{cov}^{\text{RN}}(C_n, C_k),
\end{aligned}$$

where to derive the second equality we added zero terms $\Big(C_n(\omega_N) - C_n(\omega_N)\Big)\Big(C_k(\omega_N) - C_k(\omega_N)\Big)\pi_N^{\text{RN}}$, $\Big(C_n(\omega_N) - C_n(\omega_N)\Big)\pi_N^{\text{RN}}$ and $\Big(C_k(\omega_N) - C_k(\omega_N)\Big)\pi_N^{\text{RN}}$ to summations, and then removed constants $C_n(\omega_N)$ and $C_k(\omega_N)$, because they do not affect covariances.

Therefore, we conclude that $J_p \Omega\, e^{rT} = \mathrm{var}^{\mathrm{RN}}[C]$. Then, by the inverse function theorem, we now find that $\Omega^{-1}\partial\tilde{v}/\partial p = \left(\mathrm{var}^{\mathrm{RN}}[C]\right)^{-1} e^{rT}$. Using the latter equality and differentiating optimal portfolios (11) and (21) with respect to p we obtain that the first of these two partial derivatives is given by Equation (24) and the second is given by:

$$\frac{\partial\theta_U^*(p)}{\partial p} = \left(\frac{1}{\gamma_I}E - \frac{\gamma_I+\gamma_U}{\gamma_I\gamma_U}(E+Q)^{-1}\right)\left(\mathrm{var}^{\mathrm{RN}}[C]\right)^{-1} e^{rT}. \tag{A.29}$$

We note the following equation for the inverse matrix $(E+Q)^{-1}$:

$$(E+Q)^{-1} = E - \frac{\lambda\lambda^\top\Sigma_\nu^{-1}}{\gamma_U\gamma_I\left(\lambda^\top\Sigma_\nu^{-1}\lambda/\gamma_I^2 + 1/\sigma_0^2\right) + \lambda^\top\Sigma_\nu^{-1}\lambda},$$

which can be verified by multiplying both sides of the latter equation by $E+Q$. Substituting $(E+Q)^{-1}$ above into Equation (A.29), we obtain Equation (25) for $\partial\theta_U^*(p)/\partial p^\top$.

Finally, we demonstrate that $\theta_{I,m}^*(p;\varepsilon)$ is downward sloping in p_m. This result follows from the fact that matrix $(\mathrm{var}^{\mathrm{RN}}[C])^{-1}$ is positive-definite (as the inverse of a positive-definite matrix), and its element m of the diagonal is given by $e_m^\top(\mathrm{var}^{\mathrm{RN}}[C])^{-1}e_m > 0$, where $e_m = (0,0,\dots,1,\dots,0)^\top$ is a vector with m^{th} element equal to 1 and other elements equal to zero. Then, from Equation (24) it follows that $\partial\theta_{I,m}^*(p;\varepsilon)/\partial p_m < 0$. ■

Proof of Lemma 3. From Condition (26) we have that $b_n = \lambda_0 + C(\omega_n)^\top\lambda$, which we substitute into the objective function (3) of the informed investor. After some algebra, we rewrite investor I's objective function as follows:

$$\begin{aligned}\mathbb{E}\Big[-e^{-\gamma_I W_{I,T}}|\varepsilon,p\Big] &= -\frac{\sum_{j=1}^N \exp\{a_j + b_j\varepsilon - \gamma_I(C(\omega_j) - e^{rT}p)^\top\theta_I\}}{\sum_{j=1}^N \exp\{a_j + b_j\varepsilon\}}\\ &= -\exp\{(\lambda_0 + e^{rT}p^\top\lambda)\varepsilon - e^{rT}p^\top(\lambda\varepsilon - \gamma_I\theta_I)\}\frac{\sum_{j=1}^N \exp\{a_j + C(\omega_j)^\top(\lambda\varepsilon - \gamma_I\theta_I)\}}{\sum_{j=1}^N \exp\{a_j + b_j\varepsilon\}}\\ &= -\exp\{(\lambda_0 + e^{rT}p^\top\lambda)\varepsilon - e^{rT}p^\top\widehat{\theta}_I\}\frac{\sum_{j=1}^N \exp\{a_j + C(\omega_j)^\top\widehat{\theta}_I\}}{\sum_{j=1}^N \exp\{a_j + b_j\varepsilon\}},\end{aligned} \tag{A.30}$$

where $\widehat{\theta}_I = \lambda\varepsilon - \gamma_I\theta_I$. From the last line in Equation (A.30) we observe that finding optimal portfolio $\theta_I^*(p;\varepsilon)$ reduces to finding optimal $\widehat{\theta}_I^*$, which solves the optimization problem

$$\max_{\widehat{\theta}_I} e^{rT}p^\top\widehat{\theta}_I - g_I(\widehat{\theta}_I), \tag{A.31}$$

where $g_I(\widehat{\theta}_I) = \ln\Big(\sum_{i=1}^N \exp\{a_i + C(\omega_i)^\top\widehat{\theta}_I\}\Big)$. From optimization problem (A.31), we see that $\widehat{\theta}_I^*$ does not depend on shock ε. Hence, portfolio $\theta_I^*(p;\varepsilon)$ is given by Equation (27). ■

Proof of Proposition 3.

Step 1 (Portfolio of investor I). Investor I's optimization problem (A.31) yields the FOC for the optimal $\widehat{\theta}_I^* = \lambda\varepsilon - \gamma_I\theta_I^*$:

$$f_I\left(\widehat{\theta}_I^*\right) = e^{rT}p, \tag{A.32}$$

where $f_I(x) \equiv g_I'(x)$ is given by (29). Assuming that $f_I(\cdot)$ is invertible (as verified below), we find that $\widehat{\theta}_I^* = f_I^{-1}(e^{rT}p)$. Then, using $\widehat{\theta}_I^* = \lambda\varepsilon - \gamma_I\theta_I^*$, we find portfolio $\theta_I^*(p;\varepsilon)$ in (31).

Step 2 (Posterior probabilities and portfolio of investor U). Substituting investor I's portfolio (27) into the market clearing condition, we obtain: $\lambda\varepsilon/\gamma_I + \nu + \widehat{H}(p) = 0$, where $\widehat{H}(p) = -f_I^{-1}(e^{rT}p)/\gamma_I + \theta_U^*(p)$, which is analogous to condition (14) for complete markets. Hence, the posterior probabilities can be found similar to Lemma 2, and are given by Equation (16) in which $H(p)$ is replaced by $\widehat{H}(p)$. Substituting $b_n = \lambda_0 + C(\omega_n)^\top\lambda$ from Condition (26) into Equation (16) with $\widehat{H}(p)$ instead of $H(p)$, we obtain:

$$\pi_n^U(p) = \frac{1}{G_3(p)}\exp\left(a_n + \frac{1}{2}\frac{b_n^2 + 2C(\omega_n)^\top(\lambda\mu_0/\sigma_0^2 - \lambda\lambda^\top\Sigma_\nu^{-1}\widehat{H}(p)/\gamma_I)}{\lambda^\top\Sigma_\nu^{-1}\lambda/\gamma_I^2 + 1/\sigma_0^2}\right), \tag{A.33}$$

where $G_3(p)$ is a normalizing function. Substituting probabilities $\pi_n^U(p)$ into investor U's objective function (9), after some algebra, we obtain:

$$\begin{aligned}
&-\textstyle\sum_{n=1}^N \pi_n^U(p)\exp\left\{-\gamma_U\Big(W_{U,0}e^{rT} + \Big(C(\omega_n) - e^{rT}p\Big)^\top\theta_U\Big)\right\} \\
&= -\frac{1}{G_4(p)}\exp\left(\gamma_U e^{rT}p^\top\theta_U + g_U\left(\frac{\lambda\mu_0/\sigma_0^2 - \lambda\lambda^\top\Sigma_\nu^{-1}\widehat{H}(p)/\gamma_I}{\lambda^\top\Sigma_\nu^{-1}\lambda/\gamma_I^2 + 1/\sigma_0^2} - \gamma_U\theta_U\right)\right),
\end{aligned} \tag{A.34}$$

where $G_4(p)$ is some function of prices and $g_U : \mathbb{R}^{M-1} \to \mathbb{R}$ is a function given by:

$$g_U(x) = \ln\left(\sum_{j=1}^N \exp\left\{a_j + \frac{1}{2}\frac{b_j^2}{\lambda^\top\Sigma_\nu^{-1}\lambda/\gamma_I^2 + 1/\sigma_0^2} + C(\omega_j)^\top x\right\}\right).$$

From Equation (A.34), we find that investor U's optimization problem becomes

$$\min_{\theta_U}\ \gamma_U e^{rT}p^\top\theta_U + g_U\left(\frac{\lambda\mu_0/\sigma_0^2 - \lambda\lambda^\top\Sigma_\nu^{-1}\widehat{H}(p)/\gamma_I}{\lambda^\top\Sigma_\nu^{-1}\lambda/\gamma_I^2 + 1/\sigma_0^2} - \gamma_U\theta_U\right).$$

Let $f_U(x) \equiv g_U'(x)$, then the FOC for investor U's optimal portfolio θ_U^* is,

$$f_U\left(\frac{\lambda\mu_0/\sigma_0^2 - \lambda\lambda^\top\Sigma_\nu^{-1}\widehat{H}(p)/\gamma_I}{\lambda^\top\Sigma_\nu^{-1}\lambda/\gamma_I^2 + 1/\sigma_0^2} - \gamma_U\theta_U^*\right) = e^{rT}p.$$

Assuming that f_U is invertible (as verified below), and $e^{rT}p$ belongs to its range, we obtain

$$\frac{\lambda\mu_0/\sigma_0^2 - \lambda\lambda^\top\Sigma_\nu^{-1}\widehat{H}(p)/\gamma_I}{\lambda^\top\Sigma_\nu^{-1}\lambda/\gamma_I^2 + 1/\sigma_0^2} - \gamma_U\theta_U^* = f_U^{-1}\left(e^{rT}p\right). \tag{A.35}$$

Substituting for $\widehat{H}(p) = -f_I^{-1}\left(e^{rT}p\right)/\gamma_I + \theta_U^*(p)$ and factoring out $\gamma_U\theta_U^*(p)$ we have

$$\frac{\lambda\mu_0/\sigma_0^2 + \lambda\lambda^\top\Sigma_\nu^{-1}f_I^{-1}\left(e^{rT}p\right)/\gamma_I^2}{\lambda^\top\Sigma_\nu^{-1}\lambda/\gamma_I^2 + 1/\sigma_0^2} - \gamma_U\theta_U^*(p)\left(E+Q\right) = f_U^{-1}\left(e^{rT}p\right),$$

where E is the identity matrix and matrix Q is given in (A.16). Solving for $\theta_U^*(p)$ yields

$$\begin{aligned}\theta_U^*(p) &= \frac{1}{\gamma_U}\left(E+Q\right)^{-1}\left(\frac{\lambda\mu_0/\sigma_0^2 + \lambda\lambda^\top\Sigma_\nu^{-1}f_I^{-1}\left(e^{rT}p\right)/\gamma_I^2}{\lambda^\top\Sigma_\nu^{-1}\lambda/\gamma_I^2 + 1/\sigma_0^2} - f_U^{-1}\left(e^{rT}p\right)\right)\\ &= \left(E+Q\right)^{-1}\left(\frac{Qf_I^{-1}\left(e^{rT}p\right)}{\gamma_I} - \frac{f_U^{-1}\left(e^{rT}p\right)}{\gamma_U} + \frac{\mu_0/(\gamma_U\sigma_0^2)\lambda}{\lambda^\top\Sigma_\nu^{-1}\lambda/\gamma_I^2 + 1/\sigma_0^2}\right).\end{aligned} \tag{A.36}$$

Step 3 (Invertibility of $f_I(x)$ and $f_U(x)$). Functions $f_I(x)$ and $f_U(x)$ are special cases of function $f(x;t)$ in Equation (A.40) below for $t=0$ and $t = 0.5/(\lambda^\top\Sigma_\nu^{-1}\lambda/\gamma_I^2 + 1/\sigma_0^2)$. Function $f(x;t)$ has positive-definite and invertible Jacobian by Lemma A.3 below. Hence, by Lemma A.4 below, $f_I(x)$ and $f_U(x)$ are invertible on their ranges.

Step 4 (Equation for asset prices). Substituting θ_I^* and θ_U^* from Equations (31) and (32) into the market clearing condition $\theta_I^*(p;\varepsilon) + \theta_U^*(p) + \nu = 0$ yields, after some algebra, Equation (28) for price vector $P(\varepsilon,\nu)$.

Step 5 (Existence of Equilibrium). Finally, we show that there exists unique vector of prices satisfying Equation (28). Denote $x_I = f_I^{-1}\Big(e^{rT}P(\varepsilon,\nu)\Big)$ and $x_U = f_U^{-1}\Big(e^{rT}P(\varepsilon,\nu)\Big)$. Hence, $f_I(x_I) = f_U(x_U) = e^{rT}P(\varepsilon,\nu)$. From the latter equation and Equation (28) for $P(\varepsilon,\nu)$, we obtain the following system of equations for x_I and x_U:

$$\frac{x_I}{\gamma_I} + \frac{x_U}{\gamma_U} = \Big(E+Q\Big)\Big(\frac{\lambda\varepsilon}{\gamma_I} + \nu\Big) + \frac{\mu_0/(\gamma_U\sigma_0^2)\lambda}{\lambda^\top\Sigma_\nu^{-1}\lambda/\gamma_I^2 + 1/\sigma_0^2}, \tag{A.37}$$

$$f_I(x_I) = f_U(x_U). \tag{A.38}$$

From Equations (A.32) and (A.35), we note that x_I and x_U are related to portfolios:

$$x_I = \lambda\varepsilon - \gamma_I\theta_I^*, \quad x_U = \frac{\lambda\mu_0/\sigma_0^2 - \lambda\lambda^\top\Sigma_\nu^{-1}\widehat{H}(p)/\gamma_I}{\lambda^\top\Sigma_\nu^{-1}\lambda/\gamma_I^2 + 1/\sigma_0^2} - \gamma_U\theta_U^*. \tag{A.39}$$

From Equation (A.37), we find that $x_U = \bar{x} - \eta x_I$, where $\eta = \gamma_U/\gamma_I$ and $\bar{x}$ is given by

$$\bar{x} = \gamma_U\Big(E+Q\Big)\Big(\frac{\lambda\varepsilon}{\gamma_I} + \nu\Big) + \frac{\gamma_U(\mu_0/\sigma_0^2)\lambda}{\lambda^\top\Sigma_\nu^{-1}\lambda/\gamma_I^2 + 1/\sigma_0^2}.$$

Substituting $x_U = \bar{x} - \eta x_I$ into Equation (A.38), we find that x_I solves $f_I(x_I) = f_U(\bar{x} - \eta\, x_I)$. The latter equation is a special case of equation $f(x;0) = f(\bar{x} - \eta\, x; t)$ in (A.43) below with $t = 0.5/(\lambda^\top\Sigma_\nu^{-1}\lambda/\gamma_I^2 + 1/\sigma_0^2)$, because $f(x;0) = f_I(x)$, $f(x;t) = f_U(x)$. Hence, by Lemma

A.5, equation $f_I(x_I) = f_U(\bar{x} - \eta\, x_I)$ has a unique, continuous, and differentiable solution x_I. Because $P(\varepsilon, \nu) = e^{-rT} f_I(x_I)$, price exists, is unique, continuous, differentiable, and invertible on its range function of $\bar{x}$, and hence, also of sufficient statistic $\lambda\varepsilon/\gamma_I + \nu$. ■

Lemma A.3.
i) *Consider function* $f(x;t) : \mathbb{R}^{M-1} \times \mathbb{R} \to \mathbb{R}^{M-1}$ *given by:*

$$f(x;t) = \frac{\sum_{j=1}^{N} C(\omega_j) \exp\{a_j + tb_j^2 + C(\omega_j)^\top x\}}{\sum_{j=1}^{N} \exp\{a_j + tb_j^2 + C(\omega_j)^\top x\}}. \tag{A.40}$$

Then, for all x *and* t, $f(x;t)$ *has an invertible positive-definite Jacobian given by:*

$$\frac{\partial f(x;t)}{\partial x} = \mathrm{var}^{\pi}[C], \tag{A.41}$$

where $\mathrm{var}^{\pi}[C]$ *is a variance-covariance matrix under a certain probability measure* $\pi(x;t)$.
ii) *Consider function* $\widehat{f}(x;t) = f(x;0) - f(\bar{x} - \eta\, x;t)$ *for a fixed* $\bar{x}$ *and* $\eta > 0$. *Then, this function also has a positive-definite and invertible Jacobian* $\partial\widehat{f}(x;t)/\partial x$ *for all* x *and* t.

Proof of Lemma A.3. i) Differentiating function $f(x;t)$ with respect to x, we obtain:

$$\begin{aligned}\frac{\partial f(x;t)}{\partial x} &= \frac{\sum_{j=1}^{N} C(\omega_j)C(\omega_j)^\top \exp\{a_j + tb_j^2 + C(\omega_j)^\top x\}}{\sum_{j=1}^{N} \exp\{a_j + tb_j^2 + C(\omega_j)^\top x\}} \\ &- \frac{\sum_{j=1}^{N} C(\omega_j) \exp\{a_j + tb_j^2 + C(\omega_j)^\top x\}}{\sum_{j=1}^{N} \exp\{a_j + tb_j^2 + C(\omega_j)^\top x\}} \frac{\sum_{j=1}^{N} C(\omega_j)^\top \exp\{a_j + tb_j^2 + C(\omega_j)^\top x\}}{\sum_{j=1}^{N} \exp\{a_j + tb_j^2 + C(\omega_j)^\top x\}} \\ &= \mathrm{var}^{\pi}[C],\end{aligned} \tag{A.42}$$

where the variance $\mathrm{var}^{\pi}[C]$ is computed under a probability measure given by $\pi_j(x;t) = \exp\{a_j + tb_j^2 + C(\omega_j)^\top x\}/\left(\sum_{j=1}^{N} \exp\{a_j + tb_j^2 + C(\omega_j)^\top x\}\right)$.

Matrix $\mathrm{var}^{\pi}[C]$ has elements $\mathrm{cov}^{\pi}(C_i, C_j)$. It is positive-definite and invertible because all assets are non-redundant. Suppose, $\mathrm{var}^{\pi}[C]$ is not invertible. Then, the columns of this matrix are linearly dependent, and hence, there exist constants λ_m such that $\mathrm{cov}^{\pi}(\lambda_1 C_1 + \dots \lambda_{M-1} C_{M-1}, C_m) = 0$ for all $m = 1, \dots, M-1$. Multiplying the latter equalities by λ_m and summing up, we obtain $\mathrm{var}^{\pi}[\lambda_1 C_1 + \dots \lambda_{M-1} C_{M-1}] = 0$. Hence, $\lambda_1 C_1 + \dots \lambda_{M-1} C_{M-1}$ is a constant, which contradicts non-redundancy of the riskless asset.
ii) Function $\widehat{f}(x;t)$ has Jacobian $\partial f(x;0)/\partial x + \eta\, \partial f(\bar{x} - \eta\, x;t)/\partial x$, which is the sum of positive-definite and invertible matrices, and hence is positive-definite and invertible. ■

Lemma A.4 (Gale and Nikaidô). *Let* $f(x) : \mathbb{R}^{M-1} \to \mathbb{R}^{M-1}$ *be a continuous differentiable function with a positive-definite Jacobian. Then, function* $f(x)$ *is injective, that is, invertible on its range, so that* $\forall x_1, x_2 \in \mathbb{R}^{M-1}$ *such that* $f(x_1) = f(x_2)$ *we have* $x_1 = x_2$.

Proof of Lemma A.4. See the proof of Theorem 6 in Gale and Nikaidô (1965). ■

Lemma A.5 *Consider function* $f(x;t) : \mathbb{R}^{M-1} \times \mathbb{R} \to \mathbb{R}^{M-1}$ *given by Equation* (A.40). *Then, for all fixed* $\bar{x} \in \mathbb{R}^{M-1}$, $\eta > 0$ *and* $t \in \mathbb{R}$ *there exists unique* x *which solves equation*

$$f(x;0) = f(\bar{x} - \eta\, x; t). \tag{A.43}$$

Moreover, solution $x(t;\bar{x})$ *is continuous and differentiable in* t *and* $\bar{x}$.

Proof of Lemma A.5. The proof proceeds in three steps.

Step 1. Let us fix $\bar{x}$ and show that the solution of Equation (A.43) exists for all t. For $t = 0$ Equation (A.43) has solution $x_0 = \bar{x}/(1+\eta)$. Function $\widehat{f}(x;t) \equiv f(x;0) - f(\bar{x} - \eta\, x; t)$ is continuously differentiable and has an invertible Jacobian with respect to x by Lemma A.3. Hence, by the implicit function theorem (Theorem A.1 below), there exists a unique continuously differentiable function $x(t)$ that solves (A.43) in some interval $t \in (-t_-, t_+)$, where $t_\pm > 0$. Next, we show that $t_+ = +\infty$, and the proof that $t_- = +\infty$ is analogous.

Suppose, t_+ is finite. Let $(-t_-, t_+)$ be the largest open interval in which a unique solution exists. We show in steps 2 and 3 below that there exists a unique solution of equation $\widehat{f}(x;t_+) = 0$. Because $\widehat{f}(x;t)$ has a positive definite and invertible Jacobian (see Lemma A.3), by the implicit function theorem, the solution can be extended to some $t > t_+$, which contradicts the fact that $(-t_-, t_+)$ is the largest interval in which a unique solution exists. Therefore, this leads to a contradiction, and hence, $t_+ = +\infty$.

Step 2. We show that $\widehat{f}(x;t_+) = 0$ has a unique solution, which implies that $t_+ = +\infty$, as shown above. Consider a sequence $t_k \uparrow t_+$ and solutions x_k such that

$$f(x_k;0) = f(\bar{x} - \eta\, x_k; t_k). \tag{A.44}$$

Suppose, x_k are bounded by some constant A, i.e., $|x_k| < A$. Then, by Weierstrass Theorem [e.g., Rudin (1976, Theorem 2.42)], there exists a convergent subsequence such that $x_{k_n} \to x^*$ as $n \to +\infty$. Taking limit $k_n \to \infty$ in Equation (A.44), by the continuity of $f(x;t)$ we find that $\widehat{f}(x^*;t_+) = 0$. This solution is unique by Lemma A.4 because $f(x;0) - f(\bar{x} - \eta\, x; t)$ has positive-definite Jacobian by Lemma A.3. Hence, $t_+ = +\infty$.

Step 3. It remains to prove that x_k is indeed bounded. Suppose, x_k is unbounded, i.e., there exist indices k_n such that $|x_{k_n}| \to \infty$, as $k_n \to \infty$. We renumber elements k_n by k, and hence, assume that $|x_k| \to \infty$. Let $j(k) = \arg\max_j C(\omega_j)^\top x_k$. Because $j(k)$ takes only finite number of values from 1 to N, there exists index j^* such that $j^* = j(k_n)$ for an infinite sequence of $k_n \to \infty$. Without loss of generality, we assume that $j^* = 1$ (otherwise, we relabel states ω_n accordingly) and also focus on subsequence k_n and relabel its elements by k. Hence, $C(\omega_1)^\top x_k \geq C(\omega_j)^\top x_k$ for all $j = 1, \ldots, N$. Similarly, we find a subsequence x_k such that $C(\omega_1)^\top x_k \geq C(\omega_2)^\top x_k \geq C(\omega_j)^\top x_k$ for all $j = 2, \ldots, N$. Similarly, there exists x_k such that

$$C(\omega_1)^\top x_k \geq \ldots \geq C(\omega_m)^\top x_k > C(\omega_{m+1})^\top x_k \geq \ldots \geq C(\omega_N)^\top x_k, \tag{A.45}$$

for all k, where m is the fist index for which $C(\omega_m)^\top x_k - C(\omega_{m+1})^\top x_k \to +\infty$ as $k \to \infty$. The existence of such an index m is guaranteed by Lemma A.7 below.

Next, we take the limit $k \to \infty$ in Equation (A.44). Ordering (A.45) simplifies the computation of this limit. Consider the following probability measure $\pi_j(x;t)$:

$$\pi_j(x;t) = \frac{\exp\{a_j + tb_j^2 + C(\omega_j)^\top x\}}{\sum_{j=1}^N \exp\{a_j + tb_j^2 + C(\omega_j)^\top x\}}. \tag{A.46}$$

Because $0 \leq \pi_j(x_k;t) \leq 1$, by Weierstrass theorem there exists a subsequence x_k such that $\pi_j(x_k;0) \to \pi_j^+$ and $\pi_j(\bar{x} - \eta\, x_k; t_k) \to \pi_j^-$ for all $j = 1, \ldots, N$, where $\sum_{j=1}^N \pi_j^+ = \sum_{j=1}^N \pi_j^- = 1$, $0 \leq \pi_j^+ \leq 1$ and $0 \leq \pi_j^- \leq 1$. Next, we demonstrate that

$$\pi_j^+ = 0, \text{ for } j = m+1, \ldots, N, \tag{A.47}$$

$$\pi_j^- = 0, \text{ for } j = 1, \ldots, m. \tag{A.48}$$

To derive equalities (A.47) and (A.48), we use inequalities (A.45) and the fact that $C(\omega_m)^\top x_k - C(\omega_{m+1})^\top x_k \to +\infty$ as $k \to \infty$, to obtain for all $j > m$:

$$\pi_j^+ = \lim_{k\to+\infty} \pi_j(x_k;0) \leq \lim_{k\to+\infty} \frac{\exp\{a_j + C(\omega_j)^\top x_k\}}{\exp\{a_m + C(\omega_m)^\top x_k\}} \leq \lim_{k\to+\infty} \frac{\exp\{a_j + C(\omega_{m+1})^\top x_k\}}{\exp\{a_m + C(\omega_m)^\top x_k\}} = 0.$$

Similarly, for all $j \leq m$ we obtain:

$$\begin{aligned}\pi_j^- &= \lim_{k\to+\infty} \pi_j(\bar{x} - \eta\, x_k; t_k) \leq \lim_{k\to+\infty} \frac{\exp\{a_j + t_k b_j^2 + C(\omega_j)^\top \bar{x} - \eta\, C(\omega_j)^\top x_k\}}{\exp\{a_{m+1} + t_k b_{m+1}^2 + C(\omega_{m+1})^\top \bar{x} - \eta\, C(\omega_{m+1})^\top x_k\}} \\ &\leq \lim_{k\to+\infty} \frac{\exp\{a_j + t_k b_j^2 + C(\omega_j)^\top \bar{x} - \eta\, C(\omega_m)^\top x_k\}}{\exp\{a_m + t_k b_m^2 + C(\omega_{m+1})^\top \bar{x} - \eta\, C(\omega_{m+1})^\top x_k\}} = 0.\end{aligned}$$

Using Equations (A.47) and (A.48) and taking the limit $k \to +\infty$ in (A.44), we obtain:

$$\pi_{m+1}^+ C(\omega_{m+1}) + \ldots + \pi_N^+ C(\omega_N) = \pi_1^- C(\omega_1) + \ldots + \pi_m^- C(\omega_m).$$

Transposing both sides of the above equation and multiplying by x_k, we obtain:

$$\pi_{m+1}^+ C(\omega_{m+1})^\top x_k + \ldots + \pi_N^+ C(\omega_N)^\top x_k = \pi_1^- C(\omega_1)^\top x_k + \ldots + \pi_m^- C(\omega_m)^\top x_k. \tag{A.49}$$

From the fact that $\sum_{j=1}^{N} \pi_j^+ = \sum_{j=1}^{N} \pi_j^- = 1$, demonstrated above, Equations (A.47)–(A.49), and inequality (A.45), we obtain:

$$\begin{aligned}\pi_{m+1}^{+} C(\omega_{m+1})^\top x_k + \ldots + \pi_N^{+} C(\omega_N)^\top x_k \le C(\omega_{m+1})^\top x_k < C(\omega_m)^\top x_k \\ \le \pi_1^{-} C(\omega_1)^\top x_k + \ldots + \pi_m^{-} C(\omega_m)^\top x_k.\end{aligned} \tag{A.50}$$

Inequalities (A.50) contradict Equation (A.49). Consequently, x_k is bounded. Hence, as shown in step 2 above, $t_+ = +\infty$, which proves the global existence of $x(t;\bar{x})$. The continuity and differentiability of $x(t;\bar{x})$ follows from the implicit function theorem. ■

Theorem A.1 (Implicit Function Theorem). *Consider a continuously differentiable function* $\widehat{f}(x;t) : \mathbb{R}^{M-1} \times \mathbb{R} \to \mathbb{R}^{M-1}$. *Suppose,* $\widehat{f}(x_0;t_0) = 0$ *and the Jacobian* $\partial \widehat{f}(x_0;t_0)/\partial x$ *is invertible. Then, there exist open sets* U *and* V *such that* $x_0 \in U$, $t_0 \in V$, *and a unique continuously differentiable function* $x(t) : V \to U$ *such that* $\widehat{f}(x(t);t) = 0$.

Proof of Theorem A.1. This is a special case of Theorem 9.28 in Rudin (1976). ■

Lemma A.6. *Consider a sequence* x_k *such that* $|x_k| \to \infty$ *as* $k \to \infty$. *Then, there exists index* m *such that sequence* $|C(\omega_m)^\top x_k|$ *is unbounded.*

Proof of Lemma A.6. Suppose, on the contrary, there exists constant A such that $|C(\omega_m)^\top x_k| < A$ for all m and k. Because all securities are non-redundant, the matrix with columns $C(\omega_n)$, $n = 1, \ldots, N$ has rank $M-1$ and vectors $C(\omega_n)$ span $\mathbb{R}^{M-1}$. Without loss of generality, assume that the $M-1$ vectors $C(\omega_1)$,..., $C(\omega_{M-1})$ form a basis in $\mathbb{R}^{M-1}$.

Consider vector $e_l = (0, \ldots, 0, 1, 0, \ldots, 0)^\top \in \mathbb{R}^{M-1}$ with l^{th} element equal to 1 and all other elements equal to 0. Then, there exist constants $\alpha_{m,l}$ such that $e_l = \alpha_{1,l} C(\omega_1) + \ldots + \alpha_{M-1,l} C(\omega_N)$. It can be easily observed that x_k is bounded, because for all l

$$|e_l^\top x_k| \le |\alpha_{1,l}||C(\omega_1)^\top x_k| + \ldots + |\alpha_{M-1,l}||C(\omega_{M-1})^\top x_k| \le A(M-1) \max_{m,l} |\alpha_{m,l}|,$$

which contradicts $|x_k| \to \infty$. Hence, $|C(\omega_m)^\top x_k|$ is unbounded for some m. ■

Lemma A.7. *Consider a sequence* x_k *such that* $|x_k| \to \infty$ *as* $k \to \infty$. *Then, there exists index* m *such that sequence* $|C(\omega_m)^\top x_k - C(\omega_{m+1})^\top x_k|$ *is unbounded.*

Proof of Lemma A.7. Suppose, on the contrary, sequence $|C(\omega_m)^\top x_k - C(\omega_{m+1})^\top x_k|$ is bounded for all m. The latter easily implies that $|C(\omega_i)^\top x_k - C(\omega_j)^\top x_k| < A$ for all i and j and some constant A. Because all assets are non-redundant, vectors $(C(\omega_n)^\top, 1)^\top \in \mathbb{R}^M$, where $n = 1, \ldots, N$, span $\mathbb{R}^M$. Without loss of generality, assume that the first M vectors $(C(\omega_n)^\top, 1)^\top$ form a basis in $\mathbb{R}^M$. Hence, there exist unique $(\alpha_1, \ldots, \alpha_M)^\top$ such that:

$$\begin{aligned} \alpha_1 C(\omega_1) + \ldots + \alpha_M C(\omega_M) = 0, \\ \alpha_1 + \ldots + \alpha_M = 1. \end{aligned} \tag{A.51}$$

We solve Equations (A.51) and for an arbitrary index m we obtain:

$$\begin{aligned} |C(\omega_m)^\top x_k| &= |(\alpha_1 + \ldots + \alpha_M) C(\omega_m)^\top x_k - (\alpha_1 C(\omega_1)^\top x_k + \ldots + \alpha_M C(\omega_M)^\top x_k)| \\ &\le |\alpha_1||C(\omega_m)^\top x_k - C(\omega_1)^\top x_k| + \ldots + |\alpha_M||C(\omega_m)^\top x_k - C(\omega_M)^\top x_k| \\ &\le A \max_l |\alpha_l|, \end{aligned}$$

contradicting the result of Lemma A.6 that $|C(\omega_m)^\top x_k|$ is unbounded for some m. ■

Lemma A.8. i) *The market is effectively complete iff there exists a portfolio that replicates* $\ln(\pi_n(\varepsilon)/\pi_n^U(p))$, *that is, there exist* ω*-independent* $\lambda_0 \in \mathbb{R}$ *and* $\hat{\lambda} \in \mathbb{R}^{M-1}$ *such that*

$$\ln\left(\frac{\pi_n(\varepsilon)}{\pi_n^U(p)}\right) = \hat{\lambda}_0 + C(\omega_n)^\top \hat{\lambda}. \tag{A.52}$$

ii) *The equilibrium Pareto efficient portfolios are given by:*

$$\theta_I^* = -\frac{\gamma_U \nu - \hat{\lambda}}{\gamma_I + \gamma_U}, \tag{A.53}$$

$$\theta_U^* = -\frac{\gamma_I \nu + \hat{\lambda}}{\gamma_I + \gamma_U}. \tag{A.54}$$

Proof of Lemma A.8. i) Suppose, the market is effectively complete. Taking logs on both sides of Pareto efficiency condition (33), and rearranging terms, we find that $\gamma_U W_{U,T,n} - \gamma_I W_{I,T,n} - \ln(\ell) = \ln(\pi_n^U(p)/\pi_n(\varepsilon))$. Hence, the log-ratio of probabilities can be replicated by a portfolio of $\gamma_U \theta_U^* - \gamma_I \theta_I^*$ units of risky assets and $\gamma_U(W_{U,0} - p^\top \theta_U^*)e^{rT} - \gamma_I(W_{I,0} - p^\top \theta_I^*)e^{rT} - \ln(\ell)$ units of bond.

Suppose, there exist $\lambda_0 \in \mathbb{R}$ and $\hat{\lambda} \in \mathbb{R}^{M-1}$ such that (A.52) holds. Hence, $\pi_n(\varepsilon) = \pi_n^U(p)\exp(\hat{\lambda}_0 + C(\omega_n)^\top \hat{\lambda})$. Substituting $\pi_n(\varepsilon)$ into investor I's optimization (3), we obtain:

$$\max_{\theta_I} \mathbb{E}\Big[-e^{-\gamma_I W_{I,T}}\Big|\varepsilon, p\Big] = \max_{\theta_I}\left[-\sum_{n=1}^N \pi_n^U(p) e^{\hat{\lambda}_0 + C(\omega_n)^\top \hat{\lambda} - \gamma_I W_{I,T,n}}\right]. \tag{A.55}$$

Substituting wealth $W_{I,T,n}$ from the budget constraint (5) into optimization (A.55), and rearranging terms, we observe that this optimization is equivalent to maximizing

$$\max_{\theta_I}\left[-\sum_{n=1}^N \pi_n^U(p) e^{-\gamma_I (C(\omega_n) - e^{rT}p)^\top(\theta_I - \hat{\lambda}/\gamma_I)}\right] = \max_{\hat{\theta}}\left[-\sum_{n=1}^N \pi_n^U(p) e^{-\gamma_U (C(\omega_n) - e^{rT}p)^\top \hat{\theta}}\right], \tag{A.56}$$

where, by a change of variable, $\hat{\theta} = (\theta_I - \hat{\lambda}/\gamma_I)(\gamma_I/\gamma_U)$. The second optimization in (A.56) is the same as that of investor U. Hence, $\theta_U^* = \hat{\theta}^* = (\theta_I^* - \hat{\lambda}/\gamma_I)(\gamma_I/\gamma_U)$, or, equivalently:

$$\gamma_I \theta_I^* - \gamma_U \theta_U^* = \hat{\lambda}. \tag{A.57}$$

Multiplying (A.57) by $(C(\omega_n) - e^{rT}p)$, we obtain $\gamma_I (C(\omega_n) - e^{rT}p)^\top \theta_I^* - \gamma_U (C(\omega_n) - e^{rT}p)^\top \theta_U^* = (C(\omega_n) - e^{rT}p)^\top \hat{\lambda}$. The latter equation and budget constraints (5) then imply $\gamma_U W_{U,T,n} - \gamma_I W_{I,T,n} - \ln(\ell) = \ln(\pi_n^U(p)/\pi_n(\varepsilon))$, where ℓ does not depend on ω_n, which is equivalent to the Pareto efficiency condition (33).

ii) The equilibrium Pareto efficient portfolios satisfy the market clearing condition (6) and efficiency condition (A.57). Solving the latter two equations with two unknowns, we obtain portfolios (A.53) and (A.54). ■

Proof of Proposition 4. i) Substituting $\pi_n(\varepsilon)$ from (1) and π_n^U from (A.33) into $\ln(\pi_n(\varepsilon)/\pi_n^U)$, and using the fact that by market clearing $\widehat{H}(p) = -(\lambda\varepsilon/\gamma_I + \nu)$, we obtain:

$$\ln\left(\frac{\pi_n(\varepsilon)}{\pi_n^U(p)}\right) = b_n\varepsilon - \frac{1}{2}\frac{b_n^2 + 2b_n\left(\mu_0/\sigma_0^2 + \lambda^\top \Sigma_\nu^{-1}\left(\lambda\varepsilon/\gamma_I + \nu\right)/\gamma_I\right)}{\lambda^\top \Sigma_\nu^{-1}\lambda/\gamma_I^2 + 1/\sigma_0^2} + \tilde{\lambda}_0, \tag{A.58}$$

where $\tilde{\lambda}_0$ does not depend on state ω_n. The log-ratio of probabilities in (A.58) is a quadratic function of b_n. Hence, by Lemma A.8, the optimal portfolios are Pareto efficient iff there exists a portfolio that replicates b_n^2.

ii) From the FOC of investor I (A.32), we find that $P(\varepsilon, \nu) = f_I(\lambda\varepsilon - \gamma_I\theta_I^*)$. Using the expression for $f_I(x)$ in (29), we obtain:

$$P(\varepsilon, \nu) = \frac{\sum_{j=1}^N C(\omega_j)\exp\{a_j + C(\omega_j)^\top(\lambda\varepsilon - \gamma_I\theta_I^*)\}}{\sum_{j=1}^N \exp\{a_j + C(\omega_j)^\top(\lambda\varepsilon - \gamma_I\theta_I^*)\}}. \tag{A.59}$$

Using θ_I^* from (A.53), and the spanning condition $b_n = \lambda_0 + C(\omega_n)^\top\lambda$, we obtain:

$$C(\omega_n)^\top(\lambda\varepsilon - \gamma_I\theta_I^*) = (b_n - \lambda_0)\varepsilon - \frac{\gamma_I C(\omega_n)^\top\hat{\lambda}}{\gamma_I + \gamma_U} + \frac{\gamma_I\gamma_U}{\gamma_I + \gamma_U}C(\omega_n)^\top\nu, \tag{A.60}$$

where $\hat{\lambda}$ is such that $\ln(\pi_n(\varepsilon)/\pi_n^U(p)) = \hat{\lambda}_0 + C(\omega_n)^\top\hat{\lambda}$. Hence, $C(\omega_n)^\top\hat{\lambda} = \ln(\pi_n(\varepsilon)/\pi_n^U(p)) - \hat{\lambda}_0$. Substituting $C(\omega_n)^\top\hat{\lambda}$ and $\ln(\pi_n(\varepsilon)/\pi_n^U(p))$ from (A.58) into (A.60), after some algebra, we find that $a_n + C(\omega_n)^\top(\lambda\varepsilon - \gamma_I\theta_I^*) = v_n + \tilde{\lambda}_0$, where v_n is given by (19) in which λ, $\nu \in \mathbb{R}^{M-1}$, $\Omega \in \mathbb{R}^{(N-1)\times(M-1)}$, and E, $Q \in \mathbb{R}^{(M-1)\times(M-1)}$, and $\tilde{\lambda}_0$ does not depend on ω_n. Substituting $a_n + C(\omega_n)^\top(\lambda\varepsilon - \gamma_I\theta_I^*)$ into (A.60), we find the price. ■

Proof of Proposition 5. By Lemma A.2, the model is a limiting case of a discrete model with parameters a_n and b_n given by Equations (7) when $N \to \infty$. Therefore, we derive prices for the discrete state-space case and then take the limit $N \to \infty$.

First, consider an effectively complete market with a quadratic derivative $C_1(\omega_n)^2$. Because $b_n = C_1(\omega_n)/\sigma_0^2$, we obtain that $\lambda = (1/\sigma_0^2, 0)^\top$, where λ is defined in Condition (26), and hence, $C_1(\omega_n)^2$ is informationally irrelevant. By Proposition 4, the asset prices are given by Equation (17) in terms of v_n in (19). Because, by assumption, there is no noise in the market for $C_1(\omega_n)^2$, we take the limit in which noise $\nu_2 \to 0$ in the equation for prices.[10] Substituting a_n and b_n from (7) and $\lambda = (1/\sigma_0^2, 0)^\top$ in Equation (19) for v_n, taking limit $\nu_2 \to 0$ and denoting $s = \varepsilon/(\gamma_I \sigma_0^2) + \nu_1$, after some algebra, we obtain:

$$
\begin{aligned}
v_n &= -\frac{C_1(\omega_n)^2}{2\sigma_{com}^2} - C_1(\omega_n)\left(\frac{\mu_0}{\sigma_{com}^2} - \frac{\gamma_I\gamma_U}{\gamma_I + \gamma_U}\left(1 + \frac{1/(\gamma_I\gamma_U\sigma_\nu^2\sigma_0^2)}{1 + 1/(\gamma_I^2\sigma_\nu^2\sigma_0^2)}\right)s\right) + \ln[\varphi_C(C_1(\omega_n))] \\
&= -\frac{\Big(C_1(\omega_n) - (\gamma_I\sigma_0^2 s - \mu_0)\Big)^2}{2\sigma_{com}^2} + \ln[\varphi_C(C_1(\omega_n))] + \frac{(\gamma_I\sigma_0^2 - \mu_0)^2}{2\sigma_{com}^2},
\end{aligned}
$$

where volatility parameter σ_{com}^2 is given in (37). Substituting v_n into Equations (18) and (17) for the risk-neutral probabilities and prices, we obtain:

$$
\begin{aligned}
P_{com}(s) &= \frac{\sum_{n=1}^N C_1(\omega_n)\exp\left(-\frac{(C_1(\omega_n) - \gamma_I\sigma_0^2 s + \mu_0)^2}{2\sigma_{com}^2}\right)\varphi_C(C_1(\omega_n))}{\sum_{n=1}^N \exp\left(-\frac{(C_1(\omega_n) - \gamma_I\sigma_0^2 s + \mu_0)^2}{2\sigma_{com}^2}\right)\varphi_C(C_1(\omega_n))} e^{-rT} \\
&= \frac{\mathbb{E}\left[C_1 \exp\left(-\frac{(C_1 - \gamma_I\sigma_0^2 s + \mu_0)^2}{2\sigma_{com}^2}\right)\right]}{\mathbb{E}\left[\exp\left(-\frac{(C_1 - \gamma_I\sigma_0^2 s + \mu_0)^2}{2\sigma_{com}^2}\right)\right]} e^{-rT} \equiv P_C(\gamma_I\sigma_0^2 s + \mu_0, \sigma_{com})e^{-rT}.
\end{aligned}
\tag{A.61}
$$

Equation (A.61) gives the price $P_{com}(s)$ in (34) for the discrete state-space, and it also holds for continuous state-space by taking the limit $N \to \infty$.

Next, we turn to the incomplete-market case with one risky asset and derive prices using $P(\varepsilon, \nu) = f_I(x_I)e^{-rT} = f_U(x_U)e^{-rT}$, which follows from FOC (A.32) and condition (A.38). Substituting $f_I(x)$ and $f_U(x)$ from Equations (29) and (30) into the equation for $P(\varepsilon, \nu)$ and observing that $\Sigma_\nu = \sigma_\nu^2$, we obtain:

$$
\begin{aligned}
P_{inc}(s)e^{rT} &= \frac{\sum_{n=1}^N C_1(\omega_n)\exp\{a_n + C_1(\omega_n)x_I\}}{\sum_{n=1}^N \exp\{a_n + C_1(\omega_n)x_I\}} \\
&= \frac{\sum_{n=1}^N C_1(\omega_n)\exp\{a_n + \frac{1}{2}\frac{b_n^2}{\lambda^2/(\gamma_I^2\sigma_\nu^2) + 1/\sigma_0^2} + C_1(\omega_n)x_U\}}{\sum_{n=1}^N \exp\{a_n + \frac{1}{2}\frac{b_n^2}{\lambda^2/(\gamma_I^2\sigma_\nu^2) + 1/\sigma_0^2} + C_1(\omega_n)x_U\}},
\end{aligned}
$$

[10] When $\nu_2 = 0$ the same prices $P(\varepsilon, \nu)$ can be obtained directly without taking the limit.

where x_I and x_U solve the system of equations (A.37) and (A.38). Substituting $\lambda = 1/\sigma_0^2$, a_n and b_n from (7) into the above equation for $P_{inc}(s)$, similarly to (A.61) we obtain:

$$P_{inc}(s) = P_C(\sigma_0^2 x_I - \mu_0, \sigma_0)e^{-rT} = P_C(\sigma_{inc}^2 x_U - \mu_0\sigma_{inc}^2/\sigma_0^2, \sigma_{inc})e^{-rT}, \qquad (A.62)$$

where volatility σ_{inc} is given in (37). Next, from Equation (A.37), we express x_U in terms of x_I and, after some algebra, we find that $\sigma_{inc}^2 x_U - \mu_0\sigma_{inc}^2/\sigma_0^2 = \gamma_I\sigma_0^2 s - \mu_0 + \gamma_U\sigma_{inc}^2(s - x_I/\gamma_I)$. Substituting the latter equality into the second equation in (A.62) and denoting $\widehat{s}(s) \equiv x_I/\gamma_I$, we obtain price $P_{inc}(s)$ in Equation (34), and also Equation (36) for $\widehat{s}(s)$.

Differentiating $P_C(\mu, \sigma)$ with respect to μ, after some algebra, we obtain:

$$\frac{\partial P_C(\mu,\sigma)}{\partial \mu} = \frac{\mathbb{E}\left[C_1^2 \exp\left(-\frac{(C_1-\mu)^2}{2\sigma^2}\right)\right]}{\sigma^2\mathbb{E}\left[\exp\left(-\frac{(C_1-\mu)^2}{2\sigma^2}\right)\right]} - \left(\frac{\mathbb{E}\left[C_1 \exp\left(-\frac{(C_1-\mu)^2}{2\sigma^2}\right)\right]}{\sigma\mathbb{E}\left[\exp\left(-\frac{(C_1-\mu)^2}{2\sigma^2}\right)\right]}\right)^2 = \frac{\widehat{\mathrm{var}}[C_1]}{\sigma^2} > 0,$$

where variance $\widehat{\mathrm{var}}[C_1]$ is calculated under the new probability measure, which is given by $\varphi_C(C_1)\exp\left(-(C_1-\mu)^2/(2\sigma^2)\right)/\mathbb{E}\left[\exp\left(-(C_1-\mu)^2/(2\sigma^2)\right)\right]$. Therefore, $P_C(\mu,\sigma)$ is an increasing function of μ, and hence, $P_{com}(s)$ is increasing in s. Differentiating Equation (36) with respect to s and solving for $\widehat{s}'(s)$, we find that:

$$\widehat{s}'(s) = \frac{\gamma_I\sigma_0^2(\partial P_C(\mu_U(s),\sigma_{inc})/\partial\mu)}{\gamma_I\sigma_0^2(\partial P_C(\mu_I(s),\sigma_0)/\partial\mu) + \gamma_U\sigma_{inc}^2(\partial P_C(\mu_U(s),\sigma_{inc})/\partial\mu)} > 0,$$

where $\mu_I(s) = \gamma_I\sigma_0^2\widehat{s}(s) - \mu_0$ and $\mu_U(s) = \gamma_I\sigma_0^2 s - \mu_0 + \gamma_U\sigma_{inc}^2(s - \widehat{s}(s))$. Hence, $P_{inc}(s) = P_C(\gamma_I\sigma_0^2\widehat{s}(s) - \mu_0, \sigma_{inc})$ is an increasing function of s. ■

Proof of Proposition 6. i) We derive Equation (41) by substituting PDF (38) into Equation (35), and then finding the expectations in Equation (35) in closed form.

ii) Rewriting the expectations under the generalized gamma distribution (39) in the pricing function (35) as integrals, after simple algebra, we obtain:

$$P_C(\mu,\sigma;k) = \frac{\displaystyle\int_0^{+\infty} C_1^k \exp\left\{-\frac{1}{2}\left(\frac{1}{\widehat{\sigma}_C^2} + \frac{1}{\sigma_0^2}\right)C_1^2 + \left(\frac{\mu}{\sigma^2} - \delta\right)C_1\right\}dC_1}{\displaystyle\int_0^{+\infty} C_1^{k-1} \exp\left\{-\frac{1}{2}\left(\frac{1}{\widehat{\sigma}_C^2} + \frac{1}{\sigma_0^2}\right)C_1^2 + \left(\frac{\mu}{\sigma^2} - \delta\right)C_1\right\}dC_1}. \qquad (A.63)$$

For the case $k = 1$ the above pricing function can be easily computed in closed form, and is given in Equation (42) for $k = 1$. For general integer $k > 1$, denote the integrals in the numerator and the denominator of (A.63) by I_k and I_{k-1}, respectively. Using integration by parts, we obtain the following recursive equation for I_k:

$$I_k = \frac{\widehat{\sigma}_C^2\sigma^2}{\widehat{\sigma}_C^2 + \sigma^2}\left[\left(\frac{\mu}{\sigma^2} - \delta\right)I_{k-1} - \int_0^{+\infty} C_1^{k-1}\left(\exp\left\{-\frac{1}{2}\left(\frac{1}{\widehat{\sigma}_C^2} + \frac{1}{\sigma_0^2}\right)C_1^2 + \left(\frac{\mu}{\sigma^2} - \delta\right)C_1\right\}\right)' dC_1\right]$$

$$= \frac{\widehat{\sigma}_C^2\sigma^2}{\widehat{\sigma}_C^2 + \sigma^2}\left(\left(\frac{\mu}{\sigma^2} - \delta\right)I_{k-1} + (k-1)I_{k-2}\right).$$

Dividing both sides by I_{k-1} and using $P_C(\mu, \sigma; k) = I_k/I_{k-1}$, we obtain Equation (42).
iii) Under the skew-normal distribution (40) the pricing function (35) becomes:

$$P_C(\mu,\sigma) = \frac{\displaystyle\int_{-\infty}^{+\infty} C_1 \exp\left\{-\frac{(C_1-\mu)^2}{2\sigma^2} - \frac{(C_1-\widehat{\mu}_C)^2}{2\widehat{\sigma}_C^2}\right\}\Phi\left(\alpha\frac{C_1-\widehat{\mu}_C}{\widehat{\sigma}_C}\right)dC_1}{\displaystyle\int_{-\infty}^{+\infty} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2} - \frac{(C_1-\widehat{\mu}_C)^2}{2\widehat{\sigma}_C^2}\right\}\Phi\left(\alpha\frac{C_1-\widehat{\mu}_C}{\widehat{\sigma}_C}\right)dC_1}. \tag{A.64}$$

Denote by J_1 and J_0 the integrals in the numerator and the denominator, respectively. After some algebra and integration by parts, we obtain:

$$J_1 = \exp\left\{-\frac{1}{2}\frac{(\mu-\widehat{\mu}_C)^2}{\widehat{\sigma}_C^2+\sigma^2} - \frac{1}{2}\frac{(\widehat{\mu}-\mu)^2}{\widehat{\sigma}_C^2/\alpha^2+\widehat{\sigma}^2}\right\}\frac{\operatorname{sgn}(\alpha)\widehat{\sigma}^3}{\sqrt{\widehat{\sigma}_C^2/\alpha^2+\widehat{\sigma}^2}} + \frac{\widehat{\sigma}_C^2\mu+\sigma^2\widehat{\mu}_C}{\widehat{\sigma}_C^2+\sigma^2}J_0,$$

$$\begin{aligned} J_0 &= \exp\left\{-\frac{1}{2}\frac{(\mu-\widehat{\mu}_C)^2}{\widehat{\sigma}_C^2+\sigma^2}\right\}\int_{-\infty}^{\infty}\exp\left\{-\frac{(x-\widehat{\mu})^2}{2\widehat{\sigma}^2}\right\}\Phi\left(\alpha\frac{x-\widehat{\mu}_C}{\widehat{\sigma}_C}\right)dx \\ &= \sqrt{2\pi}\widehat{\sigma}\exp\left\{-\frac{1}{2}\frac{(\mu-\widehat{\mu}_C)^2}{\widehat{\sigma}_C^2+\sigma^2}\right\}\Phi\left(\frac{\operatorname{sgn}(\alpha)(\widehat{\mu}-\widehat{\mu}_C)}{\sqrt{\widehat{\sigma}_C^2/\alpha^2+\widehat{\sigma}^2}}\right), \end{aligned}$$

where $\widehat{\sigma} = \widehat{\sigma}_C^2\sigma^2/(\widehat{\sigma}_C^2+\sigma^2)$, $\widehat{\mu} = (\widehat{\sigma}_C^2\mu+\sigma^2\widehat{\mu}_C)/(\widehat{\sigma}_C^2+\sigma^2)$, and the computation of J_0 uses the following integral 8.259.1 in Gradshteyn and Ryzhik (2007): $\int_{-\infty}^{\infty}\exp(-px^2)\operatorname{erf}(a+bx)dx = (\sqrt{\pi/p})\operatorname{erf}(a\sqrt{p}/\sqrt{b^2+p})$, where the error function is given by $\operatorname{erf}(x) \equiv 2\Phi(x\sqrt{2})-1$. After some algebra, we obtain $P_C(\mu,\sigma) = J_1/J_0$ given by Equation (43). ■

Proof of Lemma 4. For the case of $C_1 \sim \mathcal{N}(\mu_C, \sigma_C^2)$ from Equation (41) with $L = 1$, we observe that $P_C(\mu,\sigma) = (\mu\sigma_C^2 + \mu_C\sigma^2)/(\sigma_C^2+\sigma^2)$. Consequently, Equation (36) for $\widehat{s}(s)$ becomes linear. Solving the latter equation, we obtain:

$$\widehat{s}(s) = \frac{\sigma_C^2+\sigma_0^2}{\sigma_{com}^2+\sigma_C^2}s + \frac{\mu_0+\mu_C}{\sigma_{com}^2+\sigma_C^2}\frac{\sigma_{com}^2-\sigma_0^2}{\gamma_I\sigma_0^2}, \tag{A.65}$$

where σ_{com} is given in (37). Substituting (A.65) into (34), after some algebra, we obtain:

$$P_{inc}(s) = \frac{(\gamma_I\sigma_0^2\widehat{s}(s)-\mu_0)\sigma_C^2+\mu_C\sigma_0^2}{\sigma_C^2+\sigma_0^2} = \frac{(\gamma_I\sigma_0^2 s-\mu_0)\sigma_C^2+\mu_C\sigma_{com}^2}{\sigma_{com}^2+\sigma_0^2} = P_{com}(s), \tag{A.66}$$

and hence, the two prices coincide. Substituting σ_{com} from (37) into the above equation for $P_{com}(s)$, after some algebra, we obtain Equation (44) for the price. ■

Internet Appendix

"Multi-Asset Noisy Rational Expectations Equilibrium with Contingent Claims"

Georgy Chabaukari Kathy Yuan Konstantinos E. Zachariadis

In Section IA.1 we examine a complete market economy where probabilities of states, and distributions of shock and noise are general. In Section IA.2 we show the existence of a risk-neutral measure when the spanning condition is satisfied. Then, we study the informed investor's demand when the informational spanning condition is violated. In Section IA.3, we consider models with multi-dimensional shock ε.

IA.1 Complete Markets with General Distributions

In this section, we extend the analysis of Section 3.1 to a complete-market economy with general probabilities of states $\pi_n(\varepsilon)$, distribution $\varphi_\varepsilon(x)$ of the aggregate shock, and distribution $\varphi_\nu(z)$ of noise trader demands. We find asset prices $P(\varepsilon,\nu)$ and investor I's portfolio $\theta^*_I(p;\varepsilon)$ in closed form in terms of simple integrals, and investor U's portfolio $\theta^*_U(p)$ as a solution of a fixed-point problem. First, we derive portfolio $\theta^*_I(p;\varepsilon)$ below.

Lemma IA.1 (Investor I's optimal portfolio). *Investor I's optimal portfolio is*

$$\theta^*_I(p;\varepsilon) = \frac{1}{\gamma_I}\Omega^{-1}\left(\widetilde{\xi}(\varepsilon) - \widetilde{v}(p)\right), \tag{IA.1}$$

where $\Omega \in \mathbb{R}^{(N-1)\times(N-1)}$ is a matrix of excess payoffs with elements $\Omega_{n,k} = C_k(\omega_n) - C_k(\omega_N)$, and $\widetilde{\xi}(\varepsilon) \in \mathbb{R}^{N-1}$ and $\widetilde{v}(p) \in \mathbb{R}^{N-1}$ are the vectors of log ratios of real and risk-neutral probabilities, respectively, given by:

$$\widetilde{\xi}(\varepsilon) = \left(\ln\left(\frac{\pi_1(\varepsilon)}{\pi_N(\varepsilon)}\right), \dots, \ln\left(\frac{\pi_{N-1}(\varepsilon)}{\pi_N(\varepsilon)}\right)\right)^\top, \tag{IA.2}$$

$$\widetilde{v}(p) = \left(\ln\left(\frac{\pi^{\mathrm{RN}}_1}{\pi^{\mathrm{RN}}_N}\right), \dots, \ln\left(\frac{\pi^{\mathrm{RN}}_{N-1}}{\pi^{\mathrm{RN}}_N}\right)\right)^\top. \tag{IA.3}$$

Portfolio $\theta^*_I(p;\varepsilon)$ is still available in closed form but is no longer a linear function of ε.[1]

[1] It is easy to show that in our complete market the distribution of asset payoffs conditional on ε is from the exponential family, that is, has the following form:

$$\mathrm{Prob}(C(\omega_n)|\varepsilon) = \exp\left\{C(\omega_n)^\top\Omega^{-1}\widetilde{\xi}(\varepsilon) - \ln\left[\sum_{i=1}^{N}\exp\left(C(\omega_i)^\top\Omega^{-1}\widetilde{\xi}(\varepsilon)\right)\right]\right\}, \text{ for } n = 1,\dots,N.$$

Nonetheless, it remains separable in shock ε and prices p. Substituting $\theta_I^*(p;\varepsilon)$ into the market clearing condition (6), we obtain:

$$\frac{\Omega^{-1}\widetilde{\xi}(\varepsilon)}{\gamma_I} + \nu + \widetilde{H}(p) = 0, \tag{IA.4}$$

where $\widetilde{H}(p) = \theta_U^*(p) - \Omega^{-1}\widetilde{v}(p)/\gamma_I$. As in the main text, we focus on equilibrium prices that reveal the sufficient statistic $\Omega^{-1}\widetilde{\xi}(\varepsilon)/\gamma_I + \nu$. The derivation of equilibrium proceeds in the same way as in Section 3. Proposition IA.1 reports the result.

Proposition IA.1 (Equilibrium with $M = N$, general case). *Let probabilities $\pi_n(\varepsilon)$ and PDFs $\varphi_\varepsilon(x)$ and $\varphi_\nu(z)$ of shock ε and noise ν be continuous and bounded functions on $\mathbb{R}$ and $\mathbb{R}^{M-1}$, respectively. Then, the following statements hold.*

i) *If there exists an REE, then the vector of asset prices $P(\varepsilon,\nu)$ and risk-neutral probabilities π_n^{RN} are given by:*

$$P(\varepsilon,\nu) = \Big[\pi_1^{\text{RN}}C(\omega_1) + \pi_2^{\text{RN}}C(\omega_2) + \ldots + \pi_N^{\text{RN}}C(\omega_N)\Big]e^{-rT}, \tag{IA.5}$$

$$\pi_n^{\text{RN}} = \frac{e^{v_n}}{\sum_{j=1}^N e^{v_j}}, \tag{IA.6}$$

where probability parameters v_n are given by:

$$v_n = \frac{\gamma_I}{\gamma_I+\gamma_U}\Psi_n\left(-\frac{\Omega^{-1}\widetilde{\xi}(\varepsilon)}{\gamma_I} - \nu\right) + \frac{\gamma_I\gamma_U}{\gamma_I+\gamma_U}C(\omega_n)^\top\left(\frac{\Omega^{-1}\widetilde{\xi}(\varepsilon)}{\gamma_I} + \nu\right), \tag{IA.7}$$

where function Ψ_n is given by the following integral:

$$\Psi_n(z) = \ln\left(\int_{-\infty}^{+\infty}\pi_n(x)\varphi_\varepsilon(x)\varphi_\nu\left(-\frac{\Omega^{-1}\widetilde{\xi}(x)}{\gamma_I} - z\right)dx\right), \tag{IA.8}$$

and vector $\widetilde{\xi}(\varepsilon)$ is given by equation (IA.2). *Investor I's portfolio is given by Equation* (IA.1) *and investor U's portfolio solves the following fixed point equation:*

$$\theta_U^*(p) = \frac{1}{\gamma_U}\Omega^{-1}\left(\widetilde{\Psi}\left(\theta_U^*(p) - \frac{\Omega^{-1}\widetilde{v}(p)}{\gamma_I}\right) - \widetilde{v}(p)\right), \tag{IA.9}$$

where $\widetilde{v}(p) \in \mathbb{R}^{N-1}$ has elements $v_n - v_N$, and $\widetilde{\Psi}(z) \in \mathbb{R}^{N-1}$ has elements $\Psi_n(z) - \Psi_N(z)$.

ii) *The REE in which investor U observes only asset prices exists if and only if $\widetilde{\Psi}(s)/\gamma_U - \Omega s : \mathbb{R}^{M-1} \to \mathbb{R}^{M-1}$ is an invertible function on its range, or equivalently, the price vector $P(\varepsilon,\nu)$ is an invertible function of the sufficient statistic $\Omega^{-1}\widetilde{\xi}(\varepsilon)/\gamma_I + \nu$ on its range.*

We are grateful to Bradyn Breon-Drish for pointing this out to us.

Proposition IA.1 provides tractable equilibrium prices (IA.5). Finding these prices does not involve solving any equations but may require numerical integration. Investor U's portfolio $\theta_U^*(p)$ solves a fixed-point equation. The equilibrium is derived in terms of function $\widetilde{\Psi}(z)$, which is the vector of log ratios of posterior probabilities of investor U, as shown in the proof of Proposition IA.1 below. When shock ε has PDF (2) and ν is normally distributed, $\widetilde{\Psi}(z)$ becomes a linear function of z. Then, Equation (IA.9) for $\theta_U^*(p)$ can be solved in closed form, and the equilibrium coincides with that in Proposition 1.

Proposition IA.1 provides a necessary and sufficient condition for the existence of equilibrium in which investor U observes only the asset prices. The condition requires function $\widetilde{\Psi}(s)/\gamma_U - \Omega s$ to be invertible or, equivalently, price $P(\varepsilon, \nu)$ to be an invertible function of the sufficient statistic on its range. When this condition is satisfied, investor U can infer the sufficient statistic from the prices and calculate posterior probabilities, accordingly. When function $P(\varepsilon, \nu)$ is not invertible, observing $P(\varepsilon, \nu)$ does not reveal the sufficient statistic. This last fact is not consistent with the existence of equilibrium because observing prices reveals the sufficient statistic from the market clearing condition (IA.4). Similar non-existence of equilibrium occurs in a model with one risky asset in Breon-Drish (2010) when the asset price is a non-monotone function of the sufficient statistic. The intuition is that the informed and uninformed investors may trade assets in opposite directions, which makes prices less informative than the sufficient statistic because informed demands are offset by uninformed demands. This may happen when noise trader demands lead to higher asset prices that investor U may confuse with good news about the realization of ε.

When prices $P(\varepsilon, \nu)$ do not reveal the sufficient statistic $\Omega^{-1}\widetilde{\xi}(\varepsilon)/\gamma_I + \nu$, it can still be inferred in economies where investor U additionally observes the residual demand of the informed investor and noise traders, $\widehat{\Theta} \equiv \theta_I^*(p; \varepsilon) + \nu$. In particular, using the informed demand (IA.1), we find that $\Omega^{-1}\widetilde{\xi}(\varepsilon)/\gamma_I + \nu = \widehat{\Theta} + \Omega^{-1}\widetilde{v}(p)/\gamma_I$. Hence, the residual demand $\widehat{\Theta}$ may reveal additional information that is not in prices. The information in $\widehat{\Theta}$ is redundant only when prices reveal the sufficient statistic. In that case, $\widehat{\Theta}$ can be inferred by substituting prices p into the market clearing condition $\widehat{\Theta} + \theta_U^*(p)=0$. Such inference is not possible when prices are less informative than $\widehat{\Theta}$ because investor U's portfolio $\theta_U^*(p; \widehat{\Theta})$ becomes a function of $\widehat{\Theta}$, as shown below, and cannot be found without knowing $\widehat{\Theta}$.

The assumption that investor U observes the residual demand $\widehat{\Theta}$ is similar to that in the literature on microstructure theory [e.g., Kyle (1985)] where an uninformed market maker observes $\widehat{\Theta}$. Allowing investor U to observe $\widehat{\Theta}$ restores the existence of equilibrium, as shown in Breon-Drish (2010) in economies with one risky asset. Proposition IA.2 below derives the equilibrium in a multi-asset economy where investor U can observe the residual demand $\widehat{\Theta}$ and demonstrates its existence.

Proposition IA.2 (Equilibrium when $\theta_I^*(p;\varepsilon)+\nu$ can be observed). *If investor U can learn about shock ε both from asset prices p and the residual demand of informed and noise traders $\widehat{\Theta}=\theta_I^*(p;\varepsilon)+\nu$, then there exists unique REE with asset prices and risk-neutral probabilities given by Equations* (IA.5) *and* (IA.6). *The informed investor's portfolio is given by Equation* (IA.1), *and the uninformed investor's portfolio is given by:*

$$\theta_U^*\left(p;\widehat{\Theta}\right)=\frac{1}{\gamma_U}\Omega^{-1}\left(\widetilde{\Psi}\left(-\widehat{\Theta}-\frac{\Omega^{-1}\widetilde{v}(p)}{\gamma_I}\right)-\widetilde{v}(p)\right), \tag{IA.10}$$

where $\widetilde{\Psi}(z)$ is a vector with elements $\Psi_n(z)-\Psi_N(z)$, and $\Psi_n(z)$ are given by (IA.8).

Next, we provide an example of an economy where asset prices are available in analytic form and are non-invertible functions of the sufficient statistic. Specifically, we consider an economy where probabilities $\pi_n(\varepsilon)$ and PDF of the aggregate shock $\varphi_\varepsilon(x)$ are given by Equations (1) and (2), respectively, as in the baseline analysis. The market is complete, and investors can trade an underlying asset with payoffs $C_1(\omega_n)=b_n/\lambda_1$, $N-2$ informationally irrelevant state contingent claims, and a bond. Similar to the economy in Section 4.3, noise traders only trade in the market for the underlying asset. However, in contrast to our baseline analysis in Section 3.1, the noise trader demand ν_1 in the market for the underlying asset is non-normal, and its PDF is a mixture of normals given by:

$$\varphi_\nu(x)=\frac{w}{\sqrt{2\pi}\sigma_\nu}\exp\left(-\frac{(x-\mu_{\nu,1})^2}{2\sigma_\nu^2}\right)+\frac{1-w}{\sqrt{2\pi}\sigma_\nu}\exp\left(-\frac{(x-\mu_{\nu,2})^2}{2\sigma_\nu^2}\right), \tag{IA.11}$$

where $0\le w\le 1$, $\sigma_\nu>0$, and $\mu_{\nu,1},\mu_{\nu,2}\in\mathbb{R}$. Proposition IA.3 presents the asset prices.

Proposition IA.3 (Asset Prices when Noise is a Mixture of Normals). *Let $\pi_n(\varepsilon)$, φ_ε, and φ_ν be given by Equations* (1), (2), *and* (IA.11), *respectively. Then the price of the asset with payoff $C_1(\omega)=b/\lambda_1$ in the above economy is a function of $s=\lambda_1\varepsilon/\gamma_I+\nu_1$ and is given by Equation* (IA.5), *where risk-neutral probabilities π_n^{RN} are given by:*

$$\pi_n^{\mathrm{RN}}=\frac{e^{\widehat{v}_n}\left(we^{\widehat{v}_{1,n}}+(1-w)e^{\widehat{v}_{2,n}}\right)^{\frac{\gamma_I}{\gamma_I+\gamma_U}}}{\sum_{j=1}^N e^{\widehat{v}_j}\left(we^{\widehat{v}_{1,j}}+(1-w)e^{\widehat{v}_{2,j}}\right)^{\frac{\gamma_I}{\gamma_I+\gamma_U}}}, \tag{IA.12}$$

and parameters $\widehat{v}_n$, $\widehat{v}_{1,n}$ and $\widehat{v}_{2,n}$ are as follows:

$$\widehat{v}_n=a_n+\frac{1}{2}\frac{\gamma_I}{\gamma_I+\gamma_U}\frac{b_n^2+2(\mu_0/\sigma_0^2)b_n}{\lambda_1^2/(\sigma_\nu^2\gamma_I^2)+1/\sigma_0^2}+\frac{\gamma_I\gamma_U}{\gamma_I+\gamma_U}\left(1+\frac{\lambda_1^2/(\gamma_U\gamma_I\sigma_\nu^2)}{\lambda_1^2/(\sigma_\nu^2\gamma_I^2)+1/\sigma_0^2}\right)\frac{b_n s}{\lambda_1}, \tag{IA.13}$$

$$\widehat{v}_{l,n}=-\frac{1}{\gamma_I}\frac{\mu_{\nu,l}}{2\sigma_0^2\sigma_\nu^2}\frac{\gamma_I\left(\mu_{\nu,l}-2s\right)+2\lambda_1(\mu_0+b_n\sigma_0^2)}{\lambda_1^2/(\sigma_\nu^2\gamma_I^2)+1/\sigma_0^2}, \quad l=1,2. \tag{IA.14}$$

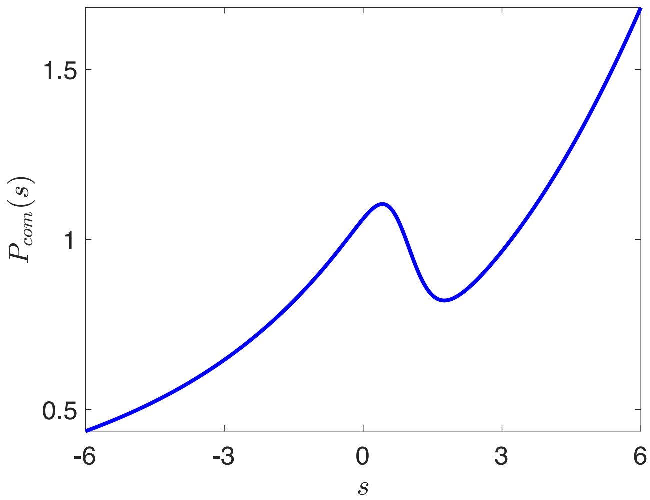

Figure IA.1: Price of the Underlying Asset when Noise is Mixture of Normals. This Figure shows the complete market price of the underlying asset $P_{com}(s)$ as a function of the sufficient statistic $s = \lambda_1 \varepsilon / \gamma_I + \nu_1$ in a model where distribution parameters a_n and b_n are given by (IA.15). Noise trader demand has mixture of normals PDF (IA.11) with weight $w = 0.5$. The other distribution and preferences parameters are: $\mu_0 = 0$, $\sigma_0 = 1$, $\sigma_\nu = 0.5$, $\widehat{\sigma}_C = 1$, $k = 3$, $\delta = 2$, $\mu_{\nu,1} = -\mu_{\nu,2} = 4$, $N = 1000$, $\gamma_I = \gamma_U = 1$, $r = 0$, and $T = 1$.

When $w = 0$ or $w = 1$ and $\mu_{\nu,1} = \mu_{\nu,2} = 0$, the model is a special case of the baseline model in Section 3.1, and the risk-neutral probability parameter $\widehat{v}_n$ in Equation (IA.13) coincides with parameter v_n in Equation (19) in the baseline model. We now look at a particular example in which the underlying asset has payoffs $C_1(\omega_n) = \underline{C}_N + (\overline{C}_N - \underline{C}_N)(n-1)/(N-1)$ and distribution parameters a_n and b_n are as follows:

$$a_n = -\frac{C_1(\omega_n)^2}{2\sigma_0^2} - \frac{\mu_0 C_1(\omega_n)}{\sigma_0^2} + \ln\left[\varphi_C(C_1(\omega_n))\right], \quad b_n = \frac{C_1(\omega_n)}{\sigma_0^2}, \tag{IA.15}$$

where $\varphi_C(x) = x^{k-1}\exp(-x^2/\widehat{\sigma}_C^2 - \delta x)/\Lambda$, $x > 0$, is a generalized gamma distribution, $\Lambda > 0$ is a normalizing constant, $k \geq 1$ is a power parameter, and δ is a drift parameter. As demonstrated in Lemma A.2 in the Appendix, in the limit $N \to \infty$, the economy with parameters (IA.15) converges to an economy where payoff C_1 has continuous PDF $\varphi_C(x)$, and ε is a signal given by $\varepsilon = C_1 + u$, where $u \sim N(\mu_0, \sigma_0^2)$. Figure IA.1 plots the price of the underlying asset $P_{com}(s)$ as a function of sufficient statistic s for specific model parameters given in the legend of the figure. The figure demonstrates that price is a non-monotone function of sufficient statistic s, and hence, the equilibrium only exists if the uninformed investor can observe the residual demand.

IA.1.1 Proofs

Proof of Lemma IA.1. Investor I's optimal portfolio in Equation (A.6) in the proof of Lemma 1 holds for general probabilities $\pi_n(\varepsilon)$. Hence, substituting $\widetilde{\xi}(\varepsilon)$ from Equation (IA.2) into Equation (A.6), we obtain Equation (IA.1). ■

Proof of Proposition IA.1. i) Suppose, the equilibrium exists. First, we find the posterior probabilities. Let $s = \Omega^{-1}\widetilde{\xi}(\varepsilon)/\gamma_I + \nu$. Similar to Equation (A.7), by Bayes rule,

$$\varphi_{\varepsilon|s}(x|y) = \frac{\varphi_\nu\Big(y - \Omega^{-1}\widetilde{\xi}(x)/\gamma_I\Big)\varphi_\varepsilon(x)}{\int_{-\infty}^{+\infty}\varphi_\nu\Big(y - \Omega^{-1}\widetilde{\xi}(x)/\gamma_I\Big)\varphi_\varepsilon(x)dx}. \tag{IA.16}$$

From the market clearing condition (IA.4) we note that in equilibrium $s = -\widetilde{H}(p)$. Then, similar to the proof of Lemma 2, investor U's posterior probabilities $\pi_n^U(p)$ are given by:

$$\begin{aligned}\pi_n^U(p) &= \int_{-\infty}^{+\infty}\pi_n(x)\varphi_{\varepsilon|s}\Big(x| - \widetilde{H}(p)\Big)dx \\ &= \frac{1}{G_1(p)}\int_{-\infty}^{+\infty}\pi_n(x)\varphi_\varepsilon(x)\varphi_\nu\left(-\frac{\Omega^{-1}\widetilde{\xi}(x)}{\gamma_I} - \widetilde{H}(p)\right)dx = \frac{\exp\Big\{\Psi_n\Big(\widetilde{H}(p)\Big)\Big\}}{G_1(p)},\end{aligned} \tag{IA.17}$$

where $G_1(p)$ is a normalizing function and $\Psi_n(\cdot)$ is given by (IA.8). The integrals in Equations (IA.16) and (IA.17) exist because $\varphi_\nu(\cdot)$ and $\varphi_\varepsilon(\cdot)$ are bounded and continuous.

Next, we derive investor U's portfolio. Equation (A.11) in the proof of Proposition 1 derives this portfolio for general posterior and risk-neutral probabilities. From Equation (IA.17), we obtain that $\ln(\pi_n^U/\pi_N^U) = \Psi_n\Big(\widetilde{H}(p)\Big) - \Psi_N\Big(\widetilde{H}(p)\Big) = \tilde{\Psi}_n\Big(\widetilde{H}(p)\Big)$. Substituting this expression in Equation (A.11), we have

$$\theta_U^*(p) = \frac{1}{\gamma_U}\Omega^{-1}\left(\tilde{\Psi}\Big(\widetilde{H}(p)\Big) - \tilde{v}(p)\right). \tag{IA.18}$$

Substituting $\widetilde{H}(p) = \theta_U^*(p) - \Omega^{-1}v(p)/\gamma_I$ into Equation (IA.18), we obtain Equation (IA.9).

Then, we find vector $\tilde{v}(p)$. Subtracting $\Omega^{-1}\tilde{v}(p)/\gamma_I$ from both sides of (IA.18), multiplying both sides by Ω, using $\widetilde{H}(p) = -\Omega^{-1}\widetilde{\xi}(\varepsilon)/\gamma_I - \nu$, and rearranging terms we obtain:

$$\frac{1}{\gamma_U}\tilde{\Psi}\left(-\frac{\Omega^{-1}\widetilde{\xi}(\varepsilon)}{\gamma_I} - \nu\right) + \Omega\left(\frac{\Omega^{-1}\widetilde{\xi}(\varepsilon)}{\gamma_I} + \nu\right) = \frac{\gamma_I + \gamma_U}{\gamma_I\gamma_U}\tilde{v}(p). \tag{IA.19}$$

After some algebra, we obtain components of vector $\tilde{v}(p) \in \mathbb{R}^{\mathbb{N}-1}$ from Equation (IA.19):

$$\begin{aligned}\tilde{v}_n = \frac{\gamma_I\gamma_U}{\gamma_I + \gamma_U}\Bigg(&\frac{1}{\gamma_U}\Psi_n\Big(-\frac{\Omega^{-1}\widetilde{\xi}(\varepsilon)}{\gamma_I} - \nu\Big) - \frac{1}{\gamma_U}\Psi_N\Big(-\frac{\Omega^{-1}\widetilde{\xi}(\varepsilon)}{\gamma_I} - \nu\Big) \\ &+ (C(\omega_n) - C(\omega_N))^\top\left(\frac{\Omega^{-1}\widetilde{\xi}(\varepsilon)}{\gamma_I} + \nu\right)\Bigg).\end{aligned}$$

where $n = 1, \ldots, N-1$. From the fact that $\tilde{v}_n = \ln(\pi_n^{\text{RN}}/\pi_N^{\text{RN}})$ we obtain:

$$\pi_n^{\text{RN}} = \begin{cases} \dfrac{e^{\widetilde{v}_n}}{1 + \sum_{i=1}^{N-1} e^{\widetilde{v}_i}}, & n = 1, \ldots, N-1, \\ \dfrac{1}{1 + \sum_{i=1}^{N-1} e^{\widetilde{v}_i}}, & n = N. \end{cases} \tag{IA.20}$$

It can be easily verified that $\widetilde{v}_n = v_n - v_N$, where v_n is defined in Equation (IA.7). Substituting $\widetilde{v}_n = v_n - v_N$ into Equation (IA.20), after simple algebra, we obtain risk-neutral probabilities (IA.6). The prices are then given by Equation (IA.5) because of the absence of arbitrage. This completes the derivation of equilibrium when it exists.

ii) Suppose, there exists an REE. First, we show that $\widetilde{\Psi}(s)/\gamma_U - \Omega s$ is invertible on its range. Suppose, it is not. Then, there exist s' and s'' such that $\widetilde{\Psi}(s')/\gamma_U - \Omega s' = \widetilde{\Psi}(s'')/\gamma_U - \Omega s''$ but $s' \neq s''$. Pick (ε', ν') and (ε'', ν'') such that $s' = \Omega^{-1}\widetilde{\xi}(\varepsilon')/\gamma_I + \nu'$ and $s'' = \Omega^{-1}\widetilde{\xi}(\varepsilon'')/\gamma_I + \nu''$. From Equation (IA.19) we observe that vector $\tilde{v}(p)$, and, hence, also the risk-neutral probabilities (IA.20), are the same for (ε', ν') and (ε'', ν''). Then, from Equation (IA.5) for $P(\varepsilon, \nu)$, we find that $P(\varepsilon', \nu') = P(\varepsilon'', \nu'')$. Hence, $P(\varepsilon, \nu)$ is not injective. In Lemma IA.2 below, we prove that the price vector is injective if an REE exists, which leads to a contradiction. Therefore, $\widetilde{\Psi}(s)/\gamma_U - \Omega s$ is invertible on its range.

Let $\widetilde{\Psi}(s)/\gamma_U - \Omega s$ be invertible on its range. Then, we show that the fixed-point equation (IA.9) has a unique solution. Subtracting $\Omega^{-1}\widetilde{v}(p)/\gamma_I$ from both sides of Equation (IA.9) and then multiplying both sides by Ω, after some algebra, we find that $\theta_U^*(p)$ satisfies

$$\frac{1}{\gamma_U}\widetilde{\Psi}\left(\theta_U^*(p) - \frac{\Omega^{-1}\widetilde{v}(p)}{\gamma_I}\right) - \Omega\left(\theta_U^*(p) - \frac{\Omega^{-1}\widetilde{v}(p)}{\gamma_I}\right) = \frac{\gamma_I + \gamma_U}{\gamma_I\gamma_U}\widetilde{v}(p). \tag{IA.21}$$

Equation (IA.21) has a unique solution $\theta_U^*(p)$, which is an implicit function of $\widetilde{v}(p)$, because $\widetilde{\Psi}(s)/\gamma_U - \Omega s$ is invertible on its range and because the right-hand side of (IA.21) belongs to this range when v_n is given by Equation (IA.7). By construction, $\theta_I^*(p;\varepsilon)$ and $\theta_U^*(p)$ are optimal. Then, it can be easily verified that the market clearing condition (IA.4) is also satisfied when the risk-neutral probabilities are given by Equation (IA.6). Hence, the equilibrium exists.

Finally, we show that $\widetilde{\Psi}(s)/\gamma_U - \Omega s$ is invertible on its range if and only if the price vector $P(\varepsilon, \nu)$ is an invertible function of the sufficient statistic on its range. Consider equation $P(\varepsilon, \nu) = p$. Next, we find unique corresponding risk-neutral probabilities π_n^{RN} and unique vector $\widetilde{v}(p) = (\ln(\pi_1^{\text{RN}}/\pi_N^{\text{RN}}), \ldots, \ln(\pi_{N-1}^{\text{RN}}/\pi_N^{\text{RN}}))^\top$ from (IA.5) and (IA.6). Then, we find unique sufficient statistic by solving Equation (IA.19), which has unique solution because $\widetilde{\Psi}(s)/\gamma_U - \Omega s$ is invertible. Hence, $P(\varepsilon, \nu)$ is an invertible function of the sufficient statistic. ■

Lemma IA.2 *If there exists an REE then the vector of risky asset prices $P(\varepsilon,\nu)$ is an injective function of the sufficient statistic $\Omega^{-1}\widetilde{\xi}(\varepsilon)/\gamma_I+\nu$.*

Proof of Lemma IA.2. First, we observe that if $P(\varepsilon',\nu')=P(\varepsilon'',\nu'')$ then $\widetilde{v}(P(\varepsilon',\nu'))=\widetilde{v}(P(\varepsilon'',\nu''))$, where $\widetilde{v}(p)=\Big(\ln(\pi_n^{\text{RN}}/\pi_N^{\text{RN}}),\ldots,\ln(\pi_{N-1}^{\text{RN}}/\pi_N^{\text{RN}})\Big)^\top$. This is because the risk-neutral probabilities are unique solutions of the risk-neutral pricing equations (8) due to the absence of arbitrage, and they uniquely determine vector $\widetilde{v}(p)$.

The rest is similar to the proof of Lemma B.1 in Breon-Drish (2010). Suppose, $P(\varepsilon,\nu)$ is not injective. Then, there exist (ε',ν') and (ε'',ν'') such that $P(\varepsilon',\nu')=P(\varepsilon'',\nu'')$ but $\Omega^{-1}\widetilde{\xi}(\varepsilon')/\gamma_I+\nu'\neq\Omega^{-1}\widetilde{\xi}(\varepsilon'')/\gamma_I+\nu''$. We denote the latter and the former statistics by s'' and s', respectively. Then, from the market clearing condition (IA.4), we obtain:

$$
\begin{aligned}
0&=s'+\theta_U^*(P(\varepsilon',\nu'))-\frac{\Omega^{-1}\widetilde{v}(P(\varepsilon',\nu'))}{\gamma_I}=s'+\theta_U^*(P(\varepsilon'',\nu''))-\frac{\Omega^{-1}\widetilde{v}(P(\varepsilon'',\nu''))}{\gamma_I}\\
&=s'-s''+s''+\theta_U^*(P(\varepsilon'',\nu''))-\frac{\Omega^{-1}\widetilde{v}(P(\varepsilon'',\nu''))}{\gamma_I}=s'-s''.
\end{aligned}
$$

Hence, $s'=s''$, which leads to a contradiction. ■

Proof of Proposition IA.2. Investor U observes prices p and the combined demand $\widehat{\Theta}=\theta_I^*(p;\varepsilon)+\nu$. Investor U then infers the sufficient statistic from the Equation $\Omega^{-1}\widetilde{\xi}(\varepsilon)/\gamma_I+\nu=\widehat{\Theta}+\Omega^{-1}\widetilde{v}(p)/\gamma_I$, which can be easily proven by substituting in $\widehat{\Theta}$ and $\theta_I^*(p;\varepsilon)$ from Equation (IA.1). Next, following similar steps as in the proof of Proposition IA.1 we find that the posterior probabilities are given by $\pi_n^U(p;\widehat{\Theta})=\exp\{\Psi_n(-\widehat{\Theta}-\Omega^{-1}\widetilde{v}(p)/\gamma_I)\}/G_1(p)$, where $\Psi_n(z)$ is given by Equation (IA.8) and $G_1(p)$ is a normalizing function. The rest of the proof follows the proof of part (i) of Proposition IA.1. The prices are not required to be invertible functions of the sufficient statistic because investor U learns the sufficient statistic from the additional variable $\widehat{\Theta}$. ■

Proof of Proposition IA.3. The sufficient statistic is given by $\Omega^{-1}\tilde{\xi}(\varepsilon)/\gamma_I+\nu$, where $\tilde{\xi}(\varepsilon)=\tilde{a}+\tilde{b}\varepsilon$, and $\tilde{a}$ and $\tilde{b}$ are vectors with elements a_n-a_N and b_n-b_N, respectively. Because noise traders only trade in the market for the underlying asset, $\nu=(\nu_1,0,\ldots,0)^\top$. Substituting $\tilde{\xi}(\varepsilon)$ and ν into the sufficient statistic we obtain: $\Omega^{-1}\tilde{\xi}(\varepsilon)/\gamma_I+\nu=se_1+\Omega^{-1}\tilde{a}/\gamma_I$, where $s=\lambda_1\varepsilon/\gamma_I+\nu_1$ is a one-dimensional statistic and $e_1=(1,0,\ldots,0)^\top$. Therefore, only the first component of vector $\Omega^{-1}\tilde{\xi}(\varepsilon)/\gamma_I+\nu$ is random, and hence, the sufficient statistic is effectively one-dimensional and given by $s+c_1$, where c_1 is the first component of vector $\Omega^{-1}\tilde{a}/\gamma_I$. Hence, we use this one-dimensional statistic in the proof.

First, we compute function $\Psi_n(z)$ in Equation (IA.8). Substituting $\pi_n(\varepsilon)$ from (1),

$\varphi_\varepsilon(x)$ from (2), and $\varphi_\nu(x)$ from (IA.11), we find that the expression under the integral in Equation (IA.8) is given by:

$$\widehat{\varphi}(x) = \frac{\exp(a_n + b_n x)}{G} \exp\left(-\frac{(x-\mu_0)^2}{2\sigma_0^2}\right)\left(w\exp\left(-\frac{\left(c_1 + z + \lambda_1 x/\gamma_I + \mu_{\nu,1}\right)^2}{2\sigma_\nu^2}\right)\right.$$
$$\left. + (1-w)\exp\left(-\frac{\left(c_1 + z + \lambda_1 x/\gamma_I + \mu_{\nu,2}\right)^2}{2\sigma_\nu^2}\right)\right),$$

where G is a constant. Integrating $\widehat{\varphi}(x)$ and taking log, we obtain $\Psi_n(z)$ as follows:

$$\begin{aligned}
\Psi_n(z) &= \ln\left(\int_{-\infty}^{+\infty}\widehat{\varphi}(x)dx\right)\\
&= g + a_n + \frac{1}{2}\frac{b_n^2 + 2b_n\left(\mu_0/\sigma_0^2 - (c_1+z)\lambda_1/(\sigma_\nu^2\gamma_I)\right)}{\lambda_1^2/(\sigma_\nu^2\gamma_I^2) + 1/\sigma_0^2}\\
&\quad + \ln\left\{w\exp\left(-\frac{\mu_{\nu,1}}{2\sigma_0^2\sigma_\nu^2\gamma_I}\frac{\gamma_I\left(\mu_{\nu,1} + 2(c_1+z)\right) + 2\lambda_1(\mu_0 + b_n\sigma_0^2)}{\lambda_1^2/(\sigma_\nu^2\gamma_I^2) + 1/\sigma_0^2}\right)\right.\\
&\quad \left. + (1-w)\exp\left(-\frac{\mu_{\nu,2}}{2\sigma_0^2\sigma_\nu^2\gamma_I}\frac{\gamma_I\left(\mu_{\nu,2} + 2(c_1+z)\right) + 2\lambda_1(\mu_0 + b_n\sigma_0^2)}{\lambda_1^2/(\sigma_\nu^2\gamma_I^2) + 1/\sigma_0^2}\right)\right\},
\end{aligned}$$

where g is a constant. Then, the parameters v_n in Equation (IA.6) are given by:

$$\begin{aligned}
v_n &= \frac{\gamma_I}{\gamma_I + \gamma_U}\Psi_n(-c_1 - s) + \frac{\gamma_U}{\gamma_I + \gamma_U}a_n + \frac{\gamma_I\gamma_U}{\gamma_I + \gamma_U}\frac{b_n s}{\lambda_1}\\
&= \widehat{g} + \widehat{v}_n + \frac{\gamma_I}{\gamma_I + \gamma_U}\ln\left(we^{\widehat{v}_{1,n}} + (1-w)e^{\widehat{v}_{2,n}}\right),
\end{aligned}$$

where $\widehat{v}_n$ and $\widehat{v}_{l,n}$ are given by Equations (IA.13) and (IA.14), respectively, and $\widehat{g}$ is a constant. Substituting v_n in Equation (IA.6) for the risk-neutral probabilities, we note that constant $\widehat{g}$ cancels out and we obtain risk-neutral probabilities (IA.12). ■

IA.2 Portfolio Choice in Incomplete Markets

Lemma IA.3 (Risk-neutral Probabilities). *There exist risk-neutral probabilities* $\pi^{\text{RN}}_{I,n}$ *in incomplete markets such that the optimal wealth of investor* I *satisfies the FOC:*

$$\gamma_I e^{-\gamma_I W_{I,T,n}} = \ell_I \frac{\pi^{\text{RN}}_{I,n} e^{-rT}}{\pi_n(\varepsilon)}. \tag{IA.22}$$

Moreover, the risk-neutral probabilities $\pi^{\text{RN}}_{I,n}$ *do not depend on* ε *if the informational spanning condition* (26) *is satisfied.*

Proof of Lemma IA.3. Due to market incompleteness, there exists a continuum of state price densities ζ such that $\mathbb{E}[\zeta_T C] = \zeta_0 p$. In incomplete markets the optimal wealth $W_{I,T}$ can be identified from the following static optimization problem:

$$\min_{\zeta_T} \max_{W_{I,T}} \mathbb{E}\left[-e^{-\gamma_I W_{I,T}}\right], \tag{IA.23}$$

subject to the following budget and asset pricing constraints:

$$\mathbb{E}[\zeta_T W_{I,T}] = W_{T,0}, \tag{IA.24}$$

$$\mathbb{E}[\zeta_T C] = p, \quad \mathbb{E}[\zeta_T] = e^{-rT}, \tag{IA.25}$$

$$\zeta_T \geq 0. \tag{IA.26}$$

First, we solve the maximization in (IA.23) subject to a static budget constraint (IA.24). After simple algebra, we find that wealth $W_{I,T}$ satisfies the FOC $\gamma_I e^{-\gamma_I, W_{I,T}} = \ell_I \zeta^*_T$, where ℓ_I is a Lagrange multiplier for (IA.24) and ζ^*_T solves the optimization problem

$$\min_{\zeta_T} \mathbb{E}[\zeta_T \ln(\zeta_T)], \tag{IA.27}$$

subject to constraints (IA.25). Recalling that SPD is given by $\zeta_T = \pi^{\text{RN}}_{I,n} e^{-rT}/\pi_n(\varepsilon)$, we obtain the FOC (IA.22). Solving problem (IA.27), we find that $\zeta^*_T = \exp(\ell_0 + C^\top \ell)$, where ℓ and ℓ_0 are the Lagrange multipliers for constraints (IA.25), respectively. The risk-neutral probabilities are then given by $\pi^{\text{RN}}_{I,n} = \pi_n(\varepsilon) e^{rT} \zeta^*_T$. Substituting $\pi_n(\varepsilon)$ and ζ^*_T into the latter equation, eliminating constant $e^{rT+\ell_0}$ and simplifying, we obtain:

$$\pi^{\text{RN}}_{I,n} = \frac{e^{a_n + b_n \varepsilon + C(\omega_n)^\top \ell}}{\sum_{i=1}^N e^{a_i + b_i \varepsilon + C(\omega_i)^\top \ell}}. \tag{IA.28}$$

If the spanning condition (26) is satisfied, substituting $b_n = \lambda_0 + C(\omega_n)^\top \lambda$ into Equation (IA.28), we obtain:

$$\pi^{\text{RN}}_{I,n} = \frac{e^{a_n + C(\omega_n)^\top \hat{\ell}}}{\sum_{i=1}^N e^{a_i + C(\omega_i)^\top \hat{\ell}}}, \tag{IA.29}$$

where $\hat{\ell} = \ell + \lambda\varepsilon$. The multipliers $\hat{\ell}$ are found from the condition $\mathbb{E}^{\text{RN}}[C]e^{-rT} = p$. Because the latter equation does not contain ε, multipliers $\hat{\ell}$ and probabilities (IA.29) also do not depend on ε. The existence of vector $\hat{\ell}$ follows from the invertibility of function $f_I(x)$ in Equation (29) demonstrated in Proposition 3.

Lemma IA.4 (Portfolio Choice and the Violation of Informational Spanning). *Suppose, Condition* (26) *is violated, and shock sensitivities, asset payoffs and prices are sufficiently small. Then, investor I's optimal portfolio is approximately given by:*

$$\theta_I^*(p,\varepsilon) = \frac{\lambda\varepsilon}{\gamma_I} - \widehat{\theta}_I^*(p) + o(|b| + |C|), \tag{IA.30}$$

where $\widehat{\theta}_I^*(p)$ *is a function of* p *and* $\lambda = \text{var}^a[C]^{-1}\,\text{cov}^a(b, C)$, *where* $\text{var}^a[\cdot]$ *and* $\text{cov}^a(\cdot,\cdot)$ *are under probability measure* $\pi_n^a = e^{a_n}/\sum_{n=1}^N e^{a_n}$.[2] *Moreover, vector* b *can be decomposed as:*

$$b = \lambda_0 + \lambda_1 C_1 + \ldots + \lambda_{M-1}C_{M-1} + \widehat{b}, \tag{IA.31}$$

where $\widehat{b}$ *is such that* $\mathbb{E}^a[\widehat{b}] = 0$ *and* $\text{cov}^a(\widehat{b}, C_m) = 0$ *for all* m, *and hence,* $\lambda_0\mathbb{I}_N + \lambda_1 C_1 + \ldots\lambda_{M-1}C_{M-1}$ *is a projection of vector* b *on the linear sub-space spanned by asset payoffs.*

Proof of Lemma IA.4. It can be easily observed that the objective function of investor I in (3) is equivalent to the following objective function under probabilities π_n^a:

$$-\sum_{n=1}^{N}\pi_n^a \exp\Big(b_n\varepsilon - \gamma_I\theta_I^\top(C(\omega_n) - pe^{rT})\Big) = -\mathbb{E}^a\Big[\exp\Big(b\varepsilon - \gamma_I\theta_I^\top(C - pe^{rT})\Big)\Big]. \tag{IA.32}$$

The existence of decomposition (IA.31) follows from the similarities between covariances and inner products, and the properties of projection operators. It can be also verified by direct calculations that $\text{cov}^a(\widehat{b}, C_m) = 0$, where $\widehat{b} \equiv b - \lambda_0 - \lambda_1 C_1 - \ldots - \lambda_{M-1}C_{M-1}$ and $\lambda = \text{var}^a[C]^{-1}\,\text{cov}^a(b, C)$. Equation (IA.31) implies that $b_n = \lambda_0 + \lambda^\top C(\omega_n) + \widehat{b}_n$. Substituting the latter equation into the objective (IA.32), after some algebra, we obtain the objective function: $-\mathbb{E}^a[\exp\big(\widehat{b}\varepsilon + \gamma_I\widehat{\theta}_I^\top(C - e^{rT}p)\big)]$, where $\widehat{\theta}_I \equiv \lambda\varepsilon/\gamma_I - \theta_I$. Approximating the expected utility by a quadratic function, we obtain:

$$\begin{aligned}&\mathbb{E}^a[\exp\big(\widehat{b}\varepsilon + \gamma_I\widehat{\theta}_I^\top(C - e^{rT}p)\big)] = \gamma_I\widehat{\theta}_I^\top\mathbb{E}^a[C - e^{rT}p]\\&+\frac{1}{2}\Big\{\gamma_I^2\widehat{\theta}_I^\top\mathbb{E}^a[(C - pe^{rT})(C - pe^{rT})^\top]\widehat{\theta}_I + 2\gamma_I\widehat{b}\,\widehat{\theta}_I^\top\,\text{cov}^a(C - pe^{rT}, \widehat{b})\Big\} + o(|b| + |C|).\end{aligned} \tag{IA.33}$$

We note, that $\text{cov}^a(C - pe^{rT}, \widehat{b}) = 0$ by the definition of $\widehat{b}$. Hence, $\widehat{\theta}_I$ solves

$$\min_{\widehat{\theta}_I}\Big\{\gamma_I\widehat{\theta}_I\mathbb{E}^a[C - e^{rT}p] + \frac{1}{2}\gamma_I^2\widehat{\theta}_I^\top\mathbb{E}^a[(C - pe^{rT})(C - pe^{rT})^\top]\widehat{\theta} + o(|b| + |C|)\Big\}. \tag{IA.34}$$

[2]By $o(|x|)$, we denote the terms that converge to zero faster than $|x|$ when $x \to 0$.

The first-order terms in the optimization (IA.34) only depend on price p. Hence, from the equation $\widehat{\theta}_I^*(p) \equiv \lambda\varepsilon/\gamma_I - \theta_I^*(p)$, we obtain Equation (IA.30).

IA.3 Models with Multi-dimensional Signals

Lemma IA.5 (Two Signals, Two Assets). *Consider an economy with two assets with payoffs C_1, C_2 that have general unconditional continuous joint PDF $\varphi_{C_1,C_2}(x_1,x_2)$. Suppose, the informed investor receives signals $\varepsilon_1 = C_1 + u_1$, $\varepsilon_2 = C_2 + u_2$, where $u_i \sim \mathcal{N}(\mu_{0,i}, \sigma_{0,i}^2)$, for $i = 1, 2$, and u_1, u_2 are independent. Then, the latter economy is a limiting case (when $N \to \infty$) of a N–state economy with state probabilities $\pi_n(\varepsilon_1, \varepsilon_2)$ and prior PDF $\varphi_{\varepsilon_1,\varepsilon_2}(x_1,x_2)$ over $\varepsilon_1, \varepsilon_2$:*

$$\pi_n(\varepsilon_1,\varepsilon_2) = \frac{\exp\left(a_n + b_{n,1}\varepsilon_1 + b_{n,2}\varepsilon_2\right)}{\sum_{j=1}^N \exp\left(a_j + b_{j,1}\varepsilon_1 + b_{j,2}\varepsilon_2\right)}, \tag{IA.35}$$

$$\varphi_{\varepsilon_1,\varepsilon_2}(x_1,x_2) = \frac{\left(\sum_{j=1}^N e^{a_j + b_{j,1}x_1 + b_{j,2}x_2}\right) e^{-0.5(x_1-\mu_{0,1})^2/\sigma_{0,1}^2 - 0.5(x_2-\mu_{0,2})^2/\sigma_{0,2}^2}}{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\sum_{j=1}^N e^{a_j + b_{j,1}x_1 + b_{j,2}x_2}\right) e^{-0.5(x_1-\mu_{0,1})^2/\sigma_{0,1}^2 - 0.5(x_2-\mu_{0,2})^2/\sigma_{0,2}^2} dx_1 dx_2}, \tag{IA.36}$$

where the distribution parameters a_n, $b_{1,n}$, and $b_{2,n}$ are given by:

$$\begin{aligned} a_n &= -\frac{C_1(\omega_n)^2}{2\sigma_{0,1}^2} - \frac{\mu_{0,1}C_1(\omega_n)}{\sigma_{0,1}^2} - \frac{C_2(\omega_n)^2}{2\sigma_{0,2}^2} - \frac{\mu_{0,2}C_2(\omega_n)}{\sigma_{0,2}^2} + \ln\left[\varphi_{C_1,C_2}\left(C_1(\omega_n), C_2(\omega_n)\right)\right], \\ b_{n,1} &= \frac{C_1(\omega_n)}{\sigma_{0,1}^2}, \quad b_{n,2} = \frac{C_2(\omega_n)}{\sigma_{0,2}^2}, \end{aligned} \tag{IA.37}$$

and $(C_1(\omega_n), C_2(\omega_n))$, for $n = 1, \ldots, N$, is a sample of N points in a rectangular grid on the range of random vector (C_1, C_2).

Proof of Lemma IA.5. Consider a discretized economy with two risky assets with payoffs $(C_1(\omega_n), C_2(\omega_n))$, for $n = 1, \ldots, N$, corresponding to points in a rectangular grid on the range of (C_1, C_2). The informed investor receives signals $\varepsilon_i = C_i(\omega) + u_i$, where $u_i \sim \mathcal{N}(\mu_{0,i}, \sigma_{0,i}^2)$, for $i = 1, 2$, and u_1, u_2 are independent. The unconditional probabilities of $(C_1(\omega_n), C_2(\omega_n))$ are given by

$$\text{Prob}(C_1(\omega_n), C_2(\omega_n)) = \varphi_{C_1,C_2}\left(C_1(\omega_n), C_2(\omega_n)\right) / \left(\sum_{n=1}^N \varphi_{C_1,C_2}\left(C_1(\omega_n), C_2(\omega_n)\right)\right). \tag{IA.38}$$

The original continuous-space economy is a limiting case of the latter economy because as $N \to \infty$, the distributions $\text{Prob}(C_1(\omega_n) \le x_1, C_2(\omega_n) \le x_2)$ and $\text{Prob}(C_1(\omega_n) \le x_1, C_2(\omega_n) \le x_2 | \varepsilon_1, \varepsilon_2)$ converge pointwise to the respective continuous-space distributions.

We show that the latter discretized economy is a special case of ours when parameters a_n, $b_{n,1}$, and $b_{n,2}$ are given by Equations (IA.37) by verifying that in our economy Equation (IA.38) holds and $\varepsilon_i = C_i(\omega) + u_i$, for $i = 1, 2$.

Consider the unconditional probability $\text{Prob}\left(C_1(\omega_n), C_2(\omega_n)\right)$, when $\pi_n(\varepsilon_1, \varepsilon_2)$ is given by Equation (IA.35) and $\varphi_{\varepsilon_1,\varepsilon_2}(x_1, x_2)$ is given by Equation (IA.36):

$$\begin{aligned}
&\text{Prob}\left(C_1(\omega_n), C_2(\omega_n)\right) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \pi_n(x_1, x_2)\varphi_{\varepsilon_1,\varepsilon_2}(x_1, x_2)dx_1dx_2 \\
&= \frac{1}{\Lambda}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\left(a_n + b_{n,1}x_1 - \frac{(x_1-\mu_{0,1})^2}{2\sigma_{0,1}^2} + b_{n,2}x_2 - \frac{(x_2-\mu_{0,2})^2}{2\sigma_{0,2}^2}\right)dx_1dx_2 \\
&= \frac{1}{\widetilde{\Lambda}} \exp\left(a_n + \frac{(\mu_{0,1}+b_{n,1}\sigma_{0,1}^2)^2}{2\sigma_{0,1}^2} + \frac{(\mu_{0,2}+b_{n,2}\sigma_{0,2}^2)^2}{2\sigma_{0,2}^2}\right),
\end{aligned}$$

where Λ and $\widetilde{\Lambda}$ are constants. Substituting a_n, $b_{n,1}$, and $b_{n,2}$ from Equations (IA.37) into the above equation, after some algebra, we verify Equation (IA.38).

Next, we verify that ε_i is given by $\varepsilon_i = C_i(\omega) + u_i$, where $u_i \sim \mathcal{N}(\mu_{0,i}, \sigma_{0,i}^2)$, for $i = 1, 2$, and u_1, u_2 are independent. Substituting a_n, $b_{n,1}$ and $b_{n,2}$ from Equation (IA.37) into Equation (IA.36) for PDF $\varphi_{\varepsilon_1,\varepsilon_2}(x_1, x_2)$, after some algebra, we obtain:

$$\varphi_{\varepsilon_1,\varepsilon_2}(x_1, x_2) = \frac{1}{G}\sum_{n=1}^{N} e^{-0.5(x_1-C_1(\omega_n)-\mu_{0,1})^2/\sigma_{0,1}^2 - 0.5(x_2-C_2(\omega_n)-\mu_{0,2})^2/\sigma_{0,2}^2}\varphi_{C_1,C_2}\left(C_1(\omega_n), C_2(\omega_n)\right),$$

where G is a normalizing constant. In the above PDF the terms inside the sum can be written as $\varphi_{\varepsilon_1|C_1}\left(x_1|C_1(\omega_n)\right)\varphi_{\varepsilon_2|C_2}\left(x_2|C_2(\omega_n)\right)\varphi_{C_1,C_2}\left(C_1(\omega_n), C_2(\omega_n)\right)$, where $\varepsilon_i = C_i(\omega) + u_i$, $u_i \sim \mathcal{N}(\mu_{0,i}, \sigma_{0,i}^2)$, $i = 1, 2$, and u_1, u_2 are independent. ■

Lemma IA.6 (Two Signals, Single Asset). *Consider an economy with one asset with payoff C_1 and general PDF $\varphi_C(x)$. The informed investor receives signals $\varepsilon_1 = C_1 + u_1$, $\varepsilon_2 = C_1 + u_2$, where $u_i \sim \mathcal{N}(\mu_{0,i}, \sigma_{0,i}^2)$, for $i = 1, 2$, and u_1, u_2 are independent. Then, the latter economy is a limiting case (when $N \to \infty$) of a N-state economy with state probabilities $\pi_n(\varepsilon)$ and prior PDF $\varphi_\varepsilon(x)$ given by Equations* (1) *and* (2)*, where*

$$a_n = -\frac{C_1(\omega_n)^2}{2\sigma_0^2} - \frac{\mu_0 C_1(\omega_n)}{\sigma_0^2} + \ln\left[\varphi_C\left(C_1(\omega_n)\right)\right], \quad b_n = \frac{C_1(\omega_n)}{\sigma_0^2}, \tag{IA.39}$$

and $\varepsilon = \left(\sigma_{0,2}^2\varepsilon_1 + \sigma_{0,1}^2\varepsilon_2\right)/(\sigma_{0,1}^2 + \sigma_{0,2}^2)$, $\mu_0 = \left(\sigma_{0,2}^2\mu_{0,1} + \sigma_{0,1}^2\mu_{0,2}\right)/(\sigma_{0,1}^2 + \sigma_{0,2}^2)$, $\sigma_0^2 = 1/\left(1/\sigma_{0,1}^2 + 1/\sigma_{0,2}^2\right)$. Hence, multiple signals ε_i can be combined into one signal ε.

Proof of Lemma IA.6. The proof is analogous to the proof of Lemma IA.5 above.